GLOBAL TIME-ANALYTIC STRONG SOLUTIONS FOR A CLASS OF 3-WAVE KINETIC EQUATIONS

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Abstract. We study a class of 3-wave kinetic equations arising in wave turbulence theory, with regularized kernels. For radial, nonnegative initial data, we construct an exact global-in-time strong solution which remains nonnegative and is analytic with respect to time. The proof combines a careful analysis of the resonant interaction surfaces with a time power-series construction and a continuation argument based on the conservation of the energy moment.

1. Introduction

Wave turbulence theory provides a statistical description of the transfer of energy across scales in weakly nonlinear dispersive wave systems. Since its early development, it has become a central framework in mathematical physics, with applications to rotating fluids, magnetohydrodynamics, Alfvén waves in the solar wind, plasma turbulence, and wave processes arising in fusion-related models. The foundations of the theory can be traced back to the pioneering work of Peierls [24], and were further developed through the seminal contributions of Benney and Saffman [4], Zakharov and Falkovich [36], Benney and Newell [3], and Hasselmann [18, 19]. These developments led to the formulation of wave kinetic equations, in particular the classical 3-wave and 4-wave kinetic equations, which describe the resonant exchange and redistribution of energy among weakly interacting modes. In recent years, the rigorous derivation of such kinetic equations from underlying dispersive wave systems has seen major progress, most notably in a series of works by Deng and Hani [8, 9, 11, 12, 10]. For broader accounts of the physical theory, its scaling laws, and its applications, we refer to [21, 25, 37].

In this work, we consider the following 3-wave kinetic equation

$$\begin{aligned} \partial_t f(t,k) &= Q_{3w}[f](k), \quad (t,k) \in (0,\infty)\times\mathbb{R}^d, \\ f(0,k) &= f_0(k), k\in\mathbb{R}^d, \quad d\geq 3 \end{aligned} \tag{1}$$

in which the collision operator is of the form

$$Q_{3w}[f](k) := \iint_{\mathbb{R}^{2d}} \Big[R_{k,k_1,k_2}[f] - R_{k_1,k,k_2}[f] - R_{k_2,k,k_1}[f]\Big]\mathrm{d}k_1\mathrm{d}k_2 \tag{2}$$

with

$$R_{k,k_1,k_2}[f] := V_{k,k_1,k_2}\delta(k-k_1-k_2)\delta(\omega_k-\omega_{k_1}-\omega_{k_2})(f_1f_2 - ff_1 - ff_2) \tag{3}$$

with the short-hand notation $f = f(t,k)$ and $f_j = f(t,k_j)$. The Dirac delta function $\delta(\cdot)$ is to ensure the following resonant conditions for the wavenumbers:

$$k = k_1 + k_2, \qquad \omega_k = \omega_{k_1} + \omega_{k_2}, \tag{4}$$

with ω_k denoting the dispersion relation of the waves.

3-wave kinetic equations appear in a broad range of physical models and have been investigated from several analytical perspectives. They arise, for instance, in the study of phonon interactions in anharmonic crystal lattices [6, 16, 17, 31], capillary wave turbulence [22], and stratified flows in geophysical fluid dynamics [17, 20]. Closely related 3-wave kinetic structures also occur in kinetic

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models connected with Bose–Einstein condensation [1, 5, 15, 13, 14, 23, 22, 27, 29]. Entropy structures, energy cascade and ergodicity of those equations have been studied in [26, 28, 30]. Alongside these analytical developments, there has been growing numerical interest in 3-wave kinetic dynamics, including recent computational and data-driven studies [2, 7, 32, 33, 34, 35].

We consider the following power-law dispersion relation

$$\omega_k = \omega(k) = |k|^\gamma, \qquad 1 < \gamma < 2. \tag{5}$$

The precise form of the collision kernel V_{k,k_1,k_2} is

$$V_{k,k_1,k_2} := \mathscr{M}(k)\,\mathscr{M}(k_1)\,\mathscr{M}(k_2), \tag{6}$$

where

$$\mathscr{M}(k) := |k|^{\gamma+d-2}\exp(-\mathscr{A}|k|), \tag{7}$$

for a constant number $\mathscr{A} > 0$.

Remark 1 (On the choice of the collision kernel). The kernel (6)–(7) should be understood as a model interaction kernel. The present choice of $\mathscr{M}(k)$ is made in order to retain the resonant 3-wave collision structure while providing a nonnegative, radial, separable, and high-frequency localized kernel. The exponential factor gives strong decay at large wave numbers, whereas the power-law prefactor controls the behavior near the origin and is compatible with the weighted estimates used below. Thus the results of this paper should be interpreted as applying to a regularized model class of 3-wave kinetic equations.

The main contribution of this paper is the construction of a global-in-time, nonnegative solution to the 3-wave kinetic equation (1) which is analytic with respect to time. The theorem is proved for a regularized radial model, whose kernel retains the essential resonant 3-wave structure while allowing uniform control of the surface collision integrals.

Definition 1 (Finite-energy strong radial solution). *Let $T \in (0,\infty]$. A function*

$$f : [0,T) \times \mathbb{R}^d \to \mathbb{R}$$

is called a finite-energy strong radial solution of the 3-wave kinetic equation

$$\partial_t f = Q_{3w}[f], \qquad f(0,\cdot) = f_0,$$

on $[0,T)$ if the following conditions hold.

(1) *For every $t \in [0,T)$, the function $f(t,\cdot)$ is radial. That is, there exists a measurable function*

$$F(t,\cdot) : [0,\infty) \to \mathbb{R}$$

such that

$$f(t,k) = F(t,|k|)$$

for a.e. $k \in \mathbb{R}^d$.

(2) *One has*

$$f \in C\big([0,T); L^\infty_{\mathrm{rad}}(\mathbb{R}^d)\big),$$

and for every $0 < T_0 < T$,

$$\sup_{0\le t\le T_0} \int_{\mathbb{R}^d} |f(t,k)|\,\omega(k)\,\mathrm{d}k < \infty.$$

Equivalently, in radial variables,

$$\sup_{0\le t\le T_0} \int_0^\infty |F(t,r)|\,r^{\gamma+d-1}\,\mathrm{d}r < \infty.$$

(3) *For every $t \in [0,T)$, the surface integrals defining $Q_{3w}[f(t)]$ are absolutely convergent for a.e. $k \in \mathbb{R}^d$, and*

$$Q_{3w}[f(t)] \in L^\infty_{\mathrm{rad}}(\mathbb{R}^d).$$

Moreover, $t \mapsto Q_{3w}[f(t)]$ belongs to $C\big([0,T); L^\infty_{\mathrm{rad}}(\mathbb{R}^d)\big)$.

(4) *The map* $t \mapsto f(t,\cdot)$ *belongs to* $C^1\big([0,T); L^\infty_{\mathrm{rad}}(\mathbb{R}^d)\big)$, *and satisfies*

$$\frac{d}{dt} f(t,\cdot) = Q_{3w}[f(t)]$$

in $L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$ *for every* $t \in [0,T)$.

(5) *One has*

$$f(0,\cdot) = f_0$$

in $L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$.

If, in addition,

$$f(t,k) \geq 0$$

for a.e. $k \in \mathbb{R}^d$ *and every* $t \in [0,T)$, *then* f *is called a nonnegative strong radial solution.*

Theorem 2. *Assume that* $d \geq 3$, $1 < \gamma < 2$, *and let*

$$\omega(k) = |k|^\gamma, \qquad \mathscr{M}(k) = |k|^{\gamma+d-2}\exp(-\mathscr{A}|k|), \qquad \mathscr{A} > 0.$$

Let the initial datum f_0 *be radial and nonnegative, and assume that*

$$M := \|f_0\|_{L^\infty(\mathbb{R}^d)} < \infty$$

and

$$\mathcal{M}_1 := \int_0^\infty f_0(r)\, r^{\gamma+d-1}\,\mathrm{d}r < \infty.$$

Then the 3-wave kinetic equation (1), *with collision operator defined by* (2), *admits an exact unique global strong solution* $f(t,k)$ *such that*

$$f(t,k) \geq 0$$

for a.e. $k \in \mathbb{R}^d$.

Moreover, $t \mapsto f(t,\cdot)$ *is real analytic as an* $L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$*-valued map on* $(0,\infty)$, *and is right-analytic at* $t=0$ *in* $L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$. *More precisely, for every* $t_0 > 0$, *there exist* $\tau_{t_0} > 0$ *and coefficients* $\mathcal{A}_n^{(t_0)} \in L^\infty_{\mathrm{rad}}(\mathbb{R}^d), n = 0,1,2,\ldots,$ *such that*

$$f(t_0+s,\cdot) = \sum_{n=0}^\infty s^n \mathcal{A}_n^{(t_0)} \quad \text{in } L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$$

for every $|s| < \tau_{t_0}$. *At* $t_0 = 0$, *there exist* $\tau_0 > 0$ *and coefficients* $\mathcal{A}_n^{(0)} \in L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$ *such that*

$$f(s,\cdot) = \sum_{n=0}^\infty s^n \mathcal{A}_n^{(0)} \quad \text{in } L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$$

for every $0 \leq s < \tau_0$.

Let us now turn to the collision operator (2). The collision operator involves surface integrals. Precisely, we introduce functions

$$\mathscr{H}_0^k(v) := \omega_{k-v} + \omega_v - \omega_k, \qquad \mathscr{H}_1^k(v) := \omega_k + \omega_v - \omega_{k+v}. \tag{8}$$

and the energy surfaces, dictated by the resonant conditions (4),

$$\mathscr{S}_k := \Big\{ v \in \mathbb{R}^d : \mathscr{H}_0^k(v) = 0 \Big\}, \quad \mathscr{S}_k' := \Big\{ v \in \mathbb{R}^d : \mathscr{H}_1^k(v) = 0 \Big\}. \tag{9}$$

The collision operator $Q_{3w}[f]$ then reduces to

$$Q_{3w}[f](k) = \int_{\mathscr{S}_k} R_{k,k-k_2,k_2}[f] \frac{d\sigma(k_2)}{|\nabla \mathscr{H}_0^k(k_2)|} - 2\int_{\mathscr{S}_k'} R_{k+k_2,k,k_2}[f] \frac{d\sigma(k_2)}{|\nabla \mathscr{H}_1^k(k_2)|}. \tag{10}$$

2. Resonance Manifolds

Our first step is to study the surface integrals.

Proposition 3 (Surface estimates on $\mathscr{S}_{k_1}$ and $\mathscr{S}'_{k_1}$)**.** *Assume that $d \geq 3$, $1 < \gamma < 2$, and*

$$\omega(k) = |k|^\gamma, \qquad \mathscr{M}(k) = |k|^{\gamma+d-2}\exp(-\mathscr{A}|k|), \qquad \mathscr{A} > 0.$$

Let $F : \mathbb{R}^d \to \mathbb{R}$ be radial, namely $F(k) = \mathcal{F}(|k|)$. Then there exist constants $\mathcal{C}_S, \mathcal{D}_S, \mathcal{C}_{S'}, \mathcal{D}_{S'} > 0$, depending only on d, γ, and $\mathscr{A}$, such that the following estimates hold uniformly for all $k_1 \in \mathbb{R}^d\backslash\{0\}$.

First,

$$\left|\int_{\mathscr{S}_{k_1}} \mathrm{d}\sigma(k_2)\, \frac{\mathscr{M}(k_1-k_2)\mathscr{M}(k_2)\mathscr{M}(k_1)F(k_2)}{|\nabla_{k_2}\mathscr{H}_0^{k_1}(k_2)|}\right| \leq \mathcal{C}_S \int_0^{|k_1|} \mathrm{d}r\, |\mathcal{F}(r)|\,\omega(r)\, r^{d-1},$$

and

$$\left|\int_{\mathscr{S}_{k_1}} \mathrm{d}\sigma(k_2)\, \frac{\mathscr{M}(k_1-k_2)\mathscr{M}(k_2)\mathscr{M}(k_1)F(k_2)}{|\nabla_{k_2}\mathscr{H}_0^{k_1}(k_2)|}\right| \leq \mathcal{D}_S \int_0^{|k_1|} \mathrm{d}r\, |\mathcal{F}(r)|\, r^{\gamma+d-1}\exp(-\mathscr{A}r).$$

Second,

$$\left|\int_{\mathscr{S}'_{k_1}} \mathrm{d}\sigma(k_2)\, \frac{\mathscr{M}(k_1+k_2)\mathscr{M}(k_2)\mathscr{M}(k_1)F(k_2)}{|\nabla_{k_2}\mathscr{H}_1^{k_1}(k_2)|}\right| \leq \mathcal{C}_{S'} \int_0^{\infty} \mathrm{d}r\, |\mathcal{F}(r)|\,\omega(r)\, r^{d-1},$$

and

$$\left|\int_{\mathscr{S}'_{k_1}} \mathrm{d}\sigma(k_2)\, \frac{\mathscr{M}(k_1+k_2)\mathscr{M}(k_2)\mathscr{M}(k_1)F(k_2)}{|\nabla_{k_2}\mathscr{H}_1^{k_1}(k_2)|}\right| \leq \mathcal{D}_{S'} \int_0^{\infty} \mathrm{d}r\, |\mathcal{F}(r)|\, r^{\gamma+d-1}\exp(-\mathscr{A}r).$$

Proof. We define

$$\begin{aligned} A &:= \int_{\mathscr{S}_{k_1}} \mathrm{d}\sigma(k_2)\, \frac{\mathscr{M}(k_1-k_2)\,\mathscr{M}(k_2)\,\mathscr{M}(k_1)\,F(k_2)}{|\nabla\mathscr{H}_0^{k_1}|}, \\ B &:= \int_{\mathscr{S}'_{k_1}} \mathrm{d}\sigma(k_2)\, \frac{\mathscr{M}(k_1+k_2)\,\mathscr{M}(k_2)\,\mathscr{M}(k_1)\,F(k_2)}{|\nabla\mathscr{H}_1^{k_1}|}. \end{aligned} \tag{11}$$

We then estimate A and B separately.

Step 1: Analyzing A.

Step 1 is divided into several substeps.

Substep 1.1. In this step, we will parametrize $\mathscr{S}_{k_1}$.

Assume first that $k_1 \neq 0$, and set

$$K := |k_1|.$$

Since the dispersion relation is radial, the surface

$$\mathscr{S}_{k_1} := \{v \in \mathbb{R}^d : \omega(k_1 - v) + \omega(v) = \omega(k_1)\}$$

is invariant under rotations fixing the axis spanned by k_1.

Let $E_1, \ldots, E_{d-1}$ be an orthonormal basis of $k_1^\perp$. For

$$q = (q_1, \ldots, q_{d-1}) \in S^{d-2} := \{q \in \mathbb{R}^{d-1} : |q| = 1\},$$

we define

$$e_q := \sum_{j=1}^{d-1} q_j E_j.$$

Then $e_q \in k_1^\perp, |e_q| = 1$. We introduce the parametrization

$$v = \xi k_1 + \eta e_q, \qquad 0 < \xi < 1, \qquad \eta \geq 0, \qquad q \in S^{d-2}.$$

Since $e_q \perp k_1$, we have

$$|v|^2 = \xi^2 K^2 + \eta^2, \qquad |k_1 - v|^2 = (1-\xi)^2 K^2 + \eta^2.$$

We define

$$\mathscr{H}_0^{k_1}(\xi,\eta) := \omega(k_1 - v) + \omega(v) - \omega(k_1).$$

Using $\omega(k) = |k|^\gamma$, we obtain

$$\mathscr{H}_0^{k_1}(\xi,\eta) = \big((1-\xi)^2K^2 + \eta^2\big)^{\gamma/2} + \big(\xi^2K^2 + \eta^2\big)^{\gamma/2} - K^\gamma.$$

We now show that for every $0 < \xi < 1$, there exists a unique $\eta = \eta_K(\xi) > 0$ such that

$$\mathscr{H}_0^{k_1}(\xi, \eta_K(\xi)) = 0.$$

Indeed, we have

$$\mathscr{H}_0^{k_1}(\xi,0) = K^\gamma\big((1-\xi)^\gamma + \xi^\gamma - 1\big) < 0,$$

because $1 < \gamma < 2$, while $\mathscr{H}_0^{k_1}(\xi,\eta) \longrightarrow +\infty$ as $\eta \to +\infty$. Thus the existence result follows by continuity. Now, we compute

$$\partial_\eta \mathscr{H}_0^{k_1}(\xi,\eta) = \gamma\eta\left[\big((1-\xi)^2K^2 + \eta^2\big)^{\gamma/2-1} + \big(\xi^2K^2 + \eta^2\big)^{\gamma/2-1}\right].$$

Hence, we conclude $\partial_\eta \mathscr{H}_0^{k_1}(\xi,\eta) > 0$ for every $\eta > 0$. Therefore $\mathscr{H}_0^{k_1}(\xi,\eta)$ is strictly increasing in η, and the zero $\eta_K(\xi)$ is unique.

Since $\partial_\eta \mathscr{H}_0^{k_1}(\xi, \eta_K(\xi)) > 0$, the implicit function theorem implies that η_K is smooth for $\xi \in (0,1)$. Moreover, by homogeneity, we may write

$$\eta_K(\xi) = K\rho_\gamma(\xi),$$

where $\rho_\gamma(\xi) > 0$ is the unique solution of

$$\big((1-\xi)^2 + \rho_\gamma(\xi)^2\big)^{\gamma/2} + \big(\xi^2 + \rho_\gamma(\xi)^2\big)^{\gamma/2} = 1.$$

Consequently, we obtain

$$\mathscr{S}_{k_1} \setminus \{0, k_1\} = \left\{\xi k_1 + K\rho_\gamma(\xi)e_q : 0 < \xi < 1,\ q \in S^{d-2}\right\}.$$

Substep 1.2.

We next record two elementary properties of the function η_K. Namely, η_K is symmetric with respect to $\xi = 1/2$, admits the continuous extension $\eta_K(0) = \eta_K(1) = 0$, and is strictly increasing on $(0, 1/2)$. Hence it is invertible on $(0, 1/2)$.

The symmetry follows from the identity

$$\mathscr{H}_0^{k_1}(\xi,\eta) = \mathscr{H}_0^{k_1}(1-\xi,\eta).$$

By uniqueness of the zero in the η-variable, we obtain $\eta_K(\xi) = \eta_K(1-\xi), 0 < \xi < 1$. Moreover, from the defining equation

$$\big((1-\xi)^2K^2 + \eta_K(\xi)^2\big)^{\gamma/2} + \big(\xi^2K^2 + \eta_K(\xi)^2\big)^{\gamma/2} = K^\gamma,$$

one sees that η_K extends continuously to the endpoints with $\eta_K(0) = \eta_K(1) = 0$.

It remains to prove monotonicity on $(0, 1/2)$. Let

$$v_\xi := \xi k_1 + \eta_K(\xi)e_q \in \mathscr{S}_{k_1}.$$

We also have

$$\mathscr{H}_0^{k_1}(\xi, \eta_K(\xi)) = 0.$$

Differentiating this identity with respect to ξ, we obtain

$$0 = \partial_\xi \mathscr{H}_0^{k_1}(\xi, \eta_K(\xi)) + \eta_K'(\xi)\partial_\eta \mathscr{H}_0^{k_1}(\xi, \eta_K(\xi)).$$

Equivalently, we have

$$0 = -K^2\frac{\omega'(|k_1 - v_\xi|)}{|k_1 - v_\xi|} + \big(K^2\xi + \eta_K(\xi)\eta_K'(\xi)\big)\left[\frac{\omega'(|k_1 - v_\xi|)}{|k_1 - v_\xi|} + \frac{\omega'(|v_\xi|)}{|v_\xi|}\right].$$

Therefore, we get

$$\eta_K(\xi)\eta_K'(\xi) = K^2 \frac{(1-\xi)\dfrac{\omega'(|k_1-v_\xi|)}{|k_1-v_\xi|} - \xi\dfrac{\omega'(|v_\xi|)}{|v_\xi|}}{\dfrac{\omega'(|k_1-v_\xi|)}{|k_1-v_\xi|} + \dfrac{\omega'(|v_\xi|)}{|v_\xi|}}.$$

We now use the power-law form $\omega(r) = r^\gamma$. We set

$$r_1 := |k_1 - v_\xi|, \qquad r_2 := |v_\xi|.$$

Then, we find

$$\frac{\omega'(r_1)}{r_1} = \gamma r_1^{\gamma-2}, \qquad \frac{\omega'(r_2)}{r_2} = \gamma r_2^{\gamma-2}.$$

For $0 < \xi < 1/2$, we have $r_1 > r_2$. Moreover,

$$\frac{r_1}{r_2} = \left(\frac{(1-\xi)^2K^2 + \eta_K(\xi)^2}{\xi^2K^2 + \eta_K(\xi)^2}\right)^{1/2} < \frac{1-\xi}{\xi}.$$

Since $1 < \gamma \le 2$, we have $0 \le 2-\gamma < 1$. Hence, we obtain

$$\frac{\dfrac{\omega'(r_2)}{r_2}}{\dfrac{\omega'(r_1)}{r_1}} = \left(\frac{r_1}{r_2}\right)^{2-\gamma} < \frac{1-\xi}{\xi}.$$

Thus, we have

$$(1-\xi)\frac{\omega'(r_1)}{r_1} - \xi\frac{\omega'(r_2)}{r_2} > 0.$$

Consequently, we have $\eta_K(\xi)\eta_K'(\xi) > 0, 0 < \xi < \frac{1}{2}$. Since $\eta_K(\xi) > 0$ for $0 < \xi < 1$, it follows that $\eta_K'(\xi) > 0, 0 < \xi < \frac{1}{2}$. Therefore η_K is strictly increasing, and hence invertible, on $(0, 1/2)$.

Substep 1.3.

We next compute the induced surface measure under the parametrization

$$v(\xi, q) = \xi k_1 + \eta_K(\xi)e_q, \qquad 0 < \xi < 1, \qquad q \in S^{d-2}.$$

Here $e_q \in k_1^\perp$, $|e_q| = 1$, and $\mathrm{d}\sigma_{d-2}(q)$ denotes the standard surface measure on S^{d-2}. We again use the notations $K := |k_1|, v_\xi := v(\xi, q)$, and introduce the abbreviations

$$r_1 := |k_1 - v_\xi|, \qquad r_2 := |v_\xi|, \qquad a := \frac{\omega'(r_1)}{r_1}, \qquad b := \frac{\omega'(r_2)}{r_2}.$$

Since

$$|v_\xi|^2 = \xi^2K^2 + \eta_K(\xi)^2, \qquad |k_1 - v_\xi|^2 = (1-\xi)^2K^2 + \eta_K(\xi)^2,$$

the identity

$$\mathscr{H}_0^{k_1}(\xi, \eta_K(\xi)) = 0$$

is equivalent to $\omega(r_1) + \omega(r_2) = \omega(K)$.

Differentiating this identity with respect to ξ, we obtain

$$0 = K^2\big[\xi b + (\xi - 1)a\big] + \eta_K(\xi)\eta_K'(\xi)(a+b).$$

From here, we find

$$\frac{1}{2}\partial_\xi \eta_K(\xi)^2 = K^2\frac{(1-\xi)a - \xi b}{a+b}.$$

Thus, we get

$$\partial_\xi \eta_K(\xi)^2 = 2K^2 \frac{(1-\xi)\dfrac{\omega'(r_1)}{r_1} - \xi\dfrac{\omega'(r_2)}{r_2}}{\dfrac{\omega'(r_1)}{r_1} + \dfrac{\omega'(r_2)}{r_2}}.$$

We now compute the surface element. Since the angular directions are orthogonal to both k_1 and e_q, the induced metric gives

$$\mathrm{d}\sigma(v) = \eta_K(\xi)^{d-2}\sqrt{K^2 + \eta_K'(\xi)^2}\,\mathrm{d}\xi\,\mathrm{d}\sigma_{d-2}(q).$$

Equivalently, we obtain

$$\mathrm{d}\sigma(v) = \eta_K(\xi)^{d-3}\left(K^2\eta_K(\xi)^2 + \frac{1}{4}\left|\partial_\xi\eta_K(\xi)^2\right|^2\right)^{1/2}\mathrm{d}\xi\,\mathrm{d}\sigma_{d-2}(q).$$

Next, we observe that

$$\nabla_v\mathscr{H}_0^{k_1} = \frac{v_\xi - k_1}{|k_1 - v_\xi|}\omega'(|k_1 - v_\xi|) + \frac{v_\xi}{|v_\xi|}\omega'(|v_\xi|).$$

Using $v_\xi = \xi k_1 + \eta_K(\xi)e_q$, we get

$$\nabla_v\mathscr{H}_0^{k_1} = \big[\xi b + (\xi-1)a\big]k_1 + \eta_K(\xi)(a+b)e_q.$$

Therefore, we find

$$|\nabla_v\mathscr{H}_0^{k_1}|^2 = K^2\big[\xi b + (\xi-1)a\big]^2 + \eta_K(\xi)^2(a+b)^2.$$

Using

$$K^2\big[\xi b + (\xi-1)a\big] = -\eta_K(\xi)\eta_K'(\xi)(a+b),$$

we obtain

$$|\nabla_v\mathscr{H}_0^{k_1}| = \frac{\left(K^2\eta_K(\xi)^2 + \frac{1}{4}\left|\partial_\xi\eta_K(\xi)^2\right|^2\right)^{1/2}}{K}(a+b).$$

Consequently, we have

$$\frac{\mathrm{d}\sigma(v)}{|\nabla_v\mathscr{H}_0^{k_1}|} = \frac{K\eta_K(\xi)^{d-3}}{a+b}\mathrm{d}\xi\,\mathrm{d}\sigma_{d-2}(q).$$

We now introduce the radial variable

$$u := |v_\xi| = r_2.$$

Since $u^2 = \xi^2K^2 + \eta_K(\xi)^2$, we have

$$\partial_\xi u^2 = 2\xi K^2 + \partial_\xi\eta_K(\xi)^2.$$

Using the formula for $\partial_\xi\eta_K(\xi)^2$, we find $\partial_\xi u^2 = \frac{2K^2a}{a+b}$. Hence

$$\mathrm{d}\xi = \frac{a+b}{K^2a}\,u\,\mathrm{d}u.$$

Therefore, we have

$$\frac{\mathrm{d}\sigma(v)}{|\nabla_v\mathscr{H}_0^{k_1}|} = \eta_K(\xi(u))^{d-3}\frac{u}{Ka}\mathrm{d}u\,\mathrm{d}\sigma_{d-2}(q).$$

Since $a = \omega'(r_1)/r_1$, this becomes

$$\frac{\mathrm{d}\sigma(v)}{|\nabla_v\mathscr{H}_0^{k_1}|} = \eta_K(\xi(u))^{d-3}\frac{u\,r_1(u)}{K\,\omega'(r_1(u))}\mathrm{d}u\,\mathrm{d}\sigma_{d-2}(q).$$

Thus, for a radial function F, we obtain

$$A = \int_0^K \mathrm{d}u\int_{S^{d-2}}\mathrm{d}\sigma_{d-2}(q)\,\eta_K(\xi(u))^{d-3}\frac{u\,r_1(u)}{K\,\omega'(r_1(u))}\mathscr{M}(k_1 - v(u,q))\mathscr{M}(v(u,q))\mathscr{M}(k_1)F(u),$$

where

$$v(u,q) = \xi(u)k_1 + \eta_K(\xi(u))e_q, \qquad u = |v(u,q)|,$$

and $r_1(u) = |k_1 - v(u,q)|$. Equivalently, since $\omega(r_1(u)) + \omega(u) = \omega(K)$, one has $r_1(u) = \omega^{-1}\big(\omega(K) - \omega(u)\big)$. For the power law $\omega(r) = r^\gamma$, this becomes

$$r_1(u) = \big(K^\gamma - u^\gamma\big)^{1/\gamma}.$$

Substep 1.4.
We now estimate A. Using the representation obtained in the previous substep, we have, for $K = |k_1| > 0$,

$$|A| \le \int_0^K \mathrm{d}u \int_{S^{d-2}} \mathrm{d}\sigma_{d-2}(q)\, \eta_K(\xi(u))^{d-3} \frac{u\, r_1(u)}{K\, \omega'(r_1(u))} \mathscr{M}(k_1 - v(u,q))\mathscr{M}(v(u,q))\mathscr{M}(k_1)|F(u)|.$$

Here $u = |v(u,q)|, r_1(u) = |k_1 - v(u,q)|$, and, on the resonant surface,

$$\omega(r_1(u)) + \omega(u) = \omega(K).$$

Since $\omega(r) = r^\gamma$, we have

$$r_1(u) = \big(K^\gamma - u^\gamma\big)^{1/\gamma}, \qquad 0 < u < K.$$

Moreover, $\omega'(r_1) = \gamma r_1^{\gamma-1}$. We obtain

$$|A| \le C_d \int_0^K \mathrm{d}u\, \eta_K(\xi(u))^{d-3} r_1(u)^d K^{\gamma+d-3} u^{\gamma+d-1} \exp\big(-\mathscr{A}(r_1(u) + K + u)\big)|F(u)|.$$

Since $0 \le u \le K$, $0 \le r_1(u) \le K$, and $0 \le \eta_K(\xi(u)) \le K$, we have, for $d \ge 3$,

$$\eta_K(\xi(u))^{d-3} r_1(u)^d K^{\gamma+d-3} \exp\big(-\mathscr{A}(r_1(u) + K)\big) \le K^{\gamma+3d-6} \exp(-\mathscr{A}K) \le C.$$

Consequently, we get

$$|A| \le \mathcal{D}_S \int_0^K \mathrm{d}u\, |F(u)|\, u^{\gamma+d-1} \exp(-\mathscr{A}u).$$

In particular, since $\exp(-\mathscr{A}u) \le 1$, we bound

$$|A| \le \mathcal{C}_S \int_0^K \mathrm{d}u\, |F(u)|\, u^{\gamma+d-1} \ = \mathcal{C}_S \int_0^K \mathrm{d}u\, |F(u)|\, \omega(u)\, u^{d-1}.$$

Thus, we have

$$|A| \le \mathcal{C}_S \int_0^{|k_1|} \mathrm{d}u\, |F(u)|\, \omega(u)\, u^{d-1},$$

and also

$$|A| \le \mathcal{D}_S \int_0^{|k_1|} \mathrm{d}u\, |F(u)|\, u^{\gamma+d-1} \exp(-\mathscr{A}u).$$

Step 2: Analyzing B.
We also divide Step 2 into smaller substeps.
Substep 2.1. We now consider the second resonant surface

$$\mathscr{S}'_{k_1} := \{v \in \mathbb{R}^d : \omega(k_1 + v) - \omega(k_1) - \omega(v) = 0\}.$$

Equivalently, up to a harmless sign in the defining function, we set

$$\mathscr{H}_1^{k_1}(v) := \omega(k_1 + v) - \omega(k_1) - \omega(v).$$

Then, we get

$$\mathscr{S}'_{k_1} = \{v \in \mathbb{R}^d : \mathscr{H}_1^{k_1}(v) = 0\}.$$

We assume first that $k_1 \ne 0$, and set

$$K := |k_1|.$$

We claim that, for $1 < \gamma < 2$, the nontrivial part of $\mathscr{S}'_{k_1}$ can be parametrized by

$$v = \xi k_1 + \eta_K(\xi) e_q, \qquad 0 < \xi < \infty, \qquad q \in S^{d-2},$$

where $e_q \in k_1^\perp$, $|e_q| = 1$, and where $\eta_K(\xi) > 0$ is uniquely determined by the resonance condition.
We first explain why only $\xi > 0$ occurs. We write, more generally,

$$v = \zeta k_1 + \eta e_q, \qquad \zeta \in \mathbb{R}, \qquad \eta \ge 0.$$

Then

$$|v|^2 = \zeta^2 K^2 + \eta^2, \qquad |k_1 + v|^2 = (1 + \zeta)^2 K^2 + \eta^2.$$

If $\zeta \leq -1/2$, then $|k_1 + v| \leq |v|$, and hence $\omega(k_1 + v) - \omega(k_1) - \omega(v) < 0$. Thus no point with $\zeta \leq -1/2$ lies on the nontrivial resonant surface.

Next suppose that $-1/2 < \zeta \leq 0$. We define

$$\Psi_\zeta(\eta) := \big((1+\zeta)^2K^2 + \eta^2\big)^{\gamma/2} - K^\gamma - \big(\zeta^2K^2 + \eta^2\big)^{\gamma/2}.$$

At $\eta = 0$, we have

$$\Psi_\zeta(0) = K^\gamma\big((1+\zeta)^\gamma - 1 - |\zeta|^\gamma\big) \leq 0,$$

with equality only at the endpoint $\zeta = 0$. Moreover, for $\eta > 0$,

$$\partial_\eta \Psi_\zeta(\eta) = \gamma\eta\left[\big((1+\zeta)^2K^2 + \eta^2\big)^{\gamma/2-1} - \big(\zeta^2K^2 + \eta^2\big)^{\gamma/2-1}\right] \leq 0,$$

because $(1+\zeta)^2K^2 + \eta^2 \geq \zeta^2K^2 + \eta^2$ and $\frac{\gamma}{2} - 1 < 0$. Hence $\Psi_\zeta(\eta) < 0$ for $\eta > 0$. Therefore, apart from the degenerate endpoint $v = 0$, no point with $\zeta \leq 0$ belongs to $\mathscr{S}'_{k_1}$. Thus the nontrivial surface is parametrized by $\xi > 0$.

We now fix $\xi > 0$ and prove the existence and uniqueness of $\eta_K(\xi) > 0$. We define

$$\Phi_\xi(\eta) := \big((1+\xi)^2K^2 + \eta^2\big)^{\gamma/2} - K^\gamma - \big(\xi^2K^2 + \eta^2\big)^{\gamma/2}.$$

The resonance condition $\omega(k_1 + v) = \omega(k_1) + \omega(v)$ is equivalent to $\Phi_\xi(\eta) = 0$. At $\eta = 0$, we have

$$\Phi_\xi(0) = K^\gamma\big((1+\xi)^\gamma - 1 - \xi^\gamma\big).$$

Since $1 < \gamma < 2$, the function

$$\xi \mapsto (1+\xi)^\gamma - \xi^\gamma$$

is strictly increasing on $(0, \infty)$ and has value 1 at $\xi = 0$. Hence, we have

$$(1+\xi)^\gamma - 1 - \xi^\gamma > 0,$$

and therefore $\Phi_\xi(0) > 0$.

On the other hand, since $1 < \gamma < 2$,

$$\big((1+\xi)^2K^2 + \eta^2\big)^{\gamma/2} - \big(\xi^2K^2 + \eta^2\big)^{\gamma/2} \longrightarrow 0 \qquad \text{as } \eta \to +\infty.$$

Consequently, we obtain

$$\lim_{\eta\to+\infty} \Phi_\xi(\eta) = -K^\gamma < 0.$$

By the continuity of the function, there exists at least one positive zero of Φ_ξ.

We compute

$$\partial_\eta \Phi_\xi(\eta) = \gamma\eta\left[\big((1+\xi)^2K^2 + \eta^2\big)^{\gamma/2-1} - \big(\xi^2K^2 + \eta^2\big)^{\gamma/2-1}\right].$$

Since $(1+\xi)^2K^2 + \eta^2 > \xi^2K^2 + \eta^2$ and $\frac{\gamma}{2} - 1 < 0$, we have

$$\big((1+\xi)^2K^2 + \eta^2\big)^{\gamma/2-1} < \big(\xi^2K^2 + \eta^2\big)^{\gamma/2-1}.$$

Therefore $\partial_\eta \Phi_\xi(\eta) < 0$for every $\eta > 0$. Thus $\eta \mapsto \Phi_\xi(\eta)$ is strictly decreasing on $(0, \infty)$. Since it starts positive at $\eta = 0$ and tends to a negative limit as $\eta \to +\infty$, it has exactly one positive zero. We denote this zero by

$$\eta = \eta_K(\xi).$$

Moreover, we have

$$\partial_\eta \Phi_\xi(\eta_K(\xi)) < 0,$$

so the implicit function theorem implies that $\xi \mapsto \eta_K(\xi)$ is smooth on $(0, \infty)$.

By homogeneity, we may write $\eta_K(\xi) = K\rho_\gamma(\xi)$, where $\rho_\gamma(\xi) > 0$ is the unique solution of

$$\big((1+\xi)^2 + \rho_\gamma(\xi)^2\big)^{\gamma/2} = 1 + \big(\xi^2 + \rho_\gamma(\xi)^2\big)^{\gamma/2}.$$

Consequently, we have

$$\mathscr{S}'_{k_1} \setminus \{0\} = \left\{\xi k_1 + K\rho_\gamma(\xi)e_q : 0 < \xi < \infty,\ q \in S^{d-2}\right\}.$$

We now compute the surface element in this parametrization. We set

$$v_\xi := \xi k_1 + \eta_K(\xi) e_q, \qquad r_+ := |k_1 + v_\xi|, \qquad r := |v_\xi|.$$

We also introduce

$$a := \frac{\omega'(r_+)}{r_+}, \qquad b := \frac{\omega'(r)}{r}.$$

On $\mathscr{S}'_{k_1}$, one has

$$\omega(r_+) = \omega(K) + \omega(r).$$

Since $\omega(r) = r^\gamma$, we have $a = \gamma r_+^{\gamma-2}, b = \gamma r^{\gamma-2}$. Moreover $r_+ > r$, and since $\gamma - 2 < 0$, it follows that $0 < a < b$. In particular, $b - a > 0$.

Differentiating the identity $\mathscr{H}_1^{k_1}(v_\xi) = 0$ with respect to ξ, we obtain

$$0 = K^2\big[(1+\xi)a - \xi b\big] + \eta_K(\xi)\eta_K'(\xi)(a-b).$$

Equivalently, we have

$$\eta_K(\xi)\eta_K'(\xi) = K^2 \frac{(1+\xi)a - \xi b}{b-a}.$$

We next compute the surface element. The induced surface measure is

$$\mathrm{d}\sigma(v) = \eta_K(\xi)^{d-2}\sqrt{K^2 + \eta_K'(\xi)^2}\, \mathrm{d}\xi \, \mathrm{d}\sigma_{d-2}(q).$$

Equivalently, we have

$$\mathrm{d}\sigma(v) = \eta_K(\xi)^{d-3}\left(K^2\eta_K(\xi)^2 + \frac{1}{4}\left|\partial_\xi \eta_K(\xi)^2\right|^2\right)^{1/2} \mathrm{d}\xi\, \mathrm{d}\sigma_{d-2}(q).$$

Moreover, we observe that

$$\nabla_v \mathscr{H}_1^{k_1}(v_\xi) = \frac{k_1 + v_\xi}{|k_1 + v_\xi|}\omega'(|k_1 + v_\xi|) - \frac{v_\xi}{|v_\xi|}\omega'(|v_\xi|).$$

Using

$$v_\xi = \xi k_1 + \eta_K(\xi) e_q,$$

this becomes

$$\nabla_v \mathscr{H}_1^{k_1}(v_\xi) = \big[(1+\xi)a - \xi b\big] k_1 + \eta_K(\xi)(a-b) e_q.$$

Therefore, we find

$$|\nabla_v \mathscr{H}_1^{k_1}(v_\xi)|^2 = K^2\big[(1+\xi)a - \xi b\big]^2 + \eta_K(\xi)^2(a-b)^2.$$

Using

$$K^2\big[(1+\xi)a - \xi b\big] = -\eta_K(\xi)\eta_K'(\xi)(a-b),$$

we obtain

$$|\nabla_v \mathscr{H}_1^{k_1}(v_\xi)| = \frac{\left(K^2\eta_K(\xi)^2 + \frac{1}{4}\left|\partial_\xi \eta_K(\xi)^2\right|^2\right)^{1/2}}{K}|b-a|.$$

Since $b - a > 0$, this gives

$$\frac{\mathrm{d}\sigma(v)}{|\nabla_v \mathscr{H}_1^{k_1}(v)|} = \frac{K\eta_K(\xi)^{d-3}}{b-a}\mathrm{d}\xi\, \mathrm{d}\sigma_{d-2}(q).$$

We now introduce the radial variable

$$u := |v_\xi| = r.$$

Since $u^2 = \xi^2 K^2 + \eta_K(\xi)^2$, we have

$$\partial_\xi u^2 = 2\xi K^2 + \partial_\xi \eta_K(\xi)^2.$$

Using the differentiated resonance equation, we obtain

$$\partial_\xi u^2 = \frac{2K^2 a}{b-a}.$$

In particular, $\partial_\xi u^2 > 0$. Thus the map $\xi \mapsto u$ is strictly increasing. Moreover, $u \to 0$ as $\xi \downarrow 0$, while $u \to +\infty$ as $\xi \to +\infty$. Hence u gives a valid change of variables from $(0,\infty)_\xi$ onto $(0,\infty)_u$.

Since

$$2u\,\mathrm{d}u = \frac{2K^2 a}{b-a}\,\mathrm{d}\xi,$$

we have

$$\mathrm{d}\xi = \frac{b-a}{K^2 a}\,u\,\mathrm{d}u.$$

Consequently, we find

$$\frac{\mathrm{d}\sigma(v)}{|\nabla_v \mathscr{H}_1^{k_1}(v)|} = \eta_K(\xi(u))^{d-3}\frac{u}{Ka}\mathrm{d}u\,\mathrm{d}\sigma_{d-2}(q).$$

Since $a = \frac{\omega'(r_+)}{r_+}$, this becomes

$$\frac{\mathrm{d}\sigma(v)}{|\nabla_v \mathscr{H}_1^{k_1}(v)|} = \eta_K(\xi(u))^{d-3}\frac{u\,r_+(u)}{K\,\omega'(r_+(u))}\mathrm{d}u\,\mathrm{d}\sigma_{d-2}(q).$$

Therefore, for a radial function $F(k) = \mathcal{F}(|k|)$, we obtain

$$\begin{aligned} B = \int_0^\infty \mathrm{d}u \int_{S^{d-2}} \mathrm{d}\sigma_{d-2}(q)\,\eta_K(\xi(u))^{d-3}\frac{u\,r_+(u)}{K\,\omega'(r_+(u))} \\ \times \mathscr{M}(k_1+v(u,q))\mathscr{M}(v(u,q))\mathscr{M}(k_1)\mathcal{F}(u), \end{aligned}$$

where

$$v(u,q) = \xi(u)k_1 + \eta_K(\xi(u))e_q, \qquad u = |v(u,q)|,$$

and $r_+(u) := |k_1 + v(u,q)|$. On the resonant surface, we have

$$\omega(r_+(u)) = \omega(K) + \omega(u).$$

Thus, we obtain

$$r_+(u) = \omega^{-1}\big(\omega(K)+\omega(u)\big).$$

For the power law $\omega(r) = r^\gamma$, this becomes

$$r_+(u) = \big(K^\gamma + u^\gamma\big)^{1/\gamma}.$$

Substep 2.2. We now estimate B. From the representation obtained in the previous substep, with $K = |k_1| > 0$, we have

$$\begin{aligned} |B| \le \int_0^\infty \mathrm{d}u \int_{S^{d-2}} \mathrm{d}\sigma_{d-2}(q)\,\eta_K(\xi(u))^{d-3}\frac{u\,r_+(u)}{K\,\omega'(r_+(u))} \\ \times \mathscr{M}(k_1+v(u,q))\mathscr{M}(v(u,q))\mathscr{M}(k_1)|F(u)|. \end{aligned}$$

On the resonant surface, $\omega(r_+(u)) = \omega(K)+\omega(u)$. Since $\omega(r) = r^\gamma$, this gives $r_+(u) = \big(K^\gamma + u^\gamma\big)^{1/\gamma}$. Moreover, we also have $\omega'(r_+) = \gamma r_+^{\gamma-1}$. Using

$$\mathscr{M}(k) = |k|^{\gamma+d-2}\exp(-\mathscr{A}|k|),$$

we obtain

$$|B| \le C_d \int_0^\infty \mathrm{d}u\,\eta_K(\xi(u))^{d-3} r_+(u)^d K^{\gamma+d-3} u^{\gamma+d-1}\exp\big(-\mathscr{A}(r_+(u)+K+u)\big)|F(u)|.$$

For $d \ge 3$, we have

$$0 \le \eta_K(\xi(u)) \le u \le r_+(u), \qquad K \le r_+(u).$$

Therefore, we obtain

$$\eta_K(\xi(u))^{d-3} r_+(u)^d K^{\gamma+d-3}\exp\big(-\mathscr{A}(r_+(u)+K)\big) \le C,$$

where C depends only on d, γ, and $\mathscr{A}$. Consequently, we have

$$|B| \le \mathcal{D}_{S'}\int_0^\infty |F(u)|\,u^{\gamma+d-1}\exp(-\mathscr{A}u)\,\mathrm{d}u.$$

Since $\exp(-\mathscr{A}u) \le 1$, we also obtain

$$|B| \le \mathcal{C}_{S'} \int_0^\infty |F(u)|\, u^{\gamma+d-1}\, \mathrm{d}u.$$

Equivalently, because $\omega(u) = u^\gamma$, we find

$$|B| \le \mathcal{C}_{S'} \int_0^\infty |F(u)|\, \omega(u)\, u^{d-1}\, \mathrm{d}u.$$

□

3. The collision operator as a bilinear map

We set

$$X := L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$$

with norm $\|f\|_X := \|f\|_{L^\infty(\mathbb{R}^d)}$. For $f, g \in X$, we define the bilinear operator

$$\mathcal{B}(f,g)(k) := \mathcal{B}_0(f,g)(k) - 2\mathcal{B}_1(f,g)(k),$$

where

$$\mathcal{B}_0(f,g)(k) := \int_{\mathscr{S}_k} \frac{V_{k,k_1,k-k_1}}{|\nabla \mathscr{H}_0^k(k_1)|} \Big[f(k_1)g(k-k_1) - f(k)g(k_1) - f(k)g(k-k_1) \Big]\, d\sigma(k_1),$$

and

$$\mathcal{B}_1(f,g)(k) := \int_{\mathscr{S}'_k} \frac{V_{k+k_2,k,k_2}}{|\nabla \mathscr{H}_1^k(k_2)|} \Big[f(k)g(k_2) - f(k)g(k+k_2) - f(k+k_2)g(k_2) \Big]\, d\sigma(k_2).$$

Then, we can write

$$Q_{3w}[f] = \mathcal{B}(f,f).$$

Lemma 4. *There exists a constant $C_B > 0$, depending only on $d, \gamma, \mathscr{A}$ and on the constants in Proposition 3, such that for all $f, g \in X$,*

$$\|\mathcal{B}(f,g)\|_X \le C_B \|f\|_X \|g\|_X.$$

Consequently,

$$\|Q_{3w}[f] - Q_{3w}[g]\|_X \le C_B \big(\|f\|_X + \|g\|_X \big) \|f - g\|_X$$

for all $f, g \in X$.

Proof. We first observe that $\mathcal{B}(f,g)$ is radial whenever f and g are radial. Indeed, for every rotation $R \in SO(d)$, we have

$$\mathscr{S}_{Rk} = R\mathscr{S}_k, \qquad \mathscr{S}'_{Rk} = R\mathscr{S}'_k,$$

and the quantities $V_{k,k_1,k_2}, |\nabla \mathscr{H}_0^k|, |\nabla \mathscr{H}_1^k|$ are invariant under simultaneous rotations of their arguments. A change of variables on the corresponding resonance surface therefore gives

$$\mathcal{B}(f,g)(Rk) = \mathcal{B}(f,g)(k).$$

Thus $\mathcal{B}(f,g) \in X$, provided the integrals are bounded.

We now estimate the S_k-part. Since $f, g \in L^\infty$, we find

$$|\mathcal{B}_0(f,g)(k)| \le 3\|f\|_X \|g\|_X \int_{\mathscr{S}_k} \frac{V_{k,k_1,k-k_1}}{|\nabla \mathscr{H}_0^k(k_1)|}\, d\sigma(k_1).$$

Applying Proposition 3 with the radial function $F \equiv 1$, and using the symmetry $k_1 \leftrightarrow k - k_1$ on $\mathscr{S}_k$, we obtain

$$\int_{\mathscr{S}_k} \frac{V_{k,k_1,k-k_1}}{|\nabla \mathscr{H}_0^k(k_1)|}\, d\sigma(k_1) \le \mathcal{D}_S \int_0^{|k|} r^{\gamma+d-1} e^{-\mathscr{A} r}\, dr \le \mathcal{D}_S G,$$

where

$$G := \int_0^\infty r^{\gamma+d-1} e^{-\mathscr{A} r}\, dr < \infty.$$

Therefore, we can bound

$$|\mathcal{B}_0(f,g)(k)| \le 3\mathcal{D}_S G \|f\|_X \|g\|_X.$$

Similarly, for the S'_k-part, we have

$$|\mathcal{B}_1(f,g)(k)| \le 3\|f\|_X\|g\|_X \int_{\mathscr{S}'_k} \frac{V_{k+k_2,k,k_2}}{|\nabla \mathscr{H}_1^k(k_2)|}\, d\sigma(k_2).$$

Again by Proposition 3, with $F \equiv 1$, we get

$$\int_{\mathscr{S}'_k} \frac{V_{k+k_2,k,k_2}}{|\nabla \mathscr{H}_1^k(k_2)|}\, d\sigma(k_2) \le \mathcal{D}_{S'} G.$$

Hence, we obtain

$$|\mathcal{B}_1(f,g)(k)| \le 3\mathcal{D}_{S'} G\|f\|_X\|g\|_X.$$

Combining the estimates and recalling the factor 2 in front of $\mathcal{B}_1$, we get

$$\|\mathcal{B}(f,g)\|_X \le 3G(\mathcal{D}_S + 2\mathcal{D}_{S'})\|f\|_X\|g\|_X.$$

Thus one may take $C_B := 3G(\mathcal{D}_S + 2\mathcal{D}_{S'})$.

Finally, since

$$Q_{3w}[f] - Q_{3w}[g] = \mathcal{B}(f,f) - \mathcal{B}(g,g) = \mathcal{B}(f-g,f) + \mathcal{B}(g,f-g),$$

the Lipschitz estimate follows immediately from the bilinear bound. □

4. Global time-analytic solutions of 3-wave kinetic equations

We construct an exact strong solution of the 3-wave kinetic equation (1) in the form

$$f(t,k) \;=\; \sum_{n=0}^{\infty} t^n \mathcal{A}_n(k), \tag{12}$$

where $\mathcal{A}_0(k) = f_0(k)$.

Plugging (12) into the 3-wave kinetic equation (1), we find, by a direct computation

$$\begin{aligned}
\mathcal{A}_{n+1}(k) = {} & \frac{1}{n+1}\iint_{\mathbb{R}^{2d}} \mathrm{d}k_1\,\mathrm{d}k_2\, V_{k,k_1,k_2}\,\delta(k-k_1-k_2)\,\delta(\omega_k-\omega_{k_1}-\omega_{k_2}) \\
& \times \sum_{\substack{0\le i,j\\ i+j=n}} \Big(\mathcal{A}_i(k_1)\mathcal{A}_j(k_2) - \mathcal{A}_i(k)\mathcal{A}_j(k_1) - \mathcal{A}_i(k)\mathcal{A}_j(k_2)\Big) \\
& - \frac{2}{n+1}\iint_{\mathbb{R}^{2d}} \mathrm{d}k_1\,\mathrm{d}k_2\, V_{k_1,k,k_2}\,\delta(k_1-k-k_2)\,\delta(\omega_{k_1}-\omega_k-\omega_{k_2}) \\
& \times \sum_{\substack{0\le i,j\\ i+j=n}} \Big(\mathcal{A}_i(k)\mathcal{A}_j(k_2) - \mathcal{A}_i(k)\mathcal{A}_j(k_1) - \mathcal{A}_i(k_1)\mathcal{A}_j(k_2)\Big).
\end{aligned} \tag{13}$$

Lemma 5. *Assume that $M := \sup_{k\in\mathbb{R}^d} |f_0(k)| < \infty$ and set*

$$G := \int_0^\infty r^{\gamma+d-1}\exp(-\mathscr{A} r)\,\mathrm{d}r < \infty.$$

Then there exists a positive constant $\mathcal{C}_1$, depending only on M, G, and the constants in Proposition 3, such that for every $n = 0, 1, 2, \ldots,$

$$|\mathcal{A}_n(k)| \le \mathcal{C}_1^n M$$

for a.e. $k \in \mathbb{R}^d$.

Proof. We argue by induction. The estimate is immediate for $n = 0$, since $\mathcal{A}_0 = f_0$. Assume that, for some $n \ge 0$,

$$|\mathcal{A}_\ell(k)| \le \mathcal{C}_1^\ell M$$

for every $0 \le \ell \le n$ and for a.e. $k \in \mathbb{R}^d$. We prove the estimate for $\mathcal{A}_{n+1}$.

We fix indices $i, j \geq 0$ with $i + j = n$ and define

$$\mathcal{A}^1_{i,j}(k) := \iint_{\mathbb{R}^{2d}} \mathrm{d}k_1\, \mathrm{d}k_2\, V_{k,k_1,k_2}\delta(k - k_1 - k_2)\delta(\omega_k - \omega_{k_1} - \omega_{k_2}) \times \Big(\mathcal{A}_i(k_1)\mathcal{A}_j(k_2) - \mathcal{A}_i(k)\mathcal{A}_j(k_1) - \mathcal{A}_i(k)\mathcal{A}_j(k_2)\Big).$$

Using the delta constraint $k_2 = k - k_1$, we may write

$$|\mathcal{A}^1_{i,j}(k)| \leq \int_{\mathscr{S}_k} \frac{\mathrm{d}\sigma(k_1)}{|\nabla \mathscr{H}^k_0(k_1)|} V_{k,k_1,k-k_1}\Big(|\mathcal{A}_i(k_1)|\,|\mathcal{A}_j(k-k_1)| + |\mathcal{A}_i(k)|\,|\mathcal{A}_j(k_1)| + |\mathcal{A}_i(k)|\,|\mathcal{A}_j(k-k_1)|\Big).$$

By the induction hypothesis and $i + j = n$, we have

$$|\mathcal{A}_i(\cdot)|\,|\mathcal{A}_j(\cdot)| \leq \mathcal{C}^n_1 M^2.$$

Therefore, we find

$$|\mathcal{A}^1_{i,j}(k)| \leq 3\mathcal{C}^n_1 M^2 \int_{\mathscr{S}_k} \frac{\mathrm{d}\sigma(k_1)}{|\nabla \mathscr{H}^k_0(k_1)|} V_{k,k_1,k-k_1}.$$

Applying Proposition 3 gives

$$|\mathcal{A}^1_{i,j}(k)| \leq 3\mathcal{D}_S\, \mathcal{C}^n_1 M^2 G.$$

Next, we define

$$\mathcal{A}^2_{i,j}(k) := -\iint_{\mathbb{R}^{2d}} \mathrm{d}k_1\, \mathrm{d}k_2\, V_{k_1,k,k_2}\delta(k_1 - k - k_2)\delta(\omega_{k_1} - \omega_k - \omega_{k_2}) \times \Big(\mathcal{A}_i(k)\mathcal{A}_j(k_2) - \mathcal{A}_i(k)\mathcal{A}_j(k_1) - \mathcal{A}_i(k_1)\mathcal{A}_j(k_2)\Big).$$

Using the constraint $k_1 = k + k_2$, we get

$$|\mathcal{A}^2_{i,j}(k)| \leq \int_{\mathscr{S}'_k} \frac{\mathrm{d}\sigma(k_2)}{|\nabla \mathscr{H}^k_1(k_2)|} V_{k+k_2,k,k_2}\Big(|\mathcal{A}_i(k)|\,|\mathcal{A}_j(k_2)| + |\mathcal{A}_i(k)|\,|\mathcal{A}_j(k+k_2)| + |\mathcal{A}_i(k+k_2)|\,|\mathcal{A}_j(k_2)|\Big).$$

Again, by the induction hypothesis,

$$|\mathcal{A}^2_{i,j}(k)| \leq 3\mathcal{C}^n_1 M^2 \int_{\mathscr{S}'_k} \frac{\mathrm{d}\sigma(k_2)}{|\nabla \mathscr{H}^k_1(k_2)|} V_{k+k_2,k,k_2}.$$

Applying Proposition 3, yields

$$|\mathcal{A}^2_{i,j}(k)| \leq 3\mathcal{D}_{S'}\, \mathcal{C}^n_1 M^2 G.$$

We recall the recurrence relation for $\mathcal{A}_{n+1}$

$$\mathcal{A}_{n+1}(k) = \frac{1}{n+1} \sum_{\substack{i,j\geq 0\\ i+j=n}} \mathcal{A}^1_{i,j}(k) + \frac{2}{n+1} \sum_{\substack{i,j\geq 0\\ i+j=n}} \mathcal{A}^2_{i,j}(k).$$

Since there are $n + 1$ pairs (i, j) with $i + j = n$, we obtain

$$|\mathcal{A}_{n+1}(k)| \leq 3\mathcal{D}_S\, \mathcal{C}^n_1 M^2 G + 6\mathcal{D}_{S'}\, \mathcal{C}^n_1 M^2 G \leq 9\mathcal{C}\, \mathcal{C}^n_1 M^2 G,$$

where $\mathcal{C} := \max\{\mathcal{D}_S, \mathcal{D}_{S'}\}$. We choose $\mathcal{C}_1 := 9\mathcal{C}GM$. Then

$$|\mathcal{A}_{n+1}(k)| \leq \mathcal{C}^{n+1}_1 M.$$

This closes the induction and proves the lemma. □

From (12), we obtain the bound

$$\big|f(t, k)\big| \;\leq\; \sum_{n=0}^{\infty} t^n \mathcal{C}_1^{\,n}\, M \;\leq\; \frac{M}{1 - t\mathcal{C}_1}, \tag{14}$$

for a.e. $k \in \mathbb{R}^d$ and all $0 \leq t < \frac{1}{\mathcal{C}_1}$. We conclude that f is analytic in $\left[0, \dfrac{1}{\mathcal{C}_1}\right)$.

Lemma 6. *We set Let f and g be two radial local strong solutions on $[0,T]$ with the same initial datum and with*

$$\sup_{0\le t\le T}\big(\|f(t)\|_{L^\infty}+\|g(t)\|_{L^\infty}\big)<\infty.$$

Then $f=g$ on $[0,T]$.

Proof. Using the surface estimates of Proposition 3, the collision operator is locally Lipschitz in L^∞_{rad}. Hence there exists a constant $C_T>0$ such that

$$\|Q_{3w}[f(t)]-Q_{3w}[g(t)]\|_{L^\infty}\le C_T\|f(t)-g(t)\|_{L^\infty}.$$

Therefore, we find

$$\|f(t)-g(t)\|_{L^\infty}\le C_T\int_0^t\|f(s)-g(s)\|_{L^\infty}\,ds.$$

Gronwall's inequality gives $\|f(t)-g(t)\|_{L^\infty}=0$ for every $0\le t\le T$. Hence $f=g$. □

Lemma 7. *Let*

$$X:=L^\infty_{\mathrm{rad}}(\mathbb{R}^d),\qquad X_+:=\{h\in X:h\ge 0\ a.e.\}.$$

Assume that $h\in X_+$. Let $u\in C^1([0,T);X)$ be the unique strong radial solution of

$$\partial_t u=Q_{3w}[u],\qquad u(0)=h.$$

Then $u(t,\cdot)\in X_+$ for every $0\le t<T$.

Proof. We write the equation in gain-loss form. For a radial function f, we define

$$\mathcal{G}[f](k):=\int_{\mathscr{S}_k}\frac{V_{k,k_1,k-k_1}f(k_1)f(k-k_1)}{|\nabla\mathscr{H}_0^k(k_1)|}\,d\sigma(k_1)+2\int_{\mathscr{S}'_k}\frac{V_{k+k_2,k,k_2}f(k+k_2)f(k_2)}{|\nabla\mathscr{H}_1^k(k_2)|}\,d\sigma(k_2),$$

and

$$\mathcal{L}[f](k):=\int_{\mathscr{S}_k}\frac{V_{k,k_1,k-k_1}\big(f(k_1)+f(k-k_1)\big)}{|\nabla\mathscr{H}_0^k(k_1)|}\,d\sigma(k_1)+2\int_{\mathscr{S}'_k}\frac{V_{k+k_2,k,k_2}f(k_2)}{|\nabla\mathscr{H}_1^k(k_2)|}\,d\sigma(k_2),$$

and

$$\mathcal{R}[f](k):=2\int_{\mathscr{S}'_k}\frac{V_{k+k_2,k,k_2}f(k+k_2)}{|\nabla\mathscr{H}_1^k(k_2)|}\,d\sigma(k_2).$$

Then, we have

$$Q_{3w}[f](k)=\mathcal{G}[f](k)-\big(\mathcal{L}[f](k)-\mathcal{R}[f](k)\big)f(k).$$

We now set

$$\mathcal{A}[f]:=\mathcal{L}[f]-\mathcal{R}[f].$$

Thus the equation becomes

$$\partial_t f+\mathcal{A}[f]f=\mathcal{G}[f].$$

We first prove a local positivity result. By the surface estimates of Proposition 3, there exist constants $C_G,C_A>0$, depending only on $d,\gamma,\mathscr{A}$, such that for all $f,g\in X$,

$$\|\mathcal{G}[f]\|_X\le C_G\|f\|_X^2,\qquad \|\mathcal{G}[f]-\mathcal{G}[g]\|_X\le C_G\big(\|f\|_X+\|g\|_X\big)\|f-g\|_X,$$

and

$$\|\mathcal{A}[f]\|_X\le C_A\|f\|_X,\qquad \|\mathcal{A}[f]-\mathcal{A}[g]\|_X\le C_A\|f-g\|_X.$$

Moreover, if $f\in X_+$, then $\mathcal{G}[f]\ge 0$a.e.

We fix $R>0$. We claim that there exists $\tau_R>0$ such that, whenever $h\in X_+,\|h\|_X\le R$, the Cauchy problem has a unique solution on $[0,\tau_R]$, and that solution is nonnegative.

We set

$$\mathcal{E}_{T,R}:=\big\{v\in C([0,T];X):v(t)\in X_+\text{ for every }t\in[0,T],\quad \|v\|_{C([0,T];X)}\le 2R\big\}.$$

For $v\in\mathcal{E}_{T,R}$, we define

$$(\Phi v)(t,k):=h(k)\exp\left(-\int_0^t\mathcal{A}[v(s)](k)\,ds\right)+\int_0^t\mathcal{G}[v(\tau)](k)\exp\left(-\int_\tau^t\mathcal{A}[v(s)](k)\,ds\right)d\tau.$$

The exponential factors are strictly positive. Since $h \geq 0$ and $\mathcal{G}[v] \geq 0$ whenever $v \geq 0$, we have $\Phi v(t) \in X_+$ for every $t \in [0,T]$.

We next show that Φ maps $\mathcal{E}_{T,R}$ into itself for T small. If $v \in \mathcal{E}_{T,R}$, then we bound

$$\|\mathcal{A}[v(t)]\|_X \leq 2C_A R, \qquad \|\mathcal{G}[v(t)]\|_X \leq 4C_G R^2.$$

Therefore, we find

$$\|\Phi v(t)\|_X \leq \exp(2C_A RT)\big(R + 4C_G R^2 T\big).$$

Choosing $T > 0$ sufficiently small, depending only on R, we obtain

$$\exp(2C_A RT)\big(R + 4C_G R^2 T\big) \leq 2R.$$

Thus $\Phi\mathcal{E}_{T,R} \subseteq \mathcal{E}_{T,R}$. We now prove that Φ is a contraction for T still smaller. Let $v, w \in \mathcal{E}_{T,R}$. Using

$$\|\mathcal{A}[v] - \mathcal{A}[w]\|_X \leq C_A\|v - w\|_X, \quad \|\mathcal{G}[v] - \mathcal{G}[w]\|_X \leq 4C_G R\|v - w\|_X,$$

we obtain

$$\|\Phi v - \Phi w\|_{C([0,T];X)} \leq \exp(2C_A RT)\Big(C_A RT + 4C_G RT + 4C_A C_G R^2 T^2\Big)\|v - w\|_{C([0,T];X)}.$$

Choosing $T > 0$ smaller if necessary, the coefficient on the right-hand side is strictly less than 1. Therefore Φ is a contraction on $\mathcal{E}_{T,R}$.

By the Banach fixed-point theorem, Φ has a unique fixed point $u \in \mathcal{E}_{T,R}$. Since $u = \Phi u$, differentiating the variation-of-constants formula gives

$$\partial_t u + \mathcal{A}[u]u = \mathcal{G}[u],$$

or equivalently

$$\partial_t u = Q_{3w}[u], \qquad u(0) = h.$$

Moreover, because $u \in \mathcal{E}_{T,R}$, we have $u(t) \in X_+$ for every $t \in [0,T]$. This proves local positivity.

Now let $u \in C^1([0,T);X)$ be the strong radial solution with nonnegative initial datum h. We define

$$T_* := \sup\{s \in [0,T) : u(t,\cdot) \in X_+ \text{ for every } 0 \leq t \leq s\}.$$

It is clear that $T_* > 0$.

We claim that $T_* = T$. Suppose, for contradiction, that $T_* < T$. Since $u \in C([0,T);X)$, and since X_+ is a closed cone in X, we have $u(T_*,\cdot) \in X_+$. We set $h_* := u(T_*,\cdot)$. Then $h_* \in X_+$ and $h_* \in X$. Applying the local positivity result with initial datum h_*, we obtain a nonnegative strong radial solution $w \in C^1([0,\tau];X)$ of

$$\partial_t w = Q_{3w}[w], \qquad w(0) = h_*,$$

for some $\tau > 0$.

On the other hand, the shifted function $\widetilde{u}(s,\cdot) := u(T_* + s,\cdot)$ also solves the same Cauchy problem on the interval where it is defined. By local uniqueness in $X = L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$, $\widetilde{u}(s,\cdot) = w(s,\cdot)$ for $0 \leq s < \min\{\tau, T - T_*\}$. Hence $u(T_* + s,\cdot) \in X_+$ for all such s. This contradicts the definition of T_*. Therefore $T_* = T$, and the solution remains nonnegative on the whole interval of existence. □

Lemma 8. *Let $h \in X = L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$, and set $H := \|h\|_X$. If $H > 0$, then there exists a unique strong radial solution $u \in C^1((-T_h, T_h);X)$ of*

$$\partial_t u = Q_{3w}[u], \qquad u(0) = h,$$

for some $T_h \geq \frac{1}{4C_B H}$. Moreover, u is real analytic as an X-valued function of time.

More precisely, u admits the power-series representation

$$u(t) = \sum_{n=0}^{\infty} t^n A_n$$

in X for $|t| < 1/(C_B H)$, where $A_0 = h$ and

$$A_{n+1} = \frac{1}{n+1} \sum_{\substack{i,j\geq 0\\ i+j=n}} \mathcal{B}(A_i, A_j), \qquad n \geq 0.$$

The coefficients satisfy

$$A_n \in X, \qquad \|A_n\|_X \leq (C_B H)^n H.$$

If $H = 0$, then the unique solution is $u \equiv 0$.

Proof. We first construct the formal coefficients. Since $A_0 = h \in X$, and since $\mathcal{B} : X \times X \to X$ is bilinear and bounded by Lemma 4, the recursive formula implies inductively that $A_n \in X$ for every $n \geq 0$. In particular, every coefficient is radial.

We prove the coefficient estimate by induction. The estimate is immediate for $n = 0$. We assume that

$$\|A_\ell\|_X \leq (C_B H)^\ell H$$

for $0 \leq \ell \leq n$. Then we have

$$\begin{aligned}\|A_{n+1}\|_X &\leq \frac{1}{n+1} \sum_{\substack{i,j\geq 0\\ i+j=n}} \|\mathcal{B}(A_i, A_j)\|_X \leq \frac{C_B}{n+1} \sum_{\substack{i,j\geq 0\\ i+j=n}} \|A_i\|_X \|A_j\|_X \\ &\leq \frac{C_B}{n+1} \sum_{\substack{i,j\geq 0\\ i+j=n}} (C_B H)^i H (C_B H)^j H = C_B (C_B H)^n H^2 = (C_B H)^{n+1} H.\end{aligned}$$

This closes the induction.

Therefore the series $\sum_{n=0}^\infty t^n A_n$ converges absolutely in X whenever $|t| < \frac{1}{C_B H}$. Furthermore, for every $0 < \rho < 1/(C_B H)$,

$$\sum_{n=1}^\infty n\rho^{n-1} \|A_n\|_X < \infty.$$

Hence the series may be differentiated term by term for $|t| < 1/(C_B H)$, and

$$\frac{d}{dt} \sum_{n=0}^\infty t^n A_n = \sum_{n=0}^\infty (n+1) t^n A_{n+1}.$$

It remains to check that the collision operator may be applied termwise. We let

$$u(t) := \sum_{n=0}^\infty t^n A_n.$$

Since $\mathcal{B}$ is a bounded bilinear map on X, absolute convergence of the series gives

$$\mathcal{B}(u(t), u(t)) = \sum_{n=0}^\infty t^n \sum_{\substack{i,j\geq 0\\ i+j=n}} \mathcal{B}(A_i, A_j)$$

in X. By the defining recursion for A_{n+1}, this becomes

$$\mathcal{B}(u(t), u(t)) = \sum_{n=0}^\infty (n+1) t^n A_{n+1} = \partial_t u(t).$$

Since $Q_{3w}[u] = \mathcal{B}(u, u)$, we obtain $\partial_t u = Q_{3w}[u]$ in X, and $u(0) = h$.

Uniqueness follows from the Lipschitz estimate in Lemma 4. Indeed, if u and v are two solutions on a common interval and are bounded in X, then

$$\|Q_{3w}[u(t)] - Q_{3w}[v(t)]\|_X \leq C_B \big(\|u(t)\|_X + \|v(t)\|_X\big) \|u(t) - v(t)\|_X.$$

Gronwall's inequality gives $u = v$. The power-series representation proves real analyticity in time as an X-valued function. $\square$

5. Proof of Theorem 2

Let $T_{\max} \in (0,+\infty]$ be the maximal time of existence of the nonnegative strong radial solution constructed by the local theory. We will prove that

$$T_{\max} = +\infty.$$

We now set $N(t) := \|f(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$. By the local positivity result, the solution remains nonnegative on its maximal interval of existence:

$$f(t,k) \geq 0$$

for a.e. $k \in \mathbb{R}^d$ and every $0 \leq t < T_{\max}$.

By the standard energy conservation (see [22, 23]), we have

$$\int_0^\infty f(t,r)\,\omega(r)\,r^{d-1}\,\mathrm{d}r = \int_0^\infty f_0(r)\,\omega(r)\,r^{d-1}\,\mathrm{d}r = \mathcal{M}_1$$

for every $0 \leq t < T_{\max}$.

We now derive an a priori estimate for $N(t)$. We now recall

$$\partial_t f(t,k) + \big(\mathcal{L}[f](t,k) - \mathcal{R}[f](t,k)\big) f(t,k) = \mathcal{G}[f](t,k).$$

Since $f \geq 0$ and $\mathcal{L}[f] \geq 0$, we have, for a.e. k,

$$\partial_t f(t,k) \leq \mathcal{G}[f](t,k) + \mathcal{R}[f](t,k) f(t,k).$$

Therefore, we can bound

$$f(t,k) \leq f_0(k) + \int_0^t \Big(\mathcal{G}[f](\tau,k) + \mathcal{R}[f](\tau,k) f(\tau,k)\Big)\,\mathrm{d}\tau.$$

We estimate the integrand. Since $f \geq 0$, we have

$$\begin{aligned}\mathcal{G}[f](t,k) + \mathcal{R}[f](t,k)f(t,k) \ &= \int_{\mathscr{S}_k} \mathrm{d}\sigma(k_1)\, \frac{V_{k,k_1,k-k_1} f(t,k_1) f(t,k-k_1)}{|\nabla \mathscr{H}_0^k(k_1)|} \\ + 2\int_{\mathscr{S}'_k} \mathrm{d}\sigma(k_2)\, \frac{V_{k+k_2,k,k_2} f(t,k+k_2) f(t,k_2)}{|\nabla \mathscr{H}_1^k(k_2)|} &+ 2f(t,k)\int_{\mathscr{S}'_k} \mathrm{d}\sigma(k_2)\, \frac{V_{k+k_2,k,k_2} f(t,k+k_2)}{|\nabla \mathscr{H}_1^k(k_2)|}.\end{aligned}$$

Using $0 \leq f(t,\cdot) \leq N(t)$, we obtain

$$\begin{aligned}\mathcal{G}[f](t,k) + \mathcal{R}[f](t,k)f(t,k) \ &\leq N(t)\int_{\mathscr{S}_k} \mathrm{d}\sigma(k_1)\, \frac{V_{k,k_1,k-k_1} f(t,k_1)}{|\nabla \mathscr{H}_0^k(k_1)|} \\ + 2N(t)\int_{\mathscr{S}'_k} \mathrm{d}\sigma(k_2)\, \frac{V_{k+k_2,k,k_2} f(t,k_2)}{|\nabla \mathscr{H}_1^k(k_2)|} &+ 2N(t)\int_{\mathscr{S}'_k} \mathrm{d}\sigma(k_2)\, \frac{V_{k+k_2,k,k_2} f(t,k+k_2)}{|\nabla \mathscr{H}_1^k(k_2)|}.\end{aligned}$$

The first integral is controlled by Proposition 3

$$\int_{\mathscr{S}_k} \mathrm{d}\sigma(k_1)\, \frac{V_{k,k_1,k-k_1} f(t,k_1)}{|\nabla \mathscr{H}_0^k(k_1)|} \leq \mathcal{C}_S \mathcal{M}_1.$$

Similarly, we can also bound

$$\int_{\mathscr{S}'_k} \mathrm{d}\sigma(k_2)\, \frac{V_{k+k_2,k,k_2} f(t,k_2)}{|\nabla \mathscr{H}_1^k(k_2)|} \leq \mathcal{C}_{S'} \mathcal{M}_1.$$

It remains to estimate the last term. Using the parametrization of $\mathscr{S}'_k$, we write

$$u := |k_2|, \quad r_+(u) := |k+k_2|.$$

On the resonant surface, we have $\omega(r_+(u)) = \omega(|k|) + \omega(u)$. Thus, for $\omega(r) = r^\gamma$, $r_+(u) = \big(|k|^\gamma + u^\gamma\big)^{1/\gamma}$. The surface estimate gives

$$\int_{\mathscr{S}'_k} \mathrm{d}\sigma(k_2)\, \frac{V_{k+k_2,k,k_2} f(t,k+k_2)}{|\nabla \mathscr{H}_1^k(k_2)|} \ \leq\ \mathcal{C}_{S'} \int_0^\infty f(t,r_+(u))\,u^{\gamma+d-1}\,\mathrm{d}u.$$

We change variables $r = r_+(u) = \big(|k|^\gamma + u^\gamma\big)^{1/\gamma}$. Then, we find

$$u = (r^\gamma - |k|^\gamma)^{1/\gamma}, \quad \mathrm{d}u = r^{\gamma-1}(r^\gamma - |k|^\gamma)^{1/\gamma-1}\,\mathrm{d}r.$$

Hence, we obtain

$$u^{\gamma+d-1}\,\mathrm{d}u = u^d r^{\gamma-1}\,\mathrm{d}r \le r^{\gamma+d-1}\,\mathrm{d}r.$$

Therefore, we have

$$\int_0^\infty f(t, r_+(u))\, u^{\gamma+d-1}\,\mathrm{d}u \le \int_{|k|}^\infty f(t,r)\, r^{\gamma+d-1}\,\mathrm{d}r \le \mathcal{M}_1.$$

Thus the last term is bounded by $\mathcal{C}_{S'}\mathcal{M}_1$.

Combining the three estimates, we obtain

$$\mathcal{G}[f](t,k) + \mathcal{R}[f](t,k)f(t,k) \le C_0\mathcal{M}_1 N(t),$$

where $C_0 := \mathcal{C}_S + 4\mathcal{C}_{S'}$. Consequently, we have

$$f(t,k) \le f_0(k) + C_0\mathcal{M}_1 \int_0^t N(\tau)\,\mathrm{d}\tau.$$

Taking the essential supremum over k, we get

$$N(t) \le N(0) + C_0\mathcal{M}_1 \int_0^t N(\tau)\,\mathrm{d}\tau.$$

By Gronwall's inequality, we have

$$N(t) \le N(0)\exp(C_0\mathcal{M}_1 t)$$

for every $0 \le t < T_{\max}$.

If $T_{\max} < +\infty$, then the previous estimate gives

$$\limsup_{t\uparrow T_{\max}} N(t) \le N(0)\exp(C_0\mathcal{M}_1 T_{\max}) < +\infty.$$

This contradicts the blow-up alternative. Therefore, we deduce $T_{\max} = +\infty$. Hence the nonnegative strong radial solution is global in time.

Finally, we prove the asserted time analyticity. We fix $t_0 \ge 0$, and set $h := f(t_0, \cdot) \in L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$. By Lemma 8, the Cauchy problem with initial datum h has a unique L^∞_{rad}-valued analytic solution near $s = 0$. On the other hand, the shifted function $\widetilde{f}(s,\cdot) := f(t_0 + s, \cdot)$ is also a strong radial solution with the same initial datum h, for all s sufficiently small such that $t_0 + s \ge 0$. Therefore,

$$\widetilde{f}(s,\cdot) = \sum_{n=0}^\infty s^n A_n^{(t_0)}$$

in $L^\infty_{\mathrm{rad}}(\mathbb{R}^d)$ for s sufficiently small. If $t_0 > 0$, this gives a two-sided analytic expansion around t_0. If $t_0 = 0$, it gives right-analyticity at the initial time. This completes the proof of Theorem 2.

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