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Internal control of the transition kernel for stochastic lattice dynamics

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Abstract

In [4], we initiated the first study of control problems for kinetic equations arising from harmonic chains. Specifically, we developed impulsive and feedback control mechanisms for harmonic chains coupled with a point thermostat, effectively enabling control over the boundary conditions of the corresponding kinetic equations. However, the more intricate and fundamental challenge of internal control - namely, the design of control strategies that influence the collision operators within the kinetic framework - remained open.

In the present work, we address the internal control problem for stochastic lattice dynamics, with the objective of controlling the transition kernel of the limiting kinetic equation. A central innovation of our approach is the development of a novel geometric-combinatorial framework, which enables the systematic construction of control pathways within the microscopic dynamics. This methodology opens a new avenue for the internal control of kinetic equations.

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1. Introduction

During the last few years, energy transport in anharmonic chains, including the well-known Fermi-Pasta-Ulam (FPU) chain with a pinning potential and a quartic nonlinearity, has become an important topic of research (see [1,6–8]). The Hamiltonian of the FPU-chain is defined by

$$H(\beta,\alpha) := \sum_{n=-\infty}^{\infty}\left(\frac{1}{2m}\alpha_n^2+\frac{1}{2}\omega_0^2\beta_n^2\right)+\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}A(\beta_{n+1}-\beta_n)^2+B(\beta_{n+1}-\beta_n)^4\right) \tag{1}$$

where $\alpha=(\alpha_n)_{n\in\mathbb{Z}}$, $\beta=(\beta_n)_{n\in\mathbb{Z}}$, and $\beta_n\in\mathbb{R}$ is the displacement of the particle n from its equilibrium position, $\alpha_n\in\mathbb{R}$ is the canonically conjugate momentum, and m is the mass of each particle. The pinning potential, represented by the term $\omega_0^2\beta_n^2$, contributes to confining particle n. In (1), the second sum, which depends only on the displacement differences, consists of a harmonic nearest neighbor term $\frac{1}{2}A(\beta_{n+1}-\beta_n)^2$ ($A>0$) and an anharmonic term $B(\beta_{n+1}-\beta_n)^4$ ($B\geq 0$). The mathematical study of energy transport in the FPU-chain is challenging.

The authors of [1] replaced the nonlinearity in (1) by a stochastic exchange of momentum between nearest neighboring sites chosen such that the exchange conserves the local energy. In other words, they took $B=0$ and added a small perturbation to the system.

When $B = 0$, (1) corresponds to a discrete wave equation. The energy transport of the wave equation has been studied extensively using Wigner distributions (see [1,8]). When the wave propagates without "noise", the Wigner distribution is governed by the linear transport equation

$$\frac{d}{dt}W(x,k,t) + \frac{1}{2\pi}\omega'(k)\frac{\partial}{\partial x}W(x,k,t) = 0 \tag{2}$$

where $x \in \mathbb{R}$ is the position along the chain, the wave number k belongs to $\mathbb{T}$, the unit torus identified with $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with periodic endpoints. The variable t is time. The dispersion relation inferred from (1) is

$$\omega(k)^2 = \omega_0^2 + 2A(1 - \cos(2\pi k)). \tag{3}$$

Indeed, in (1), the terms involving β can be rewritten as

$$\frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\beta_m\beta_n(\omega_0^2\delta_{m-n} + 2A\delta_{m-n} - A\delta_{m-n-1} - A\delta_{m-n+1}), \tag{4}$$

where δ_n is the Kronecker delta function. The Fourier transform of (4) is

$$\frac{1}{2}\hat{\beta}(k)^2(\omega_0^2 + 2A(1 - \cos(2\pi k))).$$

The coefficient of $\hat{\beta}^2$ gives the dispersion relation (3). Equation (2) describes the energy transport because the local energy density at time t is defined either by

$$E(k,t) := \int_{\mathbb{R}} W(x,k,t)dx \quad \text{or by} \quad \bar{E}(x,t) := \int_{\mathbb{T}} W(x,k,t)dk.$$

When we take into consideration a "noise" in the discrete wave equation, we get

$$\begin{aligned} d\beta_n(t) =& \alpha_n(t)dt, \\ d\alpha_n(t) =& -\sigma \star \beta_n(t)dt + P^{\varepsilon}_{N,n}(\alpha, dt), \end{aligned} \tag{5}$$

where σ is a coupling between two particles n, n' depending only on their distance $|n - n'|$, $\sigma \star \beta_n = \sum_{n'\in\mathbb{Z}} \sigma_{n-n'}\beta_{n'}$ is the discrete convolution, and $P^{\varepsilon}_{N,n}(\alpha, dt)$ is the perturbation corresponding to the stochastic exchange of momentum between neighboring particles within distance N of the particle n conserving the local energy. With the perturbation, (2) becomes, under the kinetic limit

$$\frac{d}{dt}W(x,k,t) + \frac{1}{2\pi}\omega'(k)\frac{\partial}{\partial x}W(x,k,t) = \int_{\mathbb{T}} \mathcal{K}(k,k')(W(x,k',t) - W(x,k,t))dk, \tag{6}$$

where $\mathcal{K}$ is a *transition kernel*. We refer to [12] for related situations.

Owing to their wide array of applications, control problems in kinetic theory have attracted considerable attention in recent years, resulting in substantial advances in the field (see [2,11]).

The problem of controlling FPUT chains is also a highly important direction of research in science and technology [3,9,10,14]. Nevertheless, despite this progress, to the best of our knowledge, there is no work addressing the control of anharmonic chains through their associated kinetic equations. In our previous contribution [4], we initiated this line of research by developing impulsive and feedback control strategies for harmonic chains interacting with a point thermostat, thereby achieving control over the boundary behavior of the corresponding kinetic equations.

The much more intricate problem of internal control - where the control mechanisms alter the collision operator within the kinetic framework - has remained uncharted. The current work represents the first step toward closing this gap. Starting with results of [1] as an example, we study the internal control problem for perturbed harmonic chains, with the aim of steering the transition kernel of the limiting kinetic equation.

A key innovation of our approach lies in the development of a novel geometric-combinatorial proof technique, which enables the construction of effective control paths within the microscopic dynamics. This method lays the foundation for a general theory of internal control in kinetic equations arising from complex many-particle systems, including those governed by anharmonic interactions.

For a given target kernel $\hat{K} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, we want to find $P^{\varepsilon}_{N,n}$ that leads to (6) with $\mathcal{K} \approx \hat{K}$. For this purpose, we design stochastic exchanges of momentum between neighboring particles of distance $N \in \mathbb{N}$. This is significantly harder than the model considered in [1] where N is equal to 1. Considering a larger N increases the number of free parameters and thus the number of controls, which is a significant improvement in terms of control theory. Furthermore, the N-neighboring problem is then extended to the multi-dimensional case. To address the N-neighboring problem in the multi-dimensional case, we introduce the new concept of simple index and dual indices for the exchanges of momentum between neighboring particles.

Let us make an additional remark on the transition kernel. Since the kernel satisfies

$$\int_{\mathbb{T}} \mathcal{K}(k, k')dk' \sim |k|^2 \quad \text{for } |k| \ll 1,$$

we consider target kernels of the form

$$\sin^2(\pi k)\sin^2(\pi k')\tilde{v}(\cos(\pi k), \cos(\pi k')),$$

where $\tilde{v}$ is a polynomial with two variables. In our work, we compute and characterize the set V of possible achievable polynomials $\tilde{v}$. For each $\tilde{v} \in V$, we design a control achieving the resulting kernel.

To be more specific about the nature of our control, we set up $2N$ controlled parameters M_N. Among them we can freely choose values for up to $N - \lfloor N/2 \rfloor$ parameters. Then, there exists a perturbation and a kernel corresponding to the chosen parameters. The resulting kernel $\hat{K}_{M_N}$ yields the linear transport equation (6) for the Wigner distribution. Hence, by choosing appropriate parameters, we design an explicit control for the energy transport problem.

One of the main results of our paper is stated, in an informal way, as follows. We refer the reader to Theorem 5 in Section 3 for the complete result.

Theorem 1 *(Informal Statement of Theorem 5). We design explicit internal controls steering the transition kernel of the Boltzmann equation* (6) *that is the kinetic limit of the controlled stochastic lattice dynamics* (5)*, to some prescribed target kernel. Moreover, we give a description of a wide range of possible reachable target kernels, in form of a polynomial family.*

Thanks to the notion of a path γ introduced in Section 2.3, we also extend the kinetic limit study to multidimensional problem. For $\mathrm{d} \geq 2$, we formulate a version of equations (5) and (6) but with $\alpha_n, \beta_n \in \mathbb{R}^{\mathrm{d}}, n \in \mathbb{Z}^{\mathrm{d}}$, and $\sigma : \mathbb{T}^{\mathrm{d}} \to \mathbb{R}$. The distance on $\mathbb{Z}^{\mathrm{d}}$ is defined using the L^1-norm.

We introduce the associated controlled parameters $M_\gamma \in \mathbb{R}^{\mathrm{d}}$ for simple index and $M_\gamma^{a,b} \in \mathbb{R}$ for dual indices. We will show in Theorem 7 and Theorem 8 that we can also compute the transition kernel based on the associated controlled parameters for multidimensional problems.

Finally, to clarify our computational steps, we give an example for small distance. In the appendix, we perform detailed computations for K_{M_N} with $N = 3$ and then show explicitly the achievable target kernels.

2. The setting

2.1. Preliminaries

We normalize the mass $m = 1$ and consider the harmonic case $B = 0$. The Hamiltonian (1) is then

$$H(\beta, \alpha) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_n^2 + \frac{1}{2} \sum_{n,n'} \sigma_{n-n'} \beta_n \beta_{n'}.$$

For a function $f \in \ell^2(\mathbb{Z})$, the Fourier transform is

$$\hat{f}(k) := \sum_{n \in \mathbb{Z}} e^{-2\pi i k n} f_n, \quad \forall k \in \mathbb{T}.$$

Also, the inverse Fourier transform of a function $F \in L^2(\mathbb{T})$ is

$$\tilde{F}_n := \int_{\mathbb{T}} e^{2\pi i k n} F(k) dk.$$

We use these notations for the Fourier transform of the wave functions $\hat{\alpha}, \hat{\beta}, \hat{\phi}$, the coupling $\hat{\sigma}$, and for $\tilde{\omega}$, which denotes the inverse Fourier transform of dispersion relation. The wave ϕ is given by

$$\phi_n(t) := \frac{1}{\sqrt{2}} (\tilde{\omega} \star \beta_n(t) + i\alpha_n(t))$$

or by its Fourier transform

$$\hat{\phi}(k,t) := \frac{1}{\sqrt{2}} \left(\omega \hat{\beta}(k,t) + i\hat{\alpha}(k,t) \right). \tag{7}$$

The dispersion relation is defined by

$$\omega(k) := \sqrt{\hat{\sigma}(k)}.$$

We will specify the assumptions on the coupling σ later. An example for the dispersion relation is shown in (3).

2.1.1. The setting of [1]

Before going into more details, let us briefly report on the construction of the perturbation $P^{\varepsilon}_{N,n}$ in (5) that was done in [1]. The authors considered the 1-distance perturbation with friction. In this case, the vector field that conserves local momentum and energy is

$$\lambda_n := (\alpha_n - \alpha_{n+1})\frac{\partial}{\partial \alpha_{n-1}} + (\alpha_{n+1} - \alpha_{n-1})\frac{\partial}{\partial \alpha_n} + (\alpha_{n-1} - \alpha_n)\frac{\partial}{\partial \alpha_{n+1}}$$

and the corresponding generator of the system (5) is

$$\sum_{n\in\mathbb{Z}} \alpha_n \frac{\partial}{\partial \beta_n} - \sum_{n,n'\in\mathbb{Z}} \sigma_{n-n'}\beta_{n'}\frac{\partial}{\partial \alpha_n} + \frac{\varepsilon c}{6}\sum_{n\in\mathbb{Z}}(\lambda_n)^2,$$

where c is the friction. In (5), the perturbation used in [1] is

$$\begin{aligned}&\frac{\varepsilon c}{6}\Delta(4\alpha_n + \alpha_{n-1} + \alpha_{n+1})dt + \sqrt{\frac{\varepsilon c}{3}}\sum_{d=-1,0,1}(\lambda_{n+d}\alpha_n)Y_{n+d}(dt)\\ &\quad = \frac{\varepsilon c}{3}(\alpha_{n+2} + \alpha_{n-2} + 2\alpha_{n+1} + 2\alpha_{n-1} - 6\alpha_n) + \sqrt{\frac{\varepsilon c}{3}}\sum_{d=-1,0,1}(\lambda_{n+d}\alpha_n)Y_{n+d}(dt).\end{aligned}$$

Here, the Y_n's are independent Wiener processes; we will introduce them more formally in Sections 2.1.2 and 2.1.3.

At the limit $\varepsilon \to 0$, we get the Boltzmann transport equation (6) with the transition kernel

$$\frac{4c}{3}\left(2\sin^2(2\pi k)\sin^2(\pi k') + 2\sin^2(\pi k)\sin^2(2\pi k') - \sin^2(2\pi k)\sin^2(2\pi k')\right).$$

The purpose of our work is to design a control on this transition kernel by modifying the vector field λ_n and the perturbation $P^{\varepsilon}_{N,n}$.

2.1.2. The 1-dimensional case

The system that governs the wave is given by equation (5). The problem is one-dimensional, as we are considering $n \in \mathbb{Z}$, $\alpha_n, \beta_n \in \mathbb{R}$, and $k \in \mathbb{T}$. The underlying Hamiltonian operator is

$$O^H := \sum_{n\in\mathbb{Z}} \alpha_n \frac{\partial}{\partial \beta_n} - \sum_{n,n'\in\mathbb{Z}} \sigma_{n-n'}\beta_{n'}\frac{\partial}{\partial \alpha_n}. \tag{8}$$

Applying O^H to the wave function defined in (7), we obtain

$$O^H\hat{\phi}(k,t) = -i\omega(k)\hat{\phi}(k,t). \tag{9}$$

Now, we state the assumptions on the coupling σ.

(σ1) There exists $n \neq 0$ such that $\sigma_n \neq 0$.
(σ2) σ is an even function, i.e., $\sigma_n = \sigma_{-n}$ for all $n \in \mathbb{Z}$.
(σ3) σ decays exponentially; that is, there exist constants $C_1, C_2 > 0$ such that

$$|\sigma_n| \le C_1 e^{-C_2|n|}, \qquad \forall n \in \mathbb{Z}.$$

(σ4) The Fourier transform $\hat{\sigma}(k)$ satisfies either:

- $\hat{\sigma}(k) > 0$ for all $k \in \mathbb{T}$ (pinning case), or
- $\hat{\sigma}(k) > 0$ for all $k \in \mathbb{T} \setminus \{0\}$, $\hat{\sigma}(0) = 0$, and $\hat{\sigma}''(0) > 0$ (no-pinning case).

We denote by $\{Y_n(t), t \ge 0\}_{n\in\mathbb{Z}}$ a sequence of independent standard Wiener processes defined on a probability space equipped with the filtration $(\Omega, \mathfrak{F}_t, \mathbb{P})$. The notation $\langle\cdot\rangle_{\mu_\varepsilon}$ denotes the expectation with respect to the probability measure μ_ε, which governs the initial state of the wave. We write $\mathbb{E}_\varepsilon$ for the expectation that accounts for both the initial distribution μ_ε and the Wiener processes with probability $\mathbb{P}$.

The wave function ϕ is defined in (7), where the variables α and β are governed by a system driven by a Wiener process. We will specify this system later in (25), and begin by focusing on the initial state.

The initial state probability measure μ_ε is assumed to satisfy the following conditions:

(μ1) $\langle\phi_n(0)\rangle_{\mu_\varepsilon} = 0$, for all $n \in \mathbb{Z}$.
(μ2) $\langle\phi_n(0)\phi_{n'}(0)\rangle_{\mu_\varepsilon} = 0$, for all $n, n' \in \mathbb{Z}$.
(μ3) There exists a constant $C_3 > 0$ such that

$$\varepsilon\left\langle\|\phi(0)\|^2_{\ell^2}\right\rangle_{\mu_\varepsilon} < C_3, \quad \text{for all } \varepsilon > 0.$$

(μ4) In the no-pinning case,

$$\lim_{R\to 0}\limsup_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_{|k|<R}\left\langle|\hat{\phi}(k,0)|^2\right\rangle_{\mu_\varepsilon} dk = 0.$$

The Wigner distribution is denoted by W^ε and is defined by

$$\begin{aligned}\langle S, W^\varepsilon\rangle &:= \frac{\varepsilon}{2}\sum_{n,n'\in\mathbb{Z}}\mathbb{E}_\varepsilon\left[\phi_n(t/\varepsilon)\phi^*_{n'}(t/\varepsilon)\right]\int_{\mathbb{T}} e^{2\pi i k(n'-n)} S^*\left(\frac{\varepsilon(n+n')}{2}, k\right) dk \\ &= \frac{\varepsilon}{2}\sum_{n,n'\in\mathbb{Z}}\mathbb{E}_\varepsilon\left[\phi_n(t/\varepsilon)\phi^*_{n'}(t/\varepsilon)\right]\tilde{S}^*\left(\frac{\varepsilon(n+n')}{2}, n-n'\right),\end{aligned}$$

where S is a test function in the Schwartz space $\mathscr{S}(\mathbb{R}\times\mathbb{T})$ and $\tilde{S}$ is the inverse Fourier transform

$$\tilde{S}(x,n) = \int_{\mathbb{T}} e^{2\pi i k n} S(x,k) dk.$$

By Assumption ($\mu 3$) and the conservation of energy, we can show that the Wigner distribution W^ε is well-defined. In fact, there exists a constant $C_4 > 0$ such that

$$\left|\langle S, W^\varepsilon\rangle\right| \le \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\|\phi(t/\varepsilon)\|_{\ell^2}^2\right]\sum_{n\in\mathbb{Z}}\sup_{x\in\mathbb{R}}\left|\tilde{S}^*(x,n)\right| \le C_4.$$

We define the space $\mathcal{N}$ as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ with respect to the norm

$$\|S\|_{\mathcal{N}} := \sum_{n\in\mathbb{Z}}\sup_{x\in\mathbb{R}}\left|\tilde{S}(x,n)\right|.$$

Then the mapping $S \mapsto \langle S, W^\varepsilon\rangle$ defines a continuous linear functional on $\mathcal{N}$, and hence

$$\|W^\varepsilon\|_{\mathcal{N}'} < \infty.$$

This also implies that the family $(W^\varepsilon)_{0<\varepsilon<1}$ is sequentially weak-* compact in $\mathcal{N}'$. Therefore, by possibly passing to a subsequence, we may assume that W^ε converges in the weak-* topology. Therefore, for the initial state, we assume:

(W1) $W^\varepsilon(dx, dk, 0)$ converges (at least along a subsequence) to a non-negative measure $\mu_0(dx, dk)$.

We also define the *energy density* $E^\varepsilon(t)$ as a measure on $\mathbb{T}$ by

$$\langle S, E^\varepsilon(t)\rangle := \langle S, W^\varepsilon(t)\rangle = \frac{\varepsilon}{2}\int_{\mathbb{T}} \mathbb{E}_\varepsilon\left[|\hat{\phi}(k, t/\varepsilon)|^2\right] S(k)\, dk,$$

where $S(k)$ is a bounded real-valued test function on $\mathbb{T}$. In light of Assumption (W1), we also conclude that $E^\varepsilon(dk, 0)$ converges to a limiting measure $\nu_0(dk)$.

2.1.3. The multi-dimensional case

In the multi-dimensional setting, the particles are now located on the lattice $\mathbb{Z}^{\mathrm{d}}$, and for each $n \in \mathbb{Z}^{\mathrm{d}}$, the position and momentum vectors are $\alpha_n, \beta_n \in \mathbb{R}^{\mathrm{d}}$.

The assumptions on the coupling function σ and the initial state μ_ε in the multi-dimensional case mirror those in the one-dimensional case, with the following modifications:

- $\mathbb{Z}$ is replaced by $\mathbb{Z}^{\mathrm{d}}$,
- $\mathbb{T}$ is replaced by $\mathbb{T}^{\mathrm{d}}$,
- the condition $\hat{\sigma}''(0) > 0$ is replaced by the invertibility of $\mathrm{Hess}(\hat{\sigma})(0)$.

The assumptions on σ in the multi-dimensional case are given below:

(σ1') There exists $n \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}$ such that $\sigma_n \neq 0$.

(σ2') σ is an even function, i.e., $\sigma_n = \sigma_{-n}$ for all $n \in \mathbb{Z}^{\mathbb{d}}$.
(σ3') σ decays exponentially; that is, there exist constants $C_5, C_6 > 0$ such that

$$|\sigma_n| \le C_5 e^{-C_6|n|}, \qquad \forall n \in \mathbb{Z}^{\mathbb{d}}.$$

(σ4') One of the following holds:

- **Pinning:** $\hat{\sigma}(k) > 0$ for all $k \in \mathbb{T}^{\mathbb{d}}$.
- **No pinning:** $\hat{\sigma}(k) > 0$ for all $k \in \mathbb{T}^{\mathbb{d}} \setminus \{0\}$, $\hat{\sigma}(0) = 0$, and $\mathrm{Hess}(\hat{\sigma})(0)$ is invertible.

We recall the position and momentum functions $\alpha_n, \beta_n \in \mathbb{R}^{\mathbb{d}}$, and the wave function $\phi_n \in \mathbb{C}^{\mathbb{d}}$. We denote the a-th component of these vectors by $[\alpha_n]_a$, $[\beta_n]_a$, and $[\phi_n]_a$, respectively.

We will use the notations $\{Y_n^a(t),\ t \ge 0\}_{1 \le a \le \mathbb{d},\, n \in \mathbb{Z}^{\mathbb{d}}}$ (the simple index case discussed in Section 2.3.2) and $\{Y_{n,\gamma}^{a,b}(t),\ t \ge 0\}_{1 \le a,b \le \mathbb{d},\, n \in \mathbb{Z}^{\mathbb{d}},\, \gamma}$ (the dual index case from Section 2.3.3) to denote sequences of independent standard Wiener processes. The symbol γ denotes a path, which plays a crucial role in the control arguments for the multi-dimensional case described in Section 2.3. The remaining notations - $\mathbb{P}, \mu_\varepsilon, \mathbb{E}_\varepsilon$ - are the same as in the one-dimensional case.

We will also specify the system governing the wave, as given in equations (28) or (31). The assumptions on the initial state are as follows:

(μ1') $\langle \phi_n(0) \rangle_{\mu_\varepsilon} = 0$ for all $n \in \mathbb{Z}^{\mathbb{d}}$.
(μ2') $\langle [\phi_n(0)]_a [\phi_{n'}(0)]_b \rangle_{\mu_\varepsilon} = 0$ for all $n, n' \in \mathbb{Z}^{\mathbb{d}}$ and $1 \le a, b \le \mathbb{d}$.
(μ3') There exists a constant $C_7 > 0$ such that

$$\varepsilon^{\mathbb{d}} \left\langle \|\phi(0)\|_{\ell^2}^2 \right\rangle_{\mu_\varepsilon} < C_7, \quad \text{for all } \varepsilon > 0.$$

(μ4') If the coupling corresponds to the no-pinning case, then

$$\lim_{R \to 0} \limsup_{\varepsilon \to 0} \left(\frac{\varepsilon}{2}\right)^{\mathbb{d}} \int_{|k|<R} \left\langle \|\hat{\phi}(k,0)\|_{L^2(\mathbb{T}^{\mathbb{d}})}^2 \right\rangle_{\mu_\varepsilon} dk = 0.$$

The general Wigner distribution is defined by

$$\begin{aligned}
\langle J, W^\varepsilon(t) \rangle &:= \left(\frac{\varepsilon}{2}\right)^{\mathbb{d}} \int_{\mathbb{R}^{\mathbb{d}} \times \mathbb{T}^{\mathbb{d}}} \mathbb{E}_\varepsilon \left[\hat{\phi}\left(k - \frac{\varepsilon\xi}{2}, \frac{t}{\varepsilon}\right) \cdot S^*(\xi, k)\, \hat{\phi}\left(k + \frac{\varepsilon\xi}{2}, \frac{t}{\varepsilon}\right) \right] d\xi\, dk \\
&= \left(\frac{\varepsilon}{2}\right)^{\mathbb{d}} \sum_{n,n' \in \mathbb{Z}^{\mathbb{d}}} \sum_{1 \le a,b \le \mathbb{d}} \mathbb{E}_\varepsilon \left[[\phi_n]_a [\phi_{n'}^*]_b \right] \int_{\mathbb{T}^{\mathbb{d}}} e^{i2\pi k \cdot (n'-n)} S_{b,a}^*\left(\frac{\varepsilon(n+n')}{2}, k\right) dk \\
&= \left(\frac{\varepsilon}{2}\right)^{\mathbb{d}} \sum_{n,n' \in \mathbb{Z}^{\mathbb{d}}} \sum_{1 \le a,b \le \mathbb{d}} \mathbb{E}_\varepsilon \left[[\phi_n]_a [\phi_{n'}^*]_b \right] \tilde{S}_{b,a}^*\left(\frac{\varepsilon(n+n')}{2}, n - n'\right),
\end{aligned}$$

where $S = (S_{a,b})_{1 \le a,b \le \mathbb{d}} \in \mathcal{S}(\mathbb{R}^{\mathbb{d}} \times \mathbb{T}^{\mathbb{d}}, \mathbb{M}_{\mathbb{d}})$ is a test function valued in the space of complex $\mathbb{d} \times \mathbb{d}$ matrices.

As in the one-dimensional case, W^ε is well-defined and sequentially weak-* compact.

The Wigner distribution can be interpreted as a matrix of distributions; we write it as

$$W^\varepsilon(t) = \left(W^\varepsilon_{a,b}(t)\right)_{1\le a,b\le \mathbb{d}},$$

in the sense of distributions. See [5,13] for related formulations and interpretations.

The formulation for each entry of the Wigner distribution matrix is given by

$$W^\varepsilon_{b,a}(x,k,t) = \left(\frac{\varepsilon}{2}\right)^{\mathbb{d}} \int\limits_{\mathbb{R}^{\mathbb{d}}} e^{i2\pi x\cdot\xi}\, \mathbb{E}_\varepsilon \left[[\hat{\phi}^*]_b \left(k - \frac{\varepsilon\xi}{2}, \frac{t}{\varepsilon}\right) [\hat{\phi}]_a \left(k + \frac{\varepsilon\xi}{2}, \frac{t}{\varepsilon}\right)\right] d\xi.$$

We also have a similar assumption to (W1) in the multi-dimensional case:

(W1') For each pair (a,b) with $1 \le a, b \le \mathbb{d}$, the component $W^\varepsilon_{a,b}(dx, dk, 0)$ converges (at least along a subsequence) to a non-negative measure $(\mu_0)_{a,b}(dx, dk)$.

The Hamiltonian operator is defined as

$$O^H := \sum_{n\in\mathbb{Z}^{\mathbb{d}}} \alpha_n \cdot \nabla_{\beta_n} - \sum_{n,n'\in\mathbb{Z}^{\mathbb{d}}} \sigma(n-n')\beta_{n'} \cdot \nabla_{\alpha_n}.$$

2.2. *Control setting in the 1-dimensional case*

We now begin setting up our control framework. First, we fix a parameter $N \in \mathbb{N}$. The controlled parameters consist of the $2N$ real numbers:

$$M_N(-N),\ M_N(-N+1),\ \ldots,\ M_N(-1),\ M_N(1),\ \ldots,\ M_N(N).$$

We take a small note that we only define the parameters for integer d such that $1 \le |d| \le N$, we do not define $M_N(0)$.

We impose the following two assumptions on M_N:

$(M1)$ $M_N(d) + M_N(-d) = 0$ for all $1 \le |d| \le N$;
$(M2)$ For each $1 \le |d| \le N/2$, either $M_N(d) = 0$ or $M_N(2d) = 0$.

Assumption $(M1)$ ensures the antisymmetric (or skew-symmetric) structure of the control, while Assumption $(M2)$ addresses a technical issue by ensuring conservation properties of the perturbation.

Definition 1. Given a controlled M_N, we define the associated vector field by

$$\lambda_n^{M_N} := \sum_{\substack{d\in\mathbb{Z}\\ 1\le|d|\le N}} M_N(d)(\alpha_n - \alpha_{n+d}) \frac{\partial}{\partial \alpha_{n-d}} + \left(\sum_{\substack{d\in\mathbb{Z}\\ 1\le|d|\le N}} M_N(d)\alpha_{n+d} \right) \frac{\partial}{\partial \alpha_n}. \tag{10}$$

The corresponding controlled perturbation operator is defined as

$$O^{M_N} := \sum_{n\in\mathbb{Z}} \left(\lambda_n^{M_N}\right)^2. \tag{11}$$

We will show in Section 4 that the operator O^{M_N} conserves both the total momentum and the total energy of the system.

Definition 2. Given a controlled parameter M_N, we define the coefficient function

$$K_{M_N} : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R}$$

as follows:

- For $d = d' = 0$,

$$K_{M_N}(0,0) := 3\sum_{d=1}^{N} M_N(d)^2. \tag{12}$$

- For $1 \le |d| \le N$ and $d' = 0$,

$$K_{M_N}(d,0) = K_{M_N}(0,d) := -M_N(d)^2 - \frac{1}{2}\sum_{\substack{d_1+d_2=d\\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2). \tag{13}$$

- For $N+1 \le |d| \le 2N$ and $d' = 0$,

$$K_{M_N}(d,0) = K_{M_N}(0,d) := -\frac{1}{2}\sum_{\substack{d_1+d_2=d\\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2). \tag{14}$$

- For $0 < |d| = |d'| \le N/2$,

$$K_{M_N}(d,d') := -\frac{1}{4}\sum_{\substack{d_1+d_2=d\\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2) - M_N(d)M_N(2d). \tag{15}$$

- For $0 < |d| = |d'|$, with $N/2 < |d| \le 2N$,

$$K_{M_N}(d,d') := -\frac{1}{4}\sum_{\substack{d_1+d_2=d\\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2). \tag{16}$$

- For $1 \le |d| \le N < |d'| \le 2N$ and $|d-d'| \le N$,

$$K_{M_N}(d,d') := \frac{1}{2}M_N(d)M_N(d'-d). \tag{17}$$

- For $1 \le |d| \le N < |d'| \le 2N$ and $|d+d'| \le N$,

$$K_{M_N}(d,d') := \frac{1}{2} M_N(d) M_N(d'+d). \tag{18}$$

- For $1 \le |d'| \le N < |d| \le 2N$ and $|d'-d| \le N$,

$$K_{M_N}(d',d) := \frac{1}{2} M_N(d') M_N(d-d'). \tag{19}$$

- For $1 \le |d'| \le N < |d| \le 2N$ and $|d'+d| \le N$,

$$K_{M_N}(d',d) := \frac{1}{2} M_N(d') M_N(d+d'). \tag{20}$$

- For $1 \le |d|, |d'| \le N$ and $1 \le |d'-d|, |d'+d| \le N$,

$$\begin{aligned} K_{M_N}(d,d') = K_{M_N}(d',d) := \frac{1}{2}\Big(& M_N(d') M_N(d-d') + M_N(d) M_N(d'-d) \\ & - M_N(d') M_N(d+d') - M_N(d) M_N(d+d') \Big). \end{aligned} \tag{21}$$

- For $1 \le |d|, |d'| \le N$ and $1 \le |d'-d| \le N < |d'+d|$,

$$K_{M_N}(d,d') = K_{M_N}(d',d) := \frac{1}{2}\left(M_N(d') M_N(d-d') + M_N(d) M_N(d'-d) \right). \tag{22}$$

- For $1 \le |d|, |d'| \le N$ and $1 \le |d'+d| \le N < |d'-d|$,

$$K_{M_N}(d,d') = K_{M_N}(d',d) := -\frac{1}{2}\left(M_N(d') M_N(d+d') + M_N(d) M_N(d'+d) \right). \tag{23}$$

- For all other cases, define

$$K_{M_N}(d,d') := 0.$$

The transition kernel is derived by computing the changes in the energy $|\hat{\phi}|^2$ under the perturbed operator O^{M_N}. Intuitively, the transition kernel is the Fourier transform of $K_{M_N}(d,d')$, which has the form of a part of a convolution. Specifically, K_{M_N} is sum of some terms of the form $M_N(d_1)M_N(d_2)$ where d_1+d_2 is d or d'. Precise computation of K_{M_N} yields Definition 2.

Definition 3. We define the controlled kernel corresponding to M_N by

$$\hat{K}_{M_N}(k,k') := \sum_{d,d' \in \mathbb{Z}} e^{-2\pi i(dk+d'k')} K_{M_N}(d,d') = \sum_{|d|,|d'| \le 2N} K_{M_N}(d,d') \cos(2\pi dk) \cos(2\pi d'k'). \tag{24}$$

Referring to (5), the controlled perturbation system governing the wave is

$$\begin{aligned}\dot{\beta}_n(t) &= \alpha_n(t),\\ d\alpha_n(t) &= -\sigma \star \beta_n(t)\,dt - 2\varepsilon\left(\sum_{|d|\le 2N} K_{M_N}(d,0)\alpha_{n+d}(t)\right)dt\\ &\quad + \sqrt{2\varepsilon}\sum_{|d|\le N}\lambda_{n+d}^{M_N}\alpha_n(t)\,Y_{n+d}(dt).\end{aligned} \tag{25}$$

2.3. *Control setting in the multi-dimensional case*

2.3.1. *Notion of path*

To define a control on multi-dimensional perturbations, we introduce the notion of a *path*. As a path on the lattice resembles a chain, we only need to make a small modification in the computations to find the transition kernels. We first roughly introduce the controls based on a path γ.

- For a *simple index expansion* (see Section 2.3.2), we consider a controlled vector $M_\gamma(d) \in \mathbb{R}^{\mathbb{d}}$, for $1 \le d \le N$. Each component $[M_\gamma(d)]_a$ is then used to define a component of the vector field $\lambda_n^{M_\gamma}$.
- For a *dual indices expansion* (see Section 2.3.3), a pair of indices (a, b) is chosen first, then the controls $M_\gamma^{a,b}(d) \in \mathbb{R}$, for $1 \le d \le N$. After that, we also define a controlled vector field $\lambda_{n,\gamma}^{a,b}$.

Let us now define the notion of a path. The space $\mathbb{Z}^{\mathbb{d}}$ is partitioned into $2^{\mathbb{d}}$ regions using the $\mathbb{d}$ coordinate hyperplanes $\{n \in \mathbb{Z}^{\mathbb{d}} \mid [n]_a = 0\}$ for $a = 1, \dots, \mathbb{d}$, where $[n]_a$ denotes the a-th component of the vector n.

We index these regions using integers $p \in \{0, 1, \dots, 2^{\mathbb{d}} - 1\}$. Each index p is represented in binary as

$$p = p_1 2^{\mathbb{d}-1} + p_2 2^{\mathbb{d}-2} + \cdots + p_{\mathbb{d}} = (p_1 p_2 \dots p_{\mathbb{d}})_2,$$

where each $p_a \in \{0, 1\}$.

For each p, we define the associated *set* p as the set

$$\left\{ n \in \mathbb{Z}^{\mathbb{d}} \;\middle|\; (-1)^{p_a}[n]_a \ge 0 \text{ for all } a = 1, \dots, \mathbb{d} \right\}.$$

Note that for each $p \in \{0, 1, \dots, 2^{\mathbb{d}} - 1\}$, there exists a unique index $p' \in \{0, 1, \dots, 2^{\mathbb{d}} - 1\}$ such that the binary digits of p' are the bitwise complements of those of p-that is, $p'_a = 1 - p_a$ for each a. In other words, set p' is the reflection of set p across all coordinate hyperplanes.

Definition 4. On the set p and for a number $N \in \mathbb{N}$, we define:

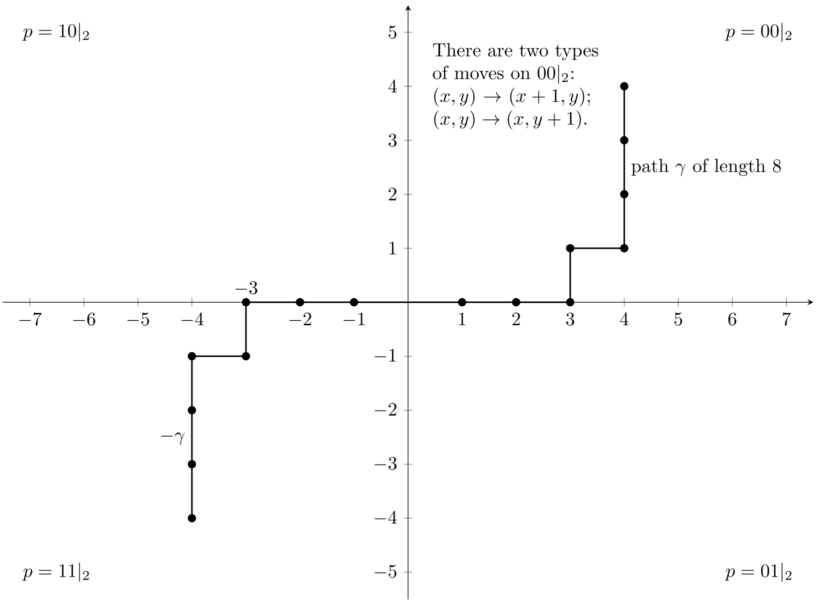

Fig. 1. A path and its opposite when $\mathbb{d} = 2$.

- A *move* on the a-th coordinate is the map

$$n \mapsto n + (-1)^{p_a} e_a,$$

where e_a is the vector in $\mathbb{Z}^{\mathbb{d}}$ with 1 at the a-th coordinate and 0 elsewhere.
- A *path* is a sequence of points obtained by making consecutive moves. We say a path is of length N if the L^1-distance between its endpoints is N. We denote by $\Gamma_{N,p}$ the family of all paths of length N on the set p starting from the origin. For each $\gamma \in \Gamma_{N,p}$, we denote by γ_d the point reached after d moves, for $d \geq 1$.
- The *symmetric* of a path γ with respect to the origin is denoted by $-\gamma$. For each $\gamma \in \Gamma_{N,p}$, we have $-\gamma \in \Gamma_{N,p'}$, where p' is the complement of p defined previously (see Fig. 1).

We note that a path that lies on an axis belongs to multiple regions. For this reason, when we mention a path, we also assume that the set p is predetermined.

Let us establish a combinatorial lemma on the number of paths passing through a given point.

Lemma 2. *Given a point $D = ([D]_a)_{1 \leq a \leq \mathbb{d}}$ in the set $p \in \{0, 1, \ldots, 2^{\mathbb{d}} - 1\}$, if*

$$\|D\|_{L^1} = |[D]_1| + \cdots + |[D]_{\mathbb{d}}| \in [1, N],$$

then the number of paths in $\Gamma_{N,p}$ passing through D is

$$\frac{\left(\|D\|_{L^1}\right)!}{(|[D]_1|)! \cdots (|[D]_{\mathbb{d}}|)!} \mathbb{d}^{N - \|D\|_{L^1}}.$$

Proof. If $[D]_a < 0$ for some $1 \le a \le \mathbb{d}$, then we use the symmetry with respect to the hyperplane $\{n \in \mathbb{Z}^{\mathbb{d}} \mid [n]_a = 0\}$ and define $\bar{p}$ by

$$\bar{p}_b = p_b \text{ for } b \neq a, \quad \text{and} \quad \bar{p}_a = 1 - p_a.$$

The point D has the symmetric $\bar{D}$ such that

$$[\bar{D}]_b = [D]_b \text{ for } b \neq a, \quad \text{and} \quad [\bar{D}]_a = -[D]_a.$$

The number of paths through D in p is equal to the number of paths through $\bar{D}$ in $\bar{p}$. Therefore, without loss of generality, we assume that D is in the set 0, i.e., $[D]_a \ge 0$ for all $1 \le a \le \mathbb{d}$.

From the origin, we have to take $[D]_1$ moves of $+e_1$, $[D]_2$ moves of $+e_2$, ..., $[D]_{\mathbb{d}}$ moves of $+e_{\mathbb{d}}$ to reach D. The number of paths of length $\|D\|_{L^1}$ having the endpoint D is equal to the number of different arrangements of those moves, which is

$$\frac{(\|D\|_{L^1})!}{([D]_1)! \cdots ([D]_{\mathbb{d}})!}.$$

For each path of length $\|D\|_{L^1}$ ending at D, we define a path of length N by adding $N - \|D\|_{L^1}$ moves picked from $+e_1, +e_2, \dots, +e_{\mathbb{d}}$. There are $\mathbb{d}^{N-\|D\|_{L^1}}$ ways to pick them. The conclusion of the lemma follows. □

2.3.2. Simple index expansion

Now, we have the framework to define the control parameters $M_\gamma(d) \in \mathbb{R}^{\mathbb{d}}$; we refer to it as parameter M for short. For each number p and path $\gamma \in \Gamma_{N,p}$, $M_\gamma(d)$ is zero if $d < 1$ or $d > N$. Similarly to $(M1)$ and $(M2)$, we impose two assumptions on the parameter M:

(M1') For p and $\gamma \in \Gamma_{N,p}$,

$$M_\gamma(d) + M_{-\gamma}(d) = 0.$$

(M2') For $D \in \mathbb{Z}^{\mathbb{d}}$ with $1 \le \|D\|_{L^1} \le N/2$,

$$\sum_{p=0}^{2^{\mathbb{d}}-1} \sum_{\substack{\gamma \in \Gamma_{N,p} \\ D \in \gamma}} M_\gamma(\|D\|_{L^1}) = 0, \quad \text{or} \quad \sum_{p=0}^{2^{\mathbb{d}}-1} \sum_{\substack{\gamma \in \Gamma_{N,p} \\ 2D \in \gamma}} M_\gamma(\|2D\|_{L^1}) = 0.$$

Similarly to $(M1)$, Assumption (M1') ensures the antisymmetry of the control. Assumption (M2') is analogous to $(M2)$ and addresses the technical issue related to the conservation of the perturbation. We interpret the second assumption as follows: when $1 \le \|D\|_{L^1} \le N/2$, either the total controlled coefficients over all the paths passing through D or over all the paths passing through $2D$ sum to zero.

Definition 5. Given p, N, $\gamma \in \Gamma_{N,p}$, and parameter M, we define the a-th component of the vector field $\lambda_n^{M_\gamma}$ by

$$[\lambda_n^{M_\gamma}]_a := \sum_{d=1}^{N}[M_\gamma(d)]_a\big([\alpha_n]_a - [\alpha_{n+\gamma_d}]_a\big)\frac{\partial}{\partial[\alpha_{n-\gamma_d}]_a} + \sum_{d=1}^{N}[M_\gamma(d)]_a[\alpha_{n+\gamma_d}]_a\frac{\partial}{\partial[\alpha_n]_a}.$$

The total vector field on M_γ is

$$\lambda_n^{M_\gamma} := \sum_{a=1}^{\mathbb{d}}[\lambda_n^{M_\gamma}]_a.$$

Then, the vector field of length N is defined by

$$\lambda_n^{M} := \sum_{\gamma}\lambda_n^{M_\gamma},$$

where the notation $\sum_\gamma$ abbreviates $\sum_{p=0}^{2^{\mathbb{d}}-1}\sum_{\gamma\in\Gamma_{N,p}}$.

The perturbed operator is defined as

$$O_s^M := \sum_n(\lambda_n^M)^2. \tag{26}$$

Definition 6. For $N\in\mathbb{N}$, a parameter M and (a,b) such that $1\le a,b\le\mathbb{d}$, we define the kernel coefficients $[K_M(D,D')]_{b,a}$ as follows:

- $D=D'=0$,

$$[K_M(0,0)]_{b,a} := \frac{3}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}=\gamma^2_{d_2}}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.$$

- For $1\le\|D\|_{L^1}\le N$,

$$\begin{aligned}[K_M(0,D)]_{b,a} = [K_M(D,0)]_{b,a} := &-\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}}[M_{\gamma^1}(\|D\|_{L^1})]_a[M_{\gamma^2}(\|D\|_{L^1})]_b\\ &-\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.\end{aligned}$$

- For $N+1\le\|D\|_{L^1}\le 2N$,

$$[K_M(0,D)]_{b,a} = [K_M(D,0)]_{b,a} := -\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.$$

- For $1 \le \|D\|_{L^1} \le N$,

$$[K_M(D,D)]_{b,a} := -\sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b - \frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2 \\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}} [M_{\gamma^1}(d_1)]_a [M_{\gamma^2}(d_2)]_b.$$

- For $1 \le \|D\|_{L^1} \le N/2$,

$$\begin{aligned}[K_M(D,-D)]_{b,a} := &\sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b \\ &- \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=D,\gamma^2_{\|2D\|_{L^1}}=2D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|2D\|_{L^1})]_b \\ &- \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|2D\|_{L^1}}=2D,\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|2D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b.\end{aligned}$$

- For $N/2 < \|D\|_{L^1} \le N$,

$$[K_M(D,-D)]_{b,a} := \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b.$$

- For $N+1 \le \|D\|_{L^1} \le 2N$,

$$[K_M(D,D)]_{b,a} := -\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2 \\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}} [M_{\gamma^1}(d_1)]_a [M_{\gamma^2}(d_2)]_b.$$

- For $1 \le \|D\|_{L^1}, \|D'\|_{L^1} \le N$ and $D \ne \pm D'$,

$$[K_M(D,D')]_{b,a} := \frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=D \\ \gamma^2_{\|D'-D\|_{L^1}}=D'-D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D'-D\|_{L^1})]_b$$

$$+\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D'-D\|_{L^1}}=D'-D\\ \gamma^2_{\|D\|_{L^1}}=D}}[M_{\gamma^1}(\|D'-D\|_{L^1})]_a[M_{\gamma^2}(\|D\|_{L^1})]_b$$

$$+\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D'\|_{L^1}}=D'\\ \gamma^2_{\|D-D'\|_{L^1}}=D-D'}}[M_{\gamma^1}(\|D'\|_{L^1})]_a[M_{\gamma^2}(\|D-D'\|_{L^1})]_b$$

$$+\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D-D'\|_{L^1}}=D-D'\\ \gamma^2_{\|D'\|_{L^1}}=D'}}[M_{\gamma^1}(\|D-D'\|_{L^1})]_a[M_{\gamma^2}(\|D'\|_{L^1})]_b$$

$$-\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D'\|_{L^1}}=D'\\ \gamma^2_{\|D\|_{L^1}}=D}}[M_{\gamma^1}(\|D'\|_{L^1})]_a[M_{\gamma^2}(\|D\|_{L^1})]_b$$

$$-\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D\|_{L^1}}=D\\ \gamma^2_{\|D'\|_{L^1}}=D'}}[M_{\gamma^1}(\|D\|_{L^1})]_a[M_{\gamma^2}(\|D'\|_{L^1})]_b.$$

- $K_M(D, D') := 0$ for other cases.

By using notion of path, the computation for the coefficient function in Definition 6 is similar to the computation for the coefficient function in Definition 2 as the purpose of our model. A key difference is that, for simple index expansion, we can no longer simplify some terms due to a loss of symmetry, and this makes the coefficients more complicated.

Definition 7. For a parameter M, our kernel is the matrix of coefficients

$$\hat{K}_M(k,k') := \left(\sum_{D,D'}[K_M(D,D')]_{a,b}\cos[2\pi(D\cdot k + D'\cdot k')]\right). \tag{27}$$

This kernel corresponds to the system

$$\begin{aligned} d\beta_n(t) &= \alpha_n(t)\,dt,\\ d\alpha_n(t) &= -\sigma\star\beta_n(t)\,dt - 2\varepsilon\left(\sum_{\|D\|_{L^1}\le N}\operatorname{diag}\left([K_M(D,0)]_{a,a}\right)\alpha_{n+D}\right)dt \end{aligned} \tag{28}$$

$$+\sum_{a=1}^{\mathbb{d}}\sum_{\|D\|_{L^1}\le N}\lambda^M_{n+D}\alpha_n\, Y^a_{n+D}(dt),$$

where the Y^a_{n+D}'s are independent standard Wiener processes.

2.3.3. Dual indices expansion

Let us consider the energy exchange between component a and component b between particles on a single path. For $a, b = 1, \dots, \mathbb{d}$ and $\gamma \in \Gamma_{N,p}$, we design a new control parameter $M^{a,b}_\gamma(d) \in \mathbb{R}$, for $1 \le d \le N$.

We impose two assumptions on the controlled parameter $M^{a,b}_\gamma$:

(M1") For a pair (a, b) and a path γ,

$$M^{a,b}_\gamma = M^{a,b}_{-\gamma} \quad \text{or} \quad M^{a,b}_\gamma = -M^{a,b}_{-\gamma}.$$

(M2") For a pair (a, b) and a path γ,

$$\sum_{1\le d_1<d_2\le N} M^{a,b}_\gamma(d_1)M^{a,b}_\gamma(d_2) = 0.$$

Definition 8. For each $M^{a,b}$ and $\gamma \in \Gamma_{N,p}$, we define the vector field

$$\begin{aligned}\lambda^{a,b}_{n,\gamma} :=& \sum_{d=1}^{N} M^{a,b}_\gamma(d)\left([\alpha_{n+\gamma_d}]_b - [\alpha_n]_b\right)\sum_{d=1}^{N} M^{a,b}_\gamma(d)\left(\frac{\partial}{\partial[\alpha_{n+\gamma_d}]_a} - \frac{\partial}{\partial[\alpha_n]_a}\right)\\ &- \sum_{d=1}^{N} M^{a,b}_\gamma(d)\left([\alpha_{n+\gamma_d}]_a - [\alpha_n]_a\right)\sum_{d=1}^{N} M^{a,b}_\gamma(d)\left(\frac{\partial}{\partial[\alpha_{n+\gamma_d}]_b} - \frac{\partial}{\partial[\alpha_n]_b}\right).\end{aligned}$$

Using this vector field, the perturbed operator is defined by

$$O^M_d := \sum_{n\in\mathbb{Z}^{\mathbb{d}}}\sum_\gamma\sum_{a,b}\left(\lambda^{a,b}_{n,\gamma}\right)^2. \tag{29}$$

In the case $a = b$, it is clear that $\lambda^{a,a}_{n,\gamma} = 0$. Thus, we only consider the case $a \ne b$.

Definition 9. For $a \ne b$, the coefficient $K^{a,b}$ is defined as follows:

- For $D = D' = 0$,

$$K^{a,b}(0,0) := \frac{1}{2}\sum_\gamma\left(\left(\sum_{d=1}^{N} M^{a,b}_\gamma(d)\right)^2 + \left(\sum_{d=1}^{N}(M^{a,b}_\gamma(d))^2\right)\right)^2.$$

- For $1 \le \|D\|_{L^1} \le N$,

$$K^{a,b}(0,D) = K^{a,b}(D,0) := -\sum_{\substack{\gamma \\ \gamma_{\|D\|_{L^1}}=D}} M_\gamma^{a,b}(\|D\|_{L^1}) \left(\sum_{d=1}^N M_\gamma^{a,b}(d)\right) \times \left(\left(\sum_{d=1}^N M_\gamma^{a,b}(d)\right)^2 + \sum_{d=1}^N (M_\gamma^{a,b}(d))^2\right) + \sum_{\substack{\gamma,d_1,d_2 \\ \gamma_{d_2}-\gamma_{d_1}=D}} M_\gamma^{a,b}(d_1) M_\gamma^{a,b}(d_2) \times \left(\left(\sum_{d=1}^N M_\gamma^{a,b}(d)\right)^2 + \sum_{d=1}^N (M_\gamma^{a,b}(d))^2\right).$$

- For $1 \le \|D\|_{L^1}, \|D'\|_{L^1} \le N$ and D, D' are both in the same set p (or D in p and D' in p'),

$$K^{a,b}(D,D') := \sum_{\substack{\gamma \\ \gamma_{\|D\|_{L^1}}=D \\ \gamma_{\|D'\|_{L^1}}=D'}} \left(\sum_{d=1}^N M_\gamma^{a,b}(d)\right)^2 M_\gamma^{a,b}(\|D\|_{L^1}) M_\gamma^{a,b}(\|D'\|_{L^1}) - \sum_{\substack{\gamma,d_1,d_2 \\ \gamma_{\|D\|_{L^1}}=D \\ \gamma_{d_2}-\gamma_{d_1}=D'}} M_\gamma^{a,b}(\|D\|_{L^1}) \left(\sum_{d=1}^N M_\gamma^{a,b}(d)\right) M_\gamma^{a,b}(d_1) M_\gamma^{a,b}(d_2).$$

  In the case where D is in set p and D' is in set p', replace $M_\gamma^{a,b}$ by $M_{-\gamma}^{a,b}$ in the expression.
- $K^{a,b}(D,D') := 0$ in all other cases.

The computation for $K^{a,b}$ in Definition 9 is similar to the computation for K_{M_N} in Definition 2. For dual indices, the vector field $\lambda_{n,\gamma}^{a,b}$ contains two controlled parameters $M_\gamma^{a,b}$'s in each term. Consequently, $K^{a,b}$ has four controlled parameters $M_\gamma^{a,b}$'s in each term as opposed to just two controlled parameters in the 1-dimensional case and the simple index case.

Definition 10. For each $a \ne b$ and parameter $M^{a,b}$, we define the dual indices kernel as

$$\hat{K}^{a,b}(k,k') := \sum_{D,D'} K^{a,b}(D,D') \cos(2\pi D \cdot k) \cos(2\pi D' \cdot k'). \tag{30}$$

This kernel $\hat{K}^{a,b}$ corresponds to the system:

$$d\beta_n(t) = \alpha_n(t)\, dt,$$

$$
\begin{aligned}
d\alpha_n(t) = &-\sigma \star \beta_n(t)\,dt - 2\varepsilon \left(\sum_{a,b} \sum_{\|D\|_{L^1} \le N} K^{a,b}(D,0)\alpha_{n+D}(t) \right) dt \\
&+ \sqrt{2\varepsilon} \sum_{a,b} \sum_{\gamma} \lambda^{a,b}_{n,\gamma} \alpha_n(t) Y^{a,b}_{n,\gamma}(dt) \\
&+ \sum_{a,b} \sum_{1 \le \|D\|_{L^1} \le N} \sum_{\gamma_{\|D\|_{L^1}} = -D} \lambda^{a,b}_{n+D,\gamma} \alpha_n(t) Y^{a,b}_{n+D,\gamma}(dt),
\end{aligned} \tag{31}
$$

where the $Y^{a,b}_{n,\gamma}$'s are independent standard Wiener processes.

3. Main results

3.1. The 1-dimensional control results

Lemma 3. *For the coupling σ and M_N satisfying Assumptions (σ1) - (σ4), (M1) and (M2), we have*

$$
\sum_{|d| \le 2N} K_{M_N}(d, d') = 0
$$

for every d' such that $|d'| \le 2N$. As a consequence,

$$
\sup_{k \in \mathbb{T}} \left| \frac{\hat{K}_{M_N}(k, k')}{\omega(k)} \right| < \infty.
$$

Using the kernel defined in (24), we also define the collision operator on the Schwartz space $\mathscr{S}(\mathbb{R} \times \mathbb{T})$ or on the space of continuous real-valued functions on $\mathbb{T}$ by

$$
O^{M_N}_{col} S(x,k) := 2 \int_{\mathbb{T}} \hat{K}_{M_N}(k,k') \big(S(x,k') - S(x,k) \big)\, dk'.
$$

Theorem 4. *Considering the wave system governed by (25), we assume that the coupling σ, the initial state, and M_N satisfy Assumptions (σ1) - (σ4), (μ1) - (μ3), (W1), (M1), and (M2). Then, $E^{\varepsilon}(t)$ subsequentially weakly-* converges as $\varepsilon \to 0$ to $\nu(t) = \nu(t, dk)$, which is the solution of*

$$
\frac{d}{dt} \langle S, \nu(t) \rangle = \langle O^{M_N}_{col} S, \nu(t) \rangle \tag{32}
$$

for every bounded real-valued function S on $\mathbb{T}$, for all $t \in [0, T]$, with $\nu(0, dk) = \nu_0(dk)$. Moreover, under Assumption (μ4) in the no-pinning case,

$$
\lim_{R \to 0} \limsup_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_{|k| < R} \mathbb{E}_{\varepsilon} \left[|\hat{\phi}(k, t/\varepsilon)|^2 \right] dk = 0, \quad \forall t \in [0, T]. \tag{33}
$$

Theorem 5. *Considering the wave system governed by (25), we assume that the coupling σ, the initial state, and parameter M_N satisfy Assumptions (σ1) - (σ4), (μ1) - (μ4), (W1), (M1), and (M2). Then, $W^\varepsilon(t)$ subsequentially weakly-* converges as $\varepsilon \to 0$ to $\mu(t) = \mu(t, dx, dk)$, which is the solution of the Boltzmann equation*

$$\frac{d}{dt}\langle S, \mu(t)\rangle = \frac{1}{2\pi}\langle \omega'(k)\frac{\partial S}{\partial x}, \mu(t)\rangle + \langle O_{col}^{M_N} S, \mu(t)\rangle$$

for any test function S in the Schwartz space, for all $t \in [0, T]$, with $\mu(0, dx, dk) = \mu_0(dx, dk)$.

Remark 6. We control the Boltzmann equation in Theorem 5 by modifying the kernel through the action of M_N. Due to the constraints on M_N, we see that we can freely choose $M_N(d)$, $N/2 < d \le N$ and then set $M_N(d) = 0$, $1 \le d \le N/2$. Therefore, we can choose $N - \lfloor N/2 \rfloor$ values for K_{M_N}.

3.2. The multi-dimensional control results

We extend Theorem 5 in two directions mentioned in Section 2.3 using similar proofs. In the case of a simple index expansion, we define

$$\begin{aligned}
[O_{col}^{M} S(x,k)]_{a,b} := & \int\limits_{\mathbb{T}^{\mathrm{d}}} \left([\hat{K}_M(k,k')]_{a,b} + [\hat{K}_M(-k,k')]_{a,b}\right) S_{a,b}(x,k')dk' \\
& - \int\limits_{\mathbb{T}^{\mathrm{d}}} \left([\hat{K}_M(k,k')]_{a,a} + [\hat{K}_M(k,k')]_{b,b}\right) S_{a,b}(x,k)dk'
\end{aligned}$$

for $S = (S_{a,b})_{1\le a,b\le \mathrm{d}}$, $S \in \mathscr{S}(\mathbb{R}^{\mathrm{d}} \times \mathbb{T}^{\mathrm{d}}, \mathbb{M}_{\mathrm{d}})$, and $\hat{K}_M$ is defined in (27). The collision operator is defined by the matrix $O_{col}^{M} S(s,k) = ([O_{col}^{M} S(x,k)]_{a,b})_{1\le a,b\le \mathrm{d}}$.

Theorem 7. *Considering the wave system governed by (28), we assume that the coupling σ, the initial state, and parameter M satisfy Assumptions (σ1') - (σ4'), (μ1') - (μ4'), (W1'), (M1'), and (M2'). Then, the Wigner distribution $W^\varepsilon(t)$ subsequentially weakly-* converges as $\varepsilon \to 0$ to $\mu(t)$, which is the solution of the Boltzmann equation*

$$\frac{d}{dt}\langle S, \mu(t)\rangle = \frac{1}{2\pi}\langle \nabla\omega(k)\cdot \nabla_x S, \mu(t)\rangle + \langle O_{col}^{M} S, \mu(t)\rangle$$

for any test function S in the Schwartz space, for all $t \in [0, T]$, with $\mu(0, dx, dk) = \mu_0(dx, dk)$.

In the case of a dual indices expansion, for any a, b, we define

$$O_{col}^{a,b} S(x,k) := \begin{cases} \sum_{a'\neq a} \int_{\mathbb{T}^{\mathrm{d}}} \hat{K}^{a,a'}(k,k')(S_{a'}(x,k') - S_{a,a}(x,k))dk', & \text{when } a = b, \\ -\int_{\mathbb{T}^{\mathrm{d}}} (\hat{K}^{a,b}(k,k') + \hat{K}^{b,a}(k,k'))(S_{a,b}(x,k') + S_{a,b}(x,k))dk', & \text{when } a \neq b, \end{cases}$$

where $\hat{K}^{a,b}$ is defined by (30). The collision operator for a dual indices expansion is defined by the matrix

$$O_{col}S := (O_{col}^{a,b}S)_{1\leq a,b\leq \mathrm{d}}.$$

Theorem 8. *Considering the wave system governed by (31), we assume that the coupling σ, the initial state, and parameter M satisfy Assumptions (σ1') - (σ4'), (μ1') - (μ4'), (W1'), (M1"), and (M2"). Then, the Wigner distribution $W^{\varepsilon}(t)$ subsequentially weakly-* converges as $\varepsilon \to 0$ to $\mu(t)$, which is the solution of the Boltzmann equation*

$$\frac{d}{dt}\langle S, \mu(t)\rangle = \frac{1}{2\pi}\langle \nabla\omega(k)\cdot\nabla_x S, \mu(t)\rangle + \langle O_{col}S, \mu(t)\rangle$$

for any test function S in the Schwartz space, for all $t \in [0, T]$, with $\mu(0, dx, dk) = \mu_0(dx, dk)$.

3.3. Achievable target kernels

In the proposition below, we prove that target kernels of the form (34) are achievable.

Proposition 9 *(Achievable Target Kernels). In the 1-dimensional case, given any N, there exist $P_{a,b}(x, y)$ for $1 \leq a \leq b \leq N - \lfloor N/2\rfloor$ and a perturbation in (25) yielding the kernel*

$$\mathcal{K}(k, k') = \sin^2(\pi k)\sin^2(\pi k')v(\cos(\pi k), \cos(\pi k')), \tag{34}$$

where $v \in V = \left\{\sum_{1\leq a\leq b\leq N-\lfloor N/2\rfloor} C_a C_b P_{a,b}(x, y) \mid C_a \in \mathbb{R}\right\}$.

Remark 10. In Proposition 9, we choose $M_N(d) \leq 0, d \leq N/2$ for simplicity, the family of polynomials $\{P_{a,b}\}$ is computable. This provides a partial description of possible target kernels. For other cases, the computation can be repeated to obtain a different family $\{P_{a,b}\}$. Thus, the form (34) represents the general form of achievable target kernel, although the range V of v is larger than what we present here.

In the multi-dimensional case, obtaining such a result is much more difficult and beyond the objective of the present article. On one hand, we retain the properties $K_M(D, D') = K_M(D', D)$ and $K_M(D, D') = K_M(-D, -D')$, which are sufficient to derive the kernel from the controlled vectors M_γ. On the other hand, when transitioning from dimension one to higher dimensions, we lose the fact that $K_M(D, D') = K_M(-D, D')$. This significantly complicates the analysis. Furthermore, the number of free parameters in the multi-dimensional case increases exponentially with respect to the dimension.

4. Conservation of momentum and energy - generating operator

4.1. The 1-dimensional case

We claim that $\lambda_n^{M_N}$ conserves local momentum and energy. Indeed, by (M1), we have

$$\lambda_n^{M_N}\left(\sum_{d'=-N}^{N}\alpha_{d'}\right) = \sum_{d\in\mathbb{Z},1\leq|d|\leq N} M_N(d)(\alpha_n - \alpha_{n+d}) + \sum_{d\in\mathbb{Z},1\leq|d|\leq N} M_N(d)\alpha_{n+d} = 0, \tag{35}$$

and

$$\lambda_n^{M_N}\left(\sum_{d'=-N}^{N}\alpha_{d'}^2\right)$$
$$=2\sum_{d\in\mathbb{Z},1\le|d|\le N}M_N(d)(\alpha_n-\alpha_{n+d})\alpha_{n-d}+2\left(\sum_{d\in\mathbb{Z},1\le|d|\le N}M_N(d)\alpha_{n+d}\right)\alpha_n=0. \quad (36)$$

The generating operator is identified using the following lemma.

Lemma 11. *The operator*

$$O:=O^H+\varepsilon O^{M_N} \quad (37)$$

is the generator of the stochastic differential equation (25)*, where* O^{M_N} *is defined in* (11).

Proof. According to (7), the process is

$$\phi_n(t)=\frac{1}{\sqrt{2}}\begin{bmatrix}\sigma\star\beta_n(t)\\ \alpha_n(t)\end{bmatrix}.$$

Let us derive (37). We write (25) as

$$d\begin{bmatrix}\beta_n(t)\\ \alpha_n(t)\end{bmatrix}=\begin{bmatrix}\alpha_n(t)\\ -\sigma\star\beta_n(t)-2\varepsilon\left(\sum_{|d|\le 2N}K_{M_N}(d,0)\alpha_{n+d}(t)\right)\end{bmatrix}dt$$
$$+\sqrt{2\varepsilon}\begin{bmatrix}0 & \dots & 0\\ \lambda_{n-N}^{M_N}\alpha_n(t) & \dots & \lambda_{n+N}^{M_N}\alpha_n(t)\end{bmatrix}d\begin{bmatrix}dY_{n-N}(t)\\ \vdots\\ dY_{n+N}(t)\end{bmatrix}. \quad (38)$$

By Itô's formula, the generator of the SDE (38) is

$$\alpha_n(t)\frac{\partial}{\partial\beta_n}-\sigma\star\beta_n(t)\frac{\partial}{\partial\alpha_n}-2\varepsilon\left(\sum_{|d|\le 2N}K_{M_N}(d,0)\alpha_{n+d}(t)\right)\frac{\partial}{\partial\alpha_n}$$
$$+\varepsilon\left[\sum_{d=-N}^{N}\left(\lambda_{n+d}^{M_N}\alpha_n(t)\right)^2\right]\frac{\partial^2}{\partial\alpha_n^2}.$$

The term $\alpha_n\frac{\partial}{\partial\beta_n}-\sigma\star\beta_n\frac{\partial}{\partial\alpha_n}$ coincides with the Hamiltonian O^H. Thus, it remains to show that

$$O^{M_N}(f(\alpha_n)) = \left[-2\left(\sum_{|d|\le 2N} K_{M_N}(d,0)\alpha_{n+d}(t)\right)\frac{\partial}{\partial\alpha_n} + \left[\sum_{d=-N}^{N}\left(\lambda_{n+d}^{M_N}\alpha_n(t)\right)^2\right]\frac{\partial^2}{\partial\alpha_n^2}\right] f(\alpha_n), \forall n\in\mathbb{Z},$$

where f is a test function which is twice differentiable and compactly supported. We compute

$$\begin{aligned}
&O^{M_N}(f(\alpha_n)) \\
&= \sum_{1\le|d|\le N}(\lambda_{n+d}^{M_N})^2 f(\alpha_n) + (\lambda_n^{M_N})^2 f(\alpha_n) \\
&= \sum_{1\le|d|\le N}\lambda_{n+d}^{M_N} M_N(d)(\alpha_{n+d}-\alpha_{n+2d})\frac{\partial f(\alpha_n)}{\partial\alpha_n} + \lambda_n^{M_N}\left(\sum_{1\le|d|\le N} M_N(d)\alpha_{n+d}\right)\frac{\partial f(\alpha_n)}{\partial\alpha_n} \\
&= \sum_{1\le|d_1|,|d_2|\le N} M_N(d_1)M_N(d_2)\alpha_{n+d_1+d_2}\frac{\partial f(\alpha_n)}{\partial\alpha_n} + \sum_{1\le|d|\le N} M_N(d)^2(\alpha_{n+d}-\alpha_n)\frac{\partial f(\alpha_n)}{\partial\alpha_n} \\
&\quad - \sum_{1\le|d|\le N} M_N(d)^2(\alpha_n-\alpha_{n-d})\frac{\partial f(\alpha_n)}{\partial\alpha_n} + \sum_{1\le|d|\le N} M_N(d)^2(\alpha_{n+d}-\alpha_{n+2d})^2\frac{\partial^2 f(\alpha_n)}{\partial\alpha_n^2} \\
&\quad + \left(\sum_{1\le|d|\le N} M_N(d)\alpha_{n+d}\right)^2\frac{\partial^2 f(\alpha_n)}{\partial\alpha_n^2}.
\end{aligned} \tag{39}$$

On the other hand, the definition (10) gives

$$\begin{aligned}
\lambda_{n+d}^{M_N}\alpha_n &= M_N(d)(\alpha_{n+d}-\alpha_{n+2d}) \text{ if } 1\le|d|\le N, \\
\lambda_n^{M_N}\alpha_n &= \sum_{1\le|d|\le N} M_N(d)\alpha_{n+d}.
\end{aligned}$$

Thus, the coefficient of the second-order derivative in (39) is

$$\sum_{d=-N}^{N}\left(\lambda_{n+d}^{M_N}\alpha_n(t)\right)^2.$$

The definitions (12), (13), (14) also give us that the coefficient of the first-order derivative in (39) is

$$-2\left(\sum_{|d|\le 2N} K_{M_N}(d,0)\alpha_{n+d}\right).$$

This finally gives the system (25). □

4.2. The simple index case

Consider the local momentum and local energy around $n \in \mathbb{Z}^{\mathbb{d}}$:

$$\sum_{n' \in \mathbb{Z}^{\mathbb{d}}, \|n-n'\|_{L^1} \leq N} \alpha_{n'} \quad \text{and} \quad \sum_{n' \in \mathbb{Z}^{\mathbb{d}}, \|n-n'\|_{L^1} \leq N} \|\alpha_{n'}\|_{L^2}^2.$$

For all p and $\gamma \in \Gamma_{N,p}$, it is sufficient to prove that:

$$\lambda_n^{M_\gamma} + \lambda_n^{M_{-\gamma}}$$

conserves the local momentum and local energy.

We will design a $2N+1$-point set using the paths γ and $-\gamma$. This set will resemble a line in the 1-dimensional case. First, we define $n+\gamma$ as the $N+1$-point set obtained by translating γ by n, i.e., $\{n, n+\gamma_1, \ldots, n+\gamma_N\}$. We also define $n-\gamma = n+(-\gamma)$. The $2N+1$-point set is then $(n+\gamma) \cup (n-\gamma)$.

If $n' \notin (n+\gamma) \cup (n-\gamma)$, then:

$$(\lambda_n^{M_\gamma} + \lambda_n^{M_{-\gamma}})\alpha_{n'} = 0 \quad \text{and} \quad (\lambda_n^{M_\gamma} + \lambda_n^{M_{-\gamma}})\|\alpha_{n'}\|_{L^2}^2 = 0.$$

Thus, we only need to check conservation for the $2N+1$ points in $\{n\} \cup (n+\gamma) \cup (n-\gamma)$. We will separate each index a from 1 to $\mathbb{d}$ and work on each index a using computations similar to those in (35) and (36).

Lemma 12. *The operator*

$$O_s := O^H + \varepsilon O_s^M$$

is the generator of the SDE (28) *where* O_s^M *is defined by* (26).

Proof. This is sufficient to check that

$$O_s^M f(\alpha_n) = -2\sum_{a=1}^{\mathbb{d}} \sum_{\|D\|_{L^1} \leq N} [K_M(D,0)]_{a,a}[\alpha_{n+D}]_a \frac{\partial f}{\partial [\alpha_n]_a}$$
$$+ \sum_{a,b} \sum_{\|D\|_{L^1} \leq N} ([\lambda_{n+D}^M]_a [\alpha_n]_a)([\lambda_{n+D}^M]_b [\alpha_n]_b) \frac{\partial^2 f}{\partial [\alpha_n]_a \partial [\alpha_n]_b}.$$

It is proved by direct computations. We see that

$$O_s^M f(\alpha_n) = \sum_{1 \leq \|D\|_{L^1} \leq N} \left(\sum_\gamma \lambda_{n+D}^M\right)^2 f(\alpha_n) + \left(\sum_\gamma \lambda_n^M\right)^2 f(\alpha_n)$$

$$= \sum_{1\le\|D\|_{L^1}\le N} \lambda^M_{n+D} \sum_{\gamma_{\|D\|_{L^1}}=D} \sum_{a=1}^{\mathbb{d}} [M_\gamma(\|D\|_{L^1})]_a ([\alpha_{n+D}]_a - [\alpha_{n+2D}]_a) \frac{\partial f}{\partial[\alpha_n]_a}$$

$$+ \lambda^M_n \sum_{\gamma} \sum_{a=1}^{\mathbb{d}} \sum_{d=1}^{N} [M_\gamma(d)]_a [\alpha_{n+\gamma_d}]_a \frac{\partial f}{\partial[\alpha_n]_a}$$

$$= \sum_{a=1}^{\mathbb{d}} \sum_{\gamma^1,\gamma^2,d_1,d_2} [M_{\gamma^1}(d_1)]_a [M_{\gamma^2}(d_2)]_a [\alpha_{n+\gamma^1_{d_1}+\gamma^2_{d_2}}]_a \frac{\partial f}{\partial[\alpha_n]_a} \tag{40}$$

$$+ \sum_{a=1}^{\mathbb{d}} \sum_{1\le\|D\|_{L^1}\le N} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_a$$

$$\times ([\alpha_{n+D}]_a - [\alpha_n]_a) \frac{\partial f}{\partial[\alpha_n]_a} \tag{41}$$

$$- \sum_{a=1}^{\mathbb{d}} \sum_{1\le\|D\|_{L^1}\le N} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_a$$

$$\times ([\alpha_n]_a - [\alpha_{n-D}]_a) \frac{\partial f}{\partial[\alpha_n]_a} \tag{42}$$

$$+ \sum_{a,b} \sum_{1\le\|D\|_{L^1}\le N} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b$$

$$\times ([\alpha_{n+D}]_a - [\alpha_{n+2D}]_a)([\alpha_{n+D}]_b - [\alpha_{n+2D}]_b) \frac{\partial^2 f}{\partial[\alpha_n]_a \partial[\alpha_n]_b} \tag{43}$$

$$+ \sum_{a,b} \left(\sum_{\gamma^1} \sum_{d_1=1}^{N} [M_{\gamma^1}(d_1)]_a [\alpha_{n+\gamma^1_{d_1}}]_a \right) \left(\sum_{\gamma^2} \sum_{d_2=1}^{N} [M_{\gamma^2}(d_2)]_b [\alpha_{n+\gamma^2_{d_2}}]_b \right)$$

$$\times \frac{\partial^2 f}{\partial[\alpha_n]_a \partial[\alpha_n]_b}. \tag{44}$$

The coefficient of $[\alpha_n]_a \frac{\partial f}{\partial[\alpha_n]_a}$ is computed from $\gamma^1_{d_1} + \gamma^2_{d_2} = 0$ in (40), and from (41) and (42). These give:

$$-2[K_M(0,0)]_{a,a}.$$

For the coefficient of $[\alpha_{n+D}]_a \frac{\partial f}{\partial[\alpha_n]_a}$, (40) gives:

$$\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}} [M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_a.$$

Additionally, (41) and (42) give:

$$2\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a[M_{\gamma^2}(\|D\|_{L^1})]_a.$$

Thus, the coefficient is:

$$-2[K_M(D,0)]_{a,a}.$$

The first-order derivative part is now complete.

For the second-order derivative part, we fix a, b and compute

$$\sum_{\|D\|_{L^1}\le N}([\lambda^M_{n+D}]_a[\alpha_n]_a)([\lambda^M_{n+D}]_b[\alpha_n]_b) =([\lambda^M_n]_a[\alpha_n]_a)([\lambda^M_n]_b[\alpha_n]_b) \tag{45}$$

$$+\sum_{1\le\|D\|_{L^1}\le N}([\lambda^M_{n+D}]_a[\alpha_n]_a)([\lambda^M_{n+D}]_b[\alpha_n]_b). \tag{46}$$

We can now observe that (45) corresponds to the coefficient in (44), and (46) corresponds to (43) for each D. □

4.3. The dual indices case

We observe that the operator (29) conserves the total momentum and total energy. It is sufficient to check that $\lambda^{a,b}_{n,\gamma}$ conserves

$$\sum_{n'\in(n+\gamma)}[\alpha_{n'}]_a,\qquad \sum_{n'\in(n+\gamma)}[\alpha_{n'}]_b,\qquad \sum_{n'\in(n+\gamma)}([\alpha_{n'}]_a^2+[\alpha_{n'}]_b^2).$$

Indeed, we have

$$\left(\sum_{d=1}^N M^{a,b}_\gamma(d)\frac{\partial}{\partial[\alpha_{n+\gamma_d}]_a}-\sum_{d=1}^N M^{a,b}_\gamma(d)\frac{\partial}{[\alpha_n]_a}\right)\sum_{n'\in\{n\}\cup(n+\gamma)}[\alpha_{n'}]_a$$
$$=\left(\sum_{d=1}^N M^{a,b}_\gamma(d)-\sum_{d=1}^N M^{a,b}_\gamma(d)\right)=0. \tag{47}$$

We prove the conservation of the b-th component of the momentum similarly to (47). For the energy, we have

$$
\begin{aligned}
&\lambda_{n,\gamma}^{a,b} \sum_{n' \in \{n\} \cup (n+\gamma)} ([\alpha_{n'}]_a^2 + [\alpha_{n'}]_b^2) \\
&= 2 \left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_{n+\gamma_d}]_b - \sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_n]_b \right) \left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_{n+\gamma_d}]_a - \sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_n]_a \right) \\
&- 2 \left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_{n+\gamma_d}]_a - \sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_n]_a \right) \left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_{n+\gamma_d}]_b - \sum_{d=1}^{N} M_\gamma^{a,b}(d) [\alpha_n]_b \right) \\
&= 0.
\end{aligned}
$$

Lemma 13. *The operator*

$$O_d := O^H + \varepsilon O_d^M$$

is the generator of the SDE (31) where O_d^M is defined by (29).

Proof. We show that

$$
\begin{aligned}
O_d^M = &- 2 \left(\sum_{a,b} \sum_{\|D\|_{L^1} \le N} K^{a,b}(D,0) \alpha_{n+D}(t) \right) \cdot \frac{\partial}{\partial \alpha_n} \\
&+ \sum_{a,b} \sum_{\gamma} \sum_{a_1,a_2 \in \{a,b\}} [(\lambda_{n,\gamma}^{a,b} \alpha_n(t))(\lambda_{n,\gamma}^{a,b} \alpha_n(t))^\top]_{a_1,a_2} \frac{\partial^2}{\partial [\alpha_n]_{a_1} \partial [\alpha_n]_{a_2}} \\
&+ \sum_{a,b} \sum_{1 \le \|D\|_{L^1} \le N} \sum_{\gamma_{\|D\|_{L^1}} = -D} \sum_{a_1,a_2 \in \{a,b\}} [(\lambda_{n+D,\gamma}^{a,b} \alpha_n(t))(\lambda_{n+D,\gamma}^{a,b} \alpha_n(t))^\top]_{a_1,a_2} \\
&\times \frac{\partial^2}{\partial [\alpha_n]_{a_1} \partial [\alpha_n]_{a_2}}
\end{aligned}
$$

The computation of $O_d^M f$ yields

$$O_d^M f(\alpha_n) = \sum_{a,b} \sum_{\gamma} \sum_{1 \le \|D'\|_{L^1} \le N} (\lambda_{n+D',\gamma}^{a,b})^2 f(\alpha_n) + \sum_{a,b} \sum_{\gamma} (\lambda_{n,\gamma}^{a,b})^2 f(\alpha_n).$$

For $D' \in \mathbb{Z}^{\mathrm{d}}$, $1 \le \|D'\|_{L^1} \le N$, we compute

$$
\begin{aligned}
\sum_{\gamma} (\lambda_{n+D',\gamma}^{a,b})^2 f(\alpha_n) = &- \sum_{\gamma_{\|D'\|_{L^1}} = -D'} \sum_{d=1}^{N} M_\gamma^{a,b}(d) ([\alpha_{n+\gamma_d+D'}]_a - [\alpha_{n+D'}]_a) \qquad (48) \\
&\times \left(\left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_\gamma^{a,b}(d))^2 \right) \right) M_\gamma^{a,b}(\|D'\|_{L^1}) \frac{\partial f(\alpha_n)}{\partial [\alpha_n]_a}
\end{aligned}
$$

$$- \sum_{\gamma_{\|D'\|_{L^1}}=-D'} \sum_{d=1}^{N} M_{\gamma}^{a,b}(d)([\alpha_{n+\gamma_d+D'}]_b - [\alpha_{n+D'}]_b)$$

$$\times \left(\left(\sum_{d=1}^{N} M_{\gamma}^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_{\gamma}^{a,b}(d))^2 \right) \right) M_{\gamma}^{a,b}(\|D'\|_{L^1}) \frac{\partial f(\alpha_n)}{\partial [\alpha_n]_b}$$

$$+ \sum_{\gamma_{\|D'\|_{L^1}}=-D'} \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d+D'}]_b - [\alpha_{n+D'}]_b) \right)^2 \left(M_{\gamma}^{a,b}(\|D'\|_{L^1}) \right)^2 \frac{\partial^2 f(\alpha_n)}{\partial [\alpha_n]_a^2}$$

$$+ \sum_{\gamma_{\|D'\|_{L^1}}=-D'} \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d+D'}]_a - [\alpha_{n+D'}]_a) \right)^2 \left(M_{\gamma}^{a,b}(\|D'\|_{L^1}) \right)^2 \frac{\partial^2 f(\alpha_n)}{\partial [\alpha_n]_b^2}$$

$$- 2 \sum_{\gamma_{\|D'\|_{L^1}}=-D'} \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d+D'}]_a - [\alpha_{n+D'}]_a) \right) \left(M_{\gamma}^{a,b}(\|D'\|_{L^1}) \right)^2$$

$$\times \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d+D'}]_b - [\alpha_{n+D'}]_b) \right) \frac{\partial^2 f(\alpha_n)}{\partial [\alpha_n]_a \partial [\alpha_n]_b}.$$

We also have

$$\sum_{\gamma} (\lambda_{n,\gamma}^{a,b})^2 f(\alpha_n) = \sum_{\gamma} \sum_{d=1}^{N} M_{\gamma}^{a,b}(d)([\alpha_{n+\gamma_d}]_a - [\alpha_n]_a) \tag{49}$$

$$\times \left(\left(\sum_{d=1}^{N} M_{\gamma}^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_{\gamma}^{a,b}(d))^2 \right) \right) \left(\sum_{d=1}^{N} M_{\gamma}^{a,b} \right) \frac{\partial f(\alpha_n)}{\partial [\alpha_n]_a}$$

$$+ \sum_{\gamma} \sum_{d=1}^{N} M_{\gamma}^{a,b}(d)([\alpha_{n+\gamma_d}]_b - [\alpha_n]_b)$$

$$\times \left(\left(\sum_{d=1}^{N} M_{\gamma}^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_{\gamma}^{a,b}(d))^2 \right) \right) \left(\sum_{d=1}^{N} M_{\gamma}^{a,b} \right) \frac{\partial f(\alpha_n)}{\partial [\alpha_n]_b}$$

$$+ \sum_{\gamma} \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d}]_b - [\alpha_n]_b) \right)^2 \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}(d) \right)^2 \frac{\partial^2 f(\alpha_n)}{\partial [\alpha_n]_a^2}$$

$$+ \sum_{\gamma} \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d}]_a - [\alpha_n]_a) \right)^2 \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}(d) \right)^2 \frac{\partial^2 f(\alpha_n)}{\partial [\alpha_n]_b^2}$$

$$- 2 \sum_{\gamma} \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}([\alpha_{n+\gamma_d}]_a - [\alpha_n]_a) \right) \left(\sum_{d=1}^{N} M_{\gamma}^{a,b}(d) \right)^2$$

$$\times \left(\sum_{d=1}^{N} M_\gamma^{a,b} ([\alpha_{n+\gamma_d}]_b - [\alpha_n]_b) \right) \frac{\partial^2 f(\alpha_n)}{\partial [\alpha_n]_a \partial [\alpha_n]_b}.$$

When $D = 0$, (49) gives

$$\sum_\gamma \left(\left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_\gamma^{a,b}(d))^2 \right) \right) \left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) \right)^2 \frac{\partial}{\partial [\alpha_n]_a},$$

and (48) gives

$$\sum_{D'} \sum_{\gamma_{\|D'\|_{L^1}} = -D'} \left(\left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_\gamma^{a,b}(d))^2 \right) \right) \left(M_\gamma^{a,b}(\|D'\|_{L^1}) \right)^2 \frac{\partial}{\partial [\alpha_n]_a}.$$

The sum of two terms in case $D = 0$ is

$$-2K^{a,b}(0,0).$$

Now, we consider $1 \le \|D\|_{L^1} \le N$, (49) gives

$$\sum_{\gamma_{\|D\|_{L^1}} = D} \left(\left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) \right)^2 + \left(\sum_{d=1}^{N} (M_\gamma^{a,b}(d))^2 \right) \right) \left(\sum_{d=1}^{N} M_\gamma^{a,b}(d) \right) M_\gamma^{a,b}(\|D\|_{L^1}) \frac{\partial}{\partial [\alpha_n]_a}.$$

Similarly, equation (48) also yields the same term when $D = D'$. For the case $D \neq D'$, equation (48) produces a non-zero contribution only if there exists d_2 such that $\gamma_{d_2} - \gamma_{\|D'\|_{L^1}} = D$. Therefore, for $1 \le \|D\|_{L^1} \le N$, we obtain the coefficient

$$2 \sum_{\substack{\gamma, d_1, d_2 \\ \gamma_{d_2} - \gamma_{d_1} = D}} M_\gamma^{i,j}(d_1) M_\gamma^{i,j}(d_2) \left(\left(\sum_{d=1}^{N} M_\gamma^{i,j}(d) \right)^2 + \left(\sum_{d=1}^{N} \left(M_\gamma^{i,j}(d) \right)^2 \right) \right).$$

Hence, we obtain the coefficient for the first-order derivative part:

$$-2K^{i,j}(D,0).$$

For the second-order derivative parts, we see that

$$\begin{aligned} \lambda_{n,\gamma}^{a,b} \alpha_n = & - \sum_{d=1}^{N} M_\gamma^{a,b}(d) \left([\alpha_{n+\gamma_d}]_b - [\alpha_n]_b \right) \sum_{d=1}^{N} M_\gamma^{a,b}(d) e_a \\ & + \sum_{d=1}^{N} M_\gamma^{a,b}(d) \left([\alpha_{n+\gamma_d}]_a - [\alpha_n]_a \right) \sum_{d=1}^{N} M_\gamma^{a,b}(d) e_b. \end{aligned}$$

Hence, $(\lambda^{a,b}_{n,\gamma}\alpha_n)(\lambda^{a,b}_{n,\gamma}\alpha_n)^\top$ matches the second-order derivative part of (49). Similarly, by a direct computation, $(\lambda^{a,b}_{n+D,\gamma}\alpha_n)(\lambda^{a,b}_{n+D,\gamma}\alpha_n)^\top$ matches the second-order derivative part of (48). □

5. Computation of the kernel

5.1. The 1-dimensional case

In this section, we show how to compute the kernel K_{M_N} defined by (12) and (23). We start with the computation of the generator over the energy density $O|\hat{\phi}|^2$, where O is defined in (37). From (9), we get $O^H|\hat{\phi}|^2 = 0$. Therefore, we focus on $O^{M_N}|\hat{\phi}|^2$. Since O^{M_N} is a second-order operator, we have:

$$O^{M_N}|\hat{\phi}|^2 = (O^{M_N}\hat{\phi})\hat{\phi}^* + (O^{M_N}\hat{\phi}^*)\hat{\phi} + 2\sum_n \left(\lambda_n^{M_N}\hat{\phi}(k)\right)\left(\lambda_n^{M_N}\hat{\phi}^*(k)\right).$$

The kernel arises from the expression:

$$\sum_n \left(\lambda_n^{M_N}\hat{\phi}(k)\right)\left(\lambda_n^{M_N}\hat{\phi}^*(k)\right).$$

We compute:

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \left(\lambda_n^{M_N}\phi_{n'}\right)\left(\lambda_n^{M_N}\phi^*_{n''}\right)\\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[\sum_{\substack{d\in\mathbb{Z}\\1\le|d|\le N}} M_N(d)\,(\alpha_n-\alpha_{n+d})\,\delta_{n-n',d} + \left(\sum_{\substack{d\in\mathbb{Z}\\1\le|d|\le N}} M_N(d)\alpha_{n+d}\right)\delta_{n,n'}\right]\\
&\quad\times\left[\sum_{\substack{d\in\mathbb{Z}\\1\le|d|\le N}} M_N(d)\,(\alpha_n-\alpha_{n+d})\,\delta_{n-n'',d} + \left(\sum_{\substack{d\in\mathbb{Z}\\1\le|d|\le N}} M_N(d)\alpha_{n+d}\right)\delta_{n,n''}\right]\\
&= -\frac{1}{2}\sum_{d'=-2N}^{2N}\left(\sum_{\substack{1\le|d_1|,|d_2|\le N\\d_1+d_2=d'}} M_N(d_1)M_N(d_2)\left(\alpha_{n'-d_1}-\alpha_{n'-2d_1}\right)\left(\alpha_{n'-d_1}-\alpha_{n'-d_1+d_2}\right)\right)\delta_{n'-n'',d'}
\end{aligned}\tag{50}$$

$$-\frac{1}{2}\sum_{1\le|d_1|\le N} M_N(d_1)\left(\alpha_{n'-d_1}-\alpha_{n'-2d_1}\right)\left(\sum_{\substack{d_2\in\mathbb{Z}\\1\le|d_2|\le N}} M_N(d_2)\alpha_{n'-d_1+d_2}\right)\delta_{n'-n'',d_1}\tag{51}$$

$$+\frac{1}{2}\sum_{1\le|d_1|\le N} M_N(d_1)\left(\alpha_{n'}-\alpha_{n'+d_1}\right)\left(\sum_{\substack{d_2\in\mathbb{Z}\\1\le|d_2|\le N}} M_N(d_2)\alpha_{n'+d_2}\right)\delta_{n'-n'',d_1}\tag{52}$$

$$+\frac{1}{2}\left(\sum_{\substack{d\in\mathbb{Z}\\1\le|d|\le N}} M_N(d)\alpha_{n'+d}\right)^2 \delta_{n'-n'',0}. \tag{53}$$

Let us express the sum $\sum_{n,n',n''}(\lambda_n^{M_N}\phi_{n'})(\lambda_n^{M_N}\phi^*_{n''})$ in the form:

$$\sum_{n',n'',d,d'} K_{M_N}(d,d')\alpha_{n'}\alpha_{n'+d}\delta_{n'-n'',d'}.$$

Taking into account the invariance of $n'-n''$ under translation by the same amount for both n' and n'', we cancel the term:

$$-\frac{1}{2}M_N(d_1)\alpha_{n'-d_1}\left(\sum_{d_2} M_N(d_2)\alpha_{n'-d_1+d_2}\right)$$

in (51) with the term:

$$\frac{1}{2}M_N(d_1)\alpha_{n'}\left(\sum_{d_2} M_N(d_2)\alpha_{n'+d_2}\right)$$

in (52). Thus, we obtain the kernel K_{M_N} as follows:

- If $d=d'=0$, the term (50), with $d_1=-d_2$, produces

  $$2\sum_{d=1}^{N} M_N(d)^2,$$

  and the term (53) gives

  $$\sum_{d=1}^{N} M_N(d)^2.$$

  Thus, we obtain (12).
- If $d'=0$ and $1\le|d|\le N$, the term (50), with $d_1=-d_2=d$, gives

  $$-M_N(d)^2,$$

  and the term (53) gives

  $$-\frac{1}{2}\sum_{d_1+d_2=d,1\le|d_1|,|d_2|\le N} M_N(d_1)M_N(d_2).$$

  If $1\le|d'|\le N$ and $d=0$, the term (50) gives

$$-\frac{1}{2}\sum_{d_1+d_2=d, 1\le|d_1|,|d_2|\le N} M_N(d_1)M_N(d_2),$$

the term (51), also with $d_1 = -d_2 = d$, gives

$$-\frac{1}{2}M_N(d)^2,$$

and the term (52) gives

$$-\frac{1}{2}M_N(d)^2.$$

Thus, we obtain (13).

- If $d' = 0$ and $N+1 \le |d| \le 2N$, there is no contribution from the terms (50), (51), or (52), and the term (53) gives

$$K_N(d,0) = -\frac{1}{2}\sum_{\substack{d_1+d_2=d\\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2).$$

If $N+1 \le |d'| \le 2N$ and $d = 0$, the term (50) gives

$$K_N(0,d') = -\frac{1}{2}\sum_{\substack{d_1+d_2=d'\\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2).$$

Thus, we get (14).

- If $|d'| = |d|$, the term (50) produces

$$-\frac{1}{2}\sum_{d_1+d_2=d', 1\le|d_1|,|d_2|\le N} M_N(d_1)M_N(d_2).$$

With $d_1 = -d', d_2 = 2d'$ or $d_1 = 2d', d_2 = -d'$ in the case $|d| \le N/2$, it also produces

$$-M_N(d')M_N(2d').$$

When $|d| < N/2$, the term (51), with $d_1 = d', d_2 = -2d'$, produces

$$-\frac{1}{2}M_N(d')M_N(2d'),$$

and the term (52), with $d_1 = d', d_2 = 2d'$ produces

$$-\frac{1}{2}M_N(d')M_N(2d').$$

Therefore,

$$K_{M_N}(d,d') = -\frac{1}{4} \sum_{\substack{d_1+d_2=d \\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2) - M_N(d)M_N(2d) \quad \text{if } |d| \le N/2,$$

$$K_{M_N}(d,d') = -\frac{1}{4} \sum_{\substack{d_1+d_2=d \\ 1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2) \quad \text{for other cases.}$$

- If $1 \le |d| \le N < |d'| \le 2N$ and $|d'| - |d| \le N$, then

$$K_{M_N}(d,d') = \frac{1}{2} M_N(d)M_N(d'-d) \quad \text{if } |d'-d| \le N,$$

$$K_{M_N}(d,d') = -\frac{1}{2} M_N(d)M_N(d'+d) \quad \text{if } |d'+d| \le N.$$

- If $1 \le |d'| \le N < |d| \le 2N$ and $|d| - |d'| \le N$, then

$$K_{M_N}(d,d') = \frac{1}{2} M_N(d')M_N(d-d') \quad \text{if } |d-d'| \le N,$$

$$K_{M_N}(d,d') = -\frac{1}{2} M_N(d')M_N(d+d') \quad \text{if } |d+d'| \le N.$$

- If $1 \le |d'|, |d| \le N$ and $1 \le |d-d'|, |d+d'| \le N$, we compute from (50) with $d_1 = d, d_2 = d' - d$ or $d_1 = d - d', d_2 = d$, ensuring the symmetry by including the factor $1/2$. We also compute (51) with $d_1 = d', d_2 = d - d'$, (52) with $d_1 = d', d_2 = d + d'$. The coefficient is given by

$$K_N(d,d') = \frac{1}{2}\big(M_N(d')M_N(d-d') - M_N(d')M_N(d+d') - M_N(d)M_N(d-d') \\ - M_N(d)M_N(d+d')\big).$$

- If $1 \le |d'|, |d| \le N$ and $1 \le |d-d'| \le N < |d+d'|$, we compute from (50) with $d_1 = d, d_2 = d' - d$ or $d_1 = d - d', d_2 = d$, ensuring the symmetry by including the factor $1/2$. We also compute (51) with $d_1 = d', d_2 = d - d'$. The coefficient is given by

$$K_N(d,d') = \frac{1}{2}\left(M_N(d')M_N(d-d') - M_N(d)M_N(d-d')\right).$$

- If $1 \le |d'|, |d| \le N$ and $1 \le |d+d'| \le N < |d-d'|$, we obtain

$$-\frac{1}{2} M_N(d')M_N(d+d')$$

from the previous cases of (50). We also compute from (52) with $d_1 = d', d_2 = d + d'$. The coefficient is given by

$$K_N(d,d') = \frac{1}{2}\left(-M_N(d')M_N(d+d') - M_N(d)M_N(d+d')\right).$$

- $K_N(d,d') = 0$ in all other cases.

We obtain the coefficients K_{M_N} as defined in Definition 2.

On the kernel $\widehat{K}_{M_N}$ in Definition 3, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \left(\lambda_n^{M_N}\hat{\phi}(k)\right)\left(\lambda_n^{M_N}\hat{\phi}^*(k)\right)\\
&\quad= \sum_{n',n''\in\mathbb{Z}} e^{2\pi ik(n''-n')}\sum_n \left(\lambda_n^{M_N}\phi_{n'}\right)\left(\lambda_n^{M_N}\phi_{n''}\right)\\
&\quad= \sum_{n'd,d'} K_{M_N}(-d',d)e^{-2\pi ikd}\alpha_{n'}\alpha_{n'-d'}\\
&\quad= \int_{\mathbb{T}}\sum_{d,d'} e^{-2\pi(kd+k'd')}K_{M_N}(d,d')|\hat{\alpha}(k')|^2 dk'\\
&\quad= \int_{\mathbb{T}} \hat{K}_{M_N}(k,k')|\hat{\alpha}(k')|^2 dk'\\
&\quad= \int_{\mathbb{T}} \hat{K}_{M_N}(k,k')\left(|\hat{\phi}(k')|^2 - \frac{1}{2}\left(\hat{\phi}(k')\hat{\phi}(-k') + \hat{\phi}^*(k')\hat{\phi}^*(-k')\right)\right).
\end{aligned}$$

We also compute

$$\begin{aligned}
O^{M_N}\hat{\phi}(k) &= \frac{1}{2}\sum_{n\in\mathbb{Z}} e^{-2\pi ikn}\left(-2\sum_{d\in\mathbb{Z}} K_{M_N}(d,0)\left(\phi_{n-d} - \phi^*_{n-d}\right)\right)\\
&= -\sum_{d\in\mathbb{Z}} e^{-2\pi ikd}K_{M_N}(d,0)\left(\hat{\phi}(k) - \hat{\phi}^*(-k)\right)\\
&= -\int_{\mathbb{T}} \hat{K}_{M_N}(k,k')\left(\hat{\phi}(k) - \hat{\phi}^*(-k)\right)dk'.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&(O^{M_N}\hat{\phi}(k))\hat{\phi}^*(k) + (O^{M_N}\hat{\phi}^*(k))\hat{\phi}(k)\\
&\quad= \int_{\mathbb{T}} \hat{K}_{M_N}(k,k')\left(-2|\hat{\phi}(k)|^2 + \left(\hat{\phi}(k)\hat{\phi}(-k) + \hat{\phi}^*(k)\hat{\phi}^*(-k)\right)\right)dk'.
\end{aligned}$$

We conclude that

$$O|\hat{\phi}(k)|^2 = \varepsilon O^{M_N}_{col}\left(|\hat{\phi}(k)|^2 - \frac{1}{2}\left(\hat{\phi}(k)\hat{\phi}(-k) + \hat{\phi}^*(k)\hat{\phi}^*(-k)\right)\right).$$

5.2. The simple index case

We follow the steps in the 1-dimensional case to compute the kernel. From the parameter M, we derive the controlled kernel coefficients by computing

$$\sum_{n,n',n''\in\mathbb{Z}^{\mathrm{d}}}\sum_{\gamma^1}\sum_{\gamma^2}[\lambda_n^{M_{\gamma^1}}]_a[\phi_{n'}]_a[\lambda_n^{M_{\gamma^2}}]_b[\phi^*_{n''}]_b.$$

Omitting temporarily the sum over n', n'' in the formulation (while still using the invariance of $n'-n''$ under translation), we compute

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^{\mathrm{d}}}\sum_{\gamma^1}\sum_{\gamma^2}[\lambda_n^{M_{\gamma^1}}]_a[\phi_{n'}]_a[\lambda_n^{M_{\gamma^2}}]_b[\phi^*_{n''}]_b\\
&=\frac{1}{2}\sum_{n\in\mathbb{Z}^{\mathrm{d}}}\sum_{\gamma^1}\sum_{\gamma^2}\left[\sum_{d=1}^{N}[M_{\gamma^1}(d)]_a([\alpha_n]_a-[\alpha_{n+\gamma^1_d}]_a)\delta_{n-n',\gamma^1_d}+\left(\sum_{d=1}^{N}[M_{\gamma^1}(d)]_a[\alpha_{n+\gamma^1_d}]_a\right)\delta_{n,n'}\right]\\
&\quad\times\left[\sum_{d=1}^{N}[M_{\gamma^2}(d)]_b([\alpha_n]_b-[\alpha_{n+\gamma^2_d}]_b)\delta_{n-n'',\gamma^2_d}+\left(\sum_{d=1}^{N}[M_{\gamma^2}(d)]_b[\alpha_{n+\gamma^2_d}]_b\right)\delta_{n,n''}\right]\\
&=-\frac{1}{2}\sum_{\substack{D'\in\mathbb{Z}^{\mathrm{d}}\\ \|D'\|_{L^1}\le 2N}}\left(\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D'}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b([\alpha_{n'-\gamma^1_{d_1}}]_a-[\alpha_{n'-2\gamma^1_{d_1}}]_a)\right.\\
&\qquad\left.\times\ ([\alpha_{n'-\gamma^1_{d_1}}]_b-[\alpha_{n'-\gamma^1_{d_1}+\gamma^2_{d_2}}]_b)\right)\delta_{n'-n'',D'} &(54)\\
&-\frac{1}{2}\sum_{\gamma^1,d_1}[M_{\gamma^1}(d_1)]_a([\alpha_{n'-\gamma^1_{d_1}}]_a-[\alpha_{n'-2\gamma^1_{d_1}}]_a)\left(\sum_{\gamma^2,d_2}[M_{\gamma^2}(d_2)]_b[\alpha_{n'-\gamma^1_{d_1}+\gamma^2_{d_2}}]_b\right)\delta_{n'-n'',\gamma^1_{d_1}} &(55)\\
&+\frac{1}{2}\sum_{\gamma^2,d_2}[M_{\gamma^2}(d_2)]_b([\alpha_{n'}]_b-[\alpha_{n'+\gamma^2_{d_2}}]_b)\left(\sum_{\gamma^1,d_1}[M_{\gamma^1}(d_1)]_a[\alpha_{n'+\gamma^1_{d_1}}]_a\right)\delta_{n'-n'',\gamma^2_{d_2}} &(56)\\
&+\frac{1}{2}\left(\sum_{\gamma^1}\sum_{d=1}^{N}[M_{\gamma^1}(d)]_a[\alpha_{n+\gamma^1_d}]_a\right)\left(\sum_{\gamma^2}\sum_{d=1}^{N}[M_{\gamma^2}(d)]_b[\alpha_{n+\gamma^2_d}]_b\right)\delta_{n'-n'',0}. &(57)
\end{aligned}$$

We expect to express the sum in the form

$$\sum_{D,D'}[K_M(D,D')]_{b,a}[\alpha_{n'}]_a[\alpha_{n'+D}]_b\delta_{n'-n'',D'}.$$

The matrix $K_M(D,D') = ([K_M(D,D')]_{a,b})_{1\le a,b\le \mathrm{d}}$ is the controlled coefficient matrix for the simple index case. We compute the sum based on the L^1 norm of $D, D' \in \mathbb{Z}^{\mathrm{d}}$. The kernel coefficients are obtained as follows:

- If $D = D' = 0$, we obtain from (54) and (57) the coefficient

$$[K_M(0,0)]_{b,a} = \frac{3}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}=\gamma^2_{d_2}}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.$$

- If $1 \le \|D\|_{L^1} \le N$ and $D' = 0$, we also get from (54) and (57) the coefficient

$$[K_M(D,0)]_{b,a} = -\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}}[M_{\gamma^1}(\|D\|_{L^1})]_a[M_{\gamma^2}(\|D\|_{L^1})]_b - \frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.$$

- If $1 \le \|D'\|_{L^1} \le N$ and $D = 0$, we obtain the coefficient from (54), (55), (56), which is given by

$$[K_M(0,D')]_{b,a} = -\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D'\|_{L^1}}=\gamma^2_{\|D'\|_{L^1}}=D'}}[M_{\gamma^1}(\|D'\|_{L^1})]_a[M_{\gamma^2}(\|D'\|_{L^1})]_b - \frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D'}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.$$

- If $N+1 \le \|D\|_{L^1} \le N$ and $D' = 0$, the coefficient involves only (57). The computation on the coefficient yields

$$[K_M(D,0)]_{b,a} = -\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2,d_1,d_2\\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}}[M_{\gamma^1}(d_1)]_a[M_{\gamma^2}(d_2)]_b.$$

- If $N+1 \le \|D'\|_{L^1} \le N$ and $D = 0$, the coefficient involves only (54). The computation on the coefficient yields

$$[K_M(0,D')]_{b,a} = -\frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2,d_1,d_2 \\ \gamma^1_{d_1}+\gamma^2_{d_2}=D'}} [M_{\gamma^1}(d_1)]_a [M_{\gamma^2}(d_2)]_b.$$

- If $1 \le \|D\|_{L^1} \le N$ and $D' = D$, we consider (54), (55), (56) and get

$$[K_M(D,D)]_{b,a} = - \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b - \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2,d_1,d_2 \\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}} [M_{\gamma^1}(d_1)]_a [M_{\gamma^2}(d_2)]_b.$$

- If $1 \le \|D\|_{L^1} \le N/2$ and $D' = -D$, we also consider (54), (55), (56) and get

$$\begin{aligned}[K_M(D,-D)]_{b,a} = & \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b \\ & - \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=D,\gamma^2_{2\|D\|_{L^1}}=2D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(2\|D\|_{L^1})]_b \\ & - \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{2\|D\|_{L^1}}=2D,\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(2\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^2})]_b.\end{aligned}$$

- If $N/2 < \|D\|_{L^1} \le N$ and $D' = -D$, we have

$$[K_M(D,-D)]_{b,a} = \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b.$$

- If $N + 1 \le \|D\|_{L^1} \le 2N$ and $D' = D$, we consider only (54) and get

$$[K_M(D,D)]_{b,a} = -\frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2,d_1,d_2 \\ \gamma^1_{d_1}+\gamma^2_{d_2}=D}} [M_{\gamma^1}(d_1)]_a [M_{\gamma^2}(d_2)]_b.$$

- If $1 \le \|D\|_{L^1} \le N < \|D'\|_{L^1} \le 2N$ and $1 \le \|D - D'\|_{L^1} \le N$, only (54) contributes to the coefficient, we compute

$$
\begin{aligned}
[K_M(D,D')]_{b,a} = & \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=D,\gamma^2_{\|D'-D\|_{L^1}}=D'-D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D'-D\|_{L^1})]_b \\
& + \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D'-D\|_{L^1}}=D'-D,\gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D'-D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b .
\end{aligned}
$$

- If $1 \le \|D'\|_{L^1} \le N < \|D\|_{L^1} \le 2N$ and $1 \le \|D'-D\|_{L^1} \le N$, from (55) and (56), we compute

$$
\begin{aligned}
[K_M(D,D')]_{b,a} = & \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D'\|_{L^1}}=D',\gamma^2_{\|D-D'\|_{L^1}}=D-D'}} [M_{\gamma^1}(\|D'\|_{L^1})]_a [M_{\gamma^2}(\|D-D'\|_{L^1})]_b \\
& + \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D-D'\|_{L^1}}=D-D',\gamma^2_{\|D'\|_{L^1}}=D'}} [M_{\gamma^1}(\|D-D'\|_{L^1})]_a [M_{\gamma^2}(\|D'\|_{L^1})]_b .
\end{aligned}
$$

- If $1 \le \|D\|_{L^1}, \|D'\|_{L^1} \le N$ and $D \neq \pm D'$, we consider (54), (55), (56), we obtain

$$
\begin{aligned}
[K_M(D,D')]_{b,a} = & \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D\|_{L^1}}=D \\ \gamma^2_{\|D'-D\|_{L^1}}=D'-D}} [M_{\gamma^1}(\|D\|_{L^1})]_a [M_{\gamma^2}(\|D'-D\|_{L^1})]_b \\
& + \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D'-D\|_{L^1}}=D'-D \\ \gamma^2_{\|D\|_{L^1}}=D}} [M_{\gamma^1}(\|D'-D\|_{L^1})]_a [M_{\gamma^2}(\|D\|_{L^1})]_b \\
& + \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D'\|_{L^1}}=D' \\ \gamma^2_{\|D-D'\|_{L^1}}=D-D'}} [M_{\gamma^1}(\|D'\|_{L^1})]_a [M_{\gamma^2}(\|D-D'\|_{L^1})]_b \\
& + \frac{1}{2} \sum_{\substack{\gamma^1,\gamma^2 \\ \gamma^1_{\|D-D'\|_{L^1}}=D-D' \\ \gamma^2_{\|D'\|_{L^1}}=D'}} [M_{\gamma^1}(\|D-D'\|_{L^1})]_a [M_{\gamma^2}(\|D'\|_{L^1})]_b
\end{aligned}
$$

$$-\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D'\|_{L^1}}=D'\\ \gamma^2_{\|D\|_{L^1}}=D}}[M_{\gamma^1}(\|D'\|)]_a[M_{\gamma^2}(\|D\|)]_b$$

$$-\frac{1}{2}\sum_{\substack{\gamma^1,\gamma^2\\ \gamma^1_{\|D\|_{L^1}}=D\\ \gamma^2_{\|D'\|_{L^1}}=D'}}[M_{\gamma^1}(\|D\|)]_a[M_{\gamma^2}(\|D'\|)]_b.$$

- $K_M(D, D') = 0$ in all other cases.

Using all the previous computations, we obtain the coefficients K_M as in Definition 6.

5.3. The dual indices case

As in the previous computations, the kernel coefficients arise from $\sum_\gamma \sum_{n,n',n''}(\lambda^{a,b}_{n,\gamma}\phi_{n'}) \cdot (\lambda^{a,b}_{n,\gamma}\phi^*_{n''})$ for $a \neq b$ (note that $\lambda^{a,a}_{n,\gamma} = 0$).

We have

$$\sum_n(\lambda^{a,b}_{n,\gamma}\phi_{n'}) \cdot (\lambda^{a,b}_{n,\gamma}\phi^*_{n''})$$
$$= \frac{1}{2}\left(\left(\sum_{d=1}^N M^{a,b}_\gamma(d)([\alpha_{n+\gamma_d}]_b - [\alpha_n]_b)\right)^2 + \left(\sum_{d=1}^N M^{a,b}_\gamma(d)([\alpha_{n+\gamma_d}]_a - [\alpha_n]_a)\right)^2\right) \tag{58}$$
$$\times\left(\sum_{d=1}^N M^{a,b}_\gamma(d)\delta_{n'-n,\gamma_d} - \sum_{d=1}^N M^{a,b}_\gamma(d)\delta_{n,n'}\right)\left(\sum_{d=1}^N M^{a,b}_\gamma(d)\delta_{n''-n,\gamma_d} - \sum_{d=1}^N M^{a,b}_\gamma(d)\delta_{n,n''}\right). \tag{59}$$

Thus, we get the sum

$$\sum_{n',n'',D,D'} K^{a,b}(D, D')([\alpha_{n'}]_a[\alpha_{n'+D}]_a + [\alpha_{n'}]_b[\alpha_{n'+D}]_b)\delta_{n'-n'',D'},$$

where $K^{a,b}$ is computed as follows:

- If $D = D' = 0$, we first take the coefficients of $[\alpha_n]^2_a$ and $[\alpha_{n+\gamma_d}]^2_a$ in (58). Then, we multiply these coefficients with those of the pairs $\delta_{n,n'}$ and $\delta_{n,n''}$, as well as the pairs $\delta_{n'-n,\gamma_d}$ and $\delta_{n''-n,\gamma_d}$ that have the same d in (59). This gives us the coefficient

$$K^{a,b}(0,0) = \frac{1}{2}\sum_\gamma\left(\sum_{d=1}^N M^{a,b}_\gamma(d)\right)^4 + \sum_\gamma\left(\sum_{d=1}^N M^{a,b}_\gamma(d)\right)^2\left(\sum_{d=1}^N (M^{a,b}_\gamma(d))^2\right)$$

$$+\frac{1}{2}\sum_{\gamma}\left(\sum_{d=1}^{N}(M_{\gamma}^{a,b}(d))^2\right)^2$$
$$=\frac{1}{2}\sum_{\gamma}\left(\left(\sum_{d=1}^{N}M_{\gamma}^{a,b}(d)\right)^2+\left(\sum_{d=1}^{N}(M_{\gamma}^{a,b}(d))^2\right)\right)^2.$$

- If $1 \le \|D\|_{L^1} \le N$ and $D' = 0$, we also take the coefficients of the pairs $\delta_{n,n'}$ and $\delta_{n,n''}$, as well as the pairs $\delta_{n'-n,\gamma_d}$ and $\delta_{n''-n,\gamma_d}$ that have the same d in (59). Then, we multiply them with the coefficients of $[\alpha_n]_a[\alpha_{n+D}]_a$ in (58). This yields

$$K^{a,b}(D,0) = -\sum_{\substack{\gamma\\ \gamma_{\|D\|_{L^1}}=D}} M_{\gamma}^{a,b}(\|D\|_{L^1})\left(\sum_{d=1}^{N}M_{\gamma}^{a,b}(d)\right)\left(\left(\sum_{d=1}^{N}M_{\gamma}^{a,b}(d)\right)^2\right.$$
$$\left.+\left(\sum_{d=1}^{N}(M_{\gamma}^{a,b}(d))^2\right)\right)$$
$$+\sum_{\substack{\gamma,d_1,d_2\\ \gamma_{d_2}-\gamma_{d_1}=D}} M_{\gamma}^{a,b}(d_1)M_{\gamma}^{a,b}(d_2)\left(\left(\sum_{d=1}^{N}M_{\gamma}^{a,b}(d)\right)^2+\left(\sum_{d=1}^{N}(M_{\gamma}^{a,b}(d))^2\right)\right).$$

- If $D = 0$ and $1 \le \|D'\|_{L^1} \le N$, we take the coefficients of $[\alpha_n]_a^2$ and $[\alpha_{n+\gamma_d}]_a^2$ in (58) (take the factor 1/2 for symmetry between $\gamma, -\gamma$). Then, we multiply them with the coefficients of the pairs $\delta_{n,n'}$ and $\delta_{n''-n,-D'}$, as well as the pairs $\delta_{n'-n,D'}$ and $\delta_{n,n''}$, and the pairs $\delta_{n'-n,\gamma_{d_1}}$ and $\delta_{n''-n,\gamma_{d_2}}$, where $\gamma_{d_1} - \gamma_{d_2} = D'$ in (59). This yields

$$K^{a,b}(0,D') = -\frac{1}{2}\sum_{\substack{\gamma\\ \gamma_{\|D'\|_{L^1}}=D'}}\left(\left(\sum_{d=1}^{N}M_{\gamma}^{a,b}(d)\right)^2+\left(\sum_{d=1}^{N}(M_{\gamma}^{a,b}(d))^2\right)\right)$$
$$\times\left(\sum_{d=1}^{N}M_{\gamma}^{a,b}(d)\right)M_{\gamma}^{a,b}(\|D'\|_{L^1})$$
$$-\frac{1}{2}\sum_{\substack{\gamma\\ \gamma_{\|D'\|_{L^1}}=D'}}\left(\left(\sum_{d=1}^{N}M_{-\gamma}^{a,b}(d)\right)^2+\left(\sum_{d=1}^{N}(M_{-\gamma}^{a,b}(d))^2\right)\right)$$
$$\times\left(\sum_{d=1}^{N}M_{-\gamma}^{a,b}(d)\right)M_{-\gamma}^{a,b}(\|D'\|_{L^1})$$

$$+\frac{1}{2}\sum_{\substack{\gamma,d_1,d_2\\ \gamma_{d_2}-\gamma_{d_1}=D'}}\left(\left(\sum_{d=1}^{N}M_\gamma^{a,b}(d)\right)^2+\left(\sum_{d=1}^{N}(M_\gamma^{a,b}(d))^2\right)\right)M_\gamma^{a,b}(d_1)M_\gamma^{a,b}(d_2)$$

$$+\frac{1}{2}\sum_{\substack{\gamma,d_1,d_2\\ \gamma_{d_2}-\gamma_{d_1}=D'}}\left(\left(\sum_{d=1}^{N}M_{-\gamma}^{a,b}(d)\right)^2+\left(\sum_{d=1}^{N}(M_{-\gamma}^{a,b}(d))^2\right)\right)M_\gamma^{a,b}(d_1)M_\gamma^{a,b}(d_2).$$

- If $1\le\|D\|_{L^1},\|D'\|_{L^1}\le N$ and D,D' are both on a same set p (or D is on set p and D' is on set p'), we have

$$K^{a,b}(D,D')=\sum_{\substack{\gamma\\ \gamma_{\|D\|_{L^1}}=D\\ \gamma_{\|D'\|_{L^1}}=D'}}\left(\sum_{d=1}^{N}M_\gamma^{a,b}(d)\right)^2M_\gamma^{a,b}(\|D\|_{L^1})M_\gamma^{a,b}(\|D'\|_{L^1})$$

$$-\sum_{\substack{\gamma,d_1,d_2\\ \gamma_{\|D\|_{L^1}}=D\\ \gamma_{d_2}-\gamma_{d_1}=D'}}\left(M_\gamma^{a,b}(\|D\|_{L^1})\right)\left(\sum_{d=1}^{N}M_\gamma^{a,b}(d)\right)M_\gamma^{a,b}(d_1)M_\gamma^{a,b}(d_2).$$

We change γ to $-\gamma$ for $M_\gamma^{a,b}$ in case where D is on p and D' is on p'.
- $K^{a,b}(D,D')=0$ in all other cases.

Therefore, we get the coefficients $K^{a,b}$ as in Definition 9.

6. Proofs of the main results

When computing $[O_{col}^{M}S]_{a,b}$ for simple index or $O_{col}^{a,b}S$ for dual indices, a key difference from computing $O_{col}^{M_N}$ in 1-dimensional case is the loss of symmetry (For 1-dimension, $a=b=1$). This complicates the representation of the collision operators in multi-dimension cases compared to that of the 1-dimensional case. But, the extension to the multi-dimensional cases is straightforward. We will restrict ourselves to the 1-dimensional case in the following proofs to enhance readability.

6.1. Proof of Lemma 3

Let us first establish that

$$\sum_{|d|\le 2N}K_{M_N}(d,d')=0\qquad\forall d'\in\mathbb{N},|d'|\le 2N.\tag{60}$$

When $d'=0$, we have

$$\sum_{|d|\le 2N} K_{M_N}(d,0) = \frac{3}{2}\sum_{1\le|d|\le N} M_N(d)^2 - \sum_{1\le|d|\le N} M_N(d)^2$$
$$-\frac{1}{2}\sum_{1\le|d|\le 2N}\sum_{\substack{d_1+d_2=d\\1\le|d_1|,|d_2|\le N}} M_N(d_1)M_N(d_2)$$
$$= -\frac{1}{2}\sum_{1\le|d|\le 2N}\sum_{\substack{d_1+d_2=d\\1\le|d_1|,|d_2|\le N\\d_1\ne d_2}} M_N(d_1)M_N(d_2). \tag{61}$$

The sum in (61) is zero because if (d_1,d_2) is in the sum then so is $(-d_1,d_2)$. Hence, (60) holds when $d'=0$.

Now, we consider $1\le d'\le N$ (the case $-N\le d'\le -1$ is similar). By using the change of variable $d\to -d$, we sum all coefficients of the case $1\le|d|\le N$ into

$$\sum_{1\le|d'-d|,|d|\le N}[M_N(d)M_N(d'-d)].$$

Therefore, we obtain

$$\sum_{|d|\le 2N} K_{M_N}(d,d') = -M_N(d')^2 - \sum_{d_1+d_2=d',1\le|d_1|,|d_2|\le N} M_N(d_1)M_N(d_2) - 2M_N(d')M_N(2d')$$
$$+\sum_{d=N+1}^{N+d'} M_N(d')M_N(d-d') + \sum_{1\le|d'-d|,|d|\le N} M_N(d)M_N(d'-d)$$
$$-\sum_{1\le|d'-d|,|d|\le N} M_N(d')M_N(d'-d).$$

If $d'>N/2$, for each $d_1\in[N+1,N+d']\setminus\{2d'\}$, we can choose $d_2=2d'-d_1\in[d'-N,2d'-N-1]\setminus\{0\}$. In case $d_1=2d'$, it gives $M_N(d')^2$ which cancels $-M_N(d')^2$. The remainder is

$$-\sum_{\substack{1\le|d'-d|,|d|\le N\\|2d'-d|\le N}} M_N(d')M_N(d'-d).$$

Note that if d is in the sum, then $2d'-d$ is also in the sum. Since $d'-d$ and $d'-(2d'-d)$ are opposites of one another, the sum is 0.

If $d'\le N/2$, for each $d_1\in[N+1,N+d']$, we can choose $d_2=2d'-d_1\in[d'-N,2d'-N-1]\subset[-N,-1]$. Those terms cancel each other out. The remainder is

$$-M_N(d')^2 - 2M_N(d')M_N(2d') - \sum_{1\le|d'-d|,|d|\le N|2d'-d|\le N} M_N(d')M_N(d'-d). \tag{62}$$

The term $M_N(d')M_N(2d')$ is zero due to Assumption $(M2)$. The case $d = 2d'$ produces a term that cancels with $-M_N(d')^2$. For $d \neq 2d'$, d and $2d' - d$ are opposites, leading to the result that the expression in (62) is zero.

Hence, (60) holds when $|d'| \leq N$.

Finally, we consider $N + 1 \leq d' \leq 2N$ (the case $-2N \leq d' \leq -N - 1$ is similar). We have

$$\sum_{|d|\leq 2N} K_{M_N}(d, d') = -\sum_{\substack{d_1+d_2=d' \\ 1\leq |d_i|\leq N}} M_N(d_1)M_N(d_2) + \sum_{d=d'-N}^{N} M(d)M_N(d'-d) = 0.$$

This concludes the proof of (60).

Using the fact that $K_{M_N}(0, d') = -\sum_{1\leq|d|\leq 2N} K_{M_N}(d, d')$, we can rewrite $\hat{K}_{M_N}(k, k')$ as

$$\sum_{|d'|\leq 2N} \cos(2\pi d'k') \left(\sum_{1\leq|d|\leq 2N} K_{M_N}(d, d')(\cos(2\pi dk) - 1) \right).$$

Since $|\cos(2\pi nk) - 1| \lesssim |k|$ we get $\hat{K}_{M_N}(k, k') \lesssim |k|$. Additionally, we see that $|k| \lesssim \omega(k)$. In the pinning case, we already have $|k| \lesssim 1 \lesssim \omega(k)$. If $|k|$ is not close to zero in the no-pinning case, we still have $|k| \lesssim 1 \lesssim \omega(k)$. Now, considering the no-pinning case and assuming that $|k|$ is close to zero, we have

$$\hat{\sigma}(k) = \hat{\sigma}(0) + \hat{\sigma}'(0)k + \frac{1}{2}\hat{\sigma}''(k')k^2 = \frac{1}{2}\hat{\sigma}''(k')k^2,$$

where $0 \leq |k'| < |k|$. Since $\hat{\sigma}''(0) > 0$, when $|k|$ is close to zero, we also have $\hat{\sigma}''(k') \approx 1$. Thus,

$$\omega(k) = \sqrt{\hat{\sigma}(k)} \approx |k|$$

when k is close to zero. Therefore,

$$\sup \left| \frac{\hat{K}_{M_N}(k, k')}{\omega(k)} \right| < \infty.$$

6.2. *Proof of Theorem 4*

6.2.1. *The energy transport equation* (32)

Let $k \mapsto S(k)$ be a bounded real-valued function on $\mathbb{T}$, we have

$$\frac{d}{dt}\langle S, E^\varepsilon(t)\rangle = \frac{\varepsilon}{2} \int_{\mathbb{T}} \varepsilon^{-1}\mathbb{E}_\varepsilon \left[O|\hat{\phi}(k, t/\varepsilon)|^2 \right] S(k)dk.$$

We use the techniques developed in Section 5.1 and compute

$$\sum_{n,n',n''\in\mathbb{Z}} \left(\lambda_n^{M_N}\phi_{n'}\right)\left(\lambda_n^{M_N}\phi_{n''}^*\right)\tilde{S}^*(n''-n') = \sum_{n',d,d'} K_{M_N}(d,d')\alpha_{n'}\alpha_{n'+d}\tilde{S}^*(-d')$$
$$= \int_{\mathbb{T}} |\hat{\alpha}(k)|^2 \sum_{d,d'} e^{-2\pi i k d} K_{M_N}(d,d')\tilde{S}^*(-d')dk$$
$$= \int_{\mathbb{T}^2} |\hat{\alpha}(k)|^2 \hat{K}_{M_N}(k,k')S(k')dkdk'.$$

Thus, we obtain

$$\int_{\mathbb{T}} O^{M_N}|\hat{\phi}(k)|^2 S(k)dk = \int_{\mathbb{T}} \left(|\hat{\phi}(k)|^2 - \frac{1}{2}\left(\hat{\phi}(k)\hat{\phi}(-k) + \hat{\phi}^*(k)\hat{\phi}^*(-k)\right)\right) O_{col}^{M_N} S(k)dk.$$

Therefore, the time derivative is given by

$$\frac{d}{dt}\langle S, E^{\varepsilon}(t)\rangle$$
$$= \langle O_{col}^{M_N} S, E^{\varepsilon}(t)\rangle - \frac{\varepsilon}{4}\int_{\mathbb{T}} \left(\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon) + \hat{\phi}^*(k,t/\varepsilon)\hat{\phi}^*(-k,t/\varepsilon)\right) O_{col}^{M_N} S(k)dk.$$

It is sufficient to check that

$$\lim_{\varepsilon\to} \int_0^t \frac{\varepsilon}{2}\int_{\mathbb{T}} \hat{\phi}(k,s/\varepsilon)\hat{\phi}(-k,s/\varepsilon) O_{col}^{M_N} S(k)dkds = 0, \forall t \in [0,T]. \tag{63}$$

The complex conjugate counterpart can be proved similarly. To prove (63), we compute the derivative:

$$\frac{d}{dt}\frac{\varepsilon}{2}\int_{\mathbb{T}} \hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)S(k)dk = \frac{\varepsilon}{2}\int_{\mathbb{T}} \varepsilon^{-1} O\left(\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)\right) S(k)dk.$$

The Hamiltonian operator gives

$$O^H\left(\hat{\phi}(k)\hat{\phi}(-k)\right) = -2i\omega(k)\hat{\phi}(k)\phi(\hat{-}k).$$

We use the techniques developed in Section 5.1 again and find that

$$\int_{\mathbb{T}} O^{M_N}\left(\hat{\phi}(k)\hat{\phi}(-k)\right) S(k)dk = -\frac{d}{dt}\langle S, E^{\varepsilon}(t)\rangle$$
$$+ \frac{\varepsilon}{4}\int_{\mathbb{T}^2} \hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)\hat{K}_{M_N}(k,k')S(k)dkdk'$$

$$-\frac{\varepsilon}{4}\int_{\mathbb{T}^2}\hat{\phi}^*(k,t/\varepsilon)\hat{\phi}^*(-k,t/\varepsilon)\hat{K}_{M_N}(k,k')S(k)dkdk'.$$

Hence, we get

$$\begin{aligned}\frac{d}{dt}\frac{\varepsilon}{2}\int_{\mathbb{T}}\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)S(k)dk = &-\frac{2i}{\varepsilon}\frac{\varepsilon}{2}\int_{\mathbb{T}}\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)\omega(k)S(k)dk\\ &-\frac{d}{dt}\langle S, E^\varepsilon(t)\rangle\\ &+\frac{\varepsilon}{4}\int_{\mathbb{T}^2}\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)\hat{K}_{M_N}(k,k')S(k)dkdk'\\ &-\frac{\varepsilon}{4}\int_{\mathbb{T}^2}\hat{\phi}^*(k,t/\varepsilon)\hat{\phi}^*(-k,t/\varepsilon)\hat{K}_{M_N}(k,k')S(k)dkdk'.\end{aligned}$$

Assumption ($\mu 3$) ensures that

$$\frac{\varepsilon}{2}\int_{\mathbb{T}}\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)S(k)dk,\ \frac{\varepsilon}{2}\int_{\mathbb{T}}\hat{\phi}^*(k,t/\varepsilon)\hat{\phi}^*(-k,t/\varepsilon)S(k)dk,\ \text{and } \langle S, E^\varepsilon(t)\rangle$$

are all bounded. By integrating with respect to t, we obtain

$$\lim_{\varepsilon\to 0}\int_0^t\frac{\varepsilon}{2}\int_{\mathbb{T}}\hat{\phi}(k,s/\varepsilon)\hat{\phi}(-k,s/\varepsilon)\omega(k)S(k)dkds = 0.$$

Lemma 3 implies that $|\hat{K}_{M_N}(k,k')| \lesssim \omega(k)$, and therefore we can replace S by $\frac{O_{col}^{M_N}S(k)}{\omega(k)}$. This completes the proof of equation (32).

6.2.2. Proof of (33)

From the proof of (32), we have

$$\begin{aligned}\left|\frac{d}{dt}\langle 1_{(-R,R)}, E^\varepsilon(t)\rangle\right| \le &\left|\langle O_{col}^{M_N}1_{(-R,R)}, E^\varepsilon(t)\rangle\right|\\ &+\left|\frac{\varepsilon}{4}\int_{\mathbb{T}}\Big(\hat{\phi}(k,t/\varepsilon)\hat{\phi}(-k,t/\varepsilon)+\hat{\phi}^*(k,t/\varepsilon)\hat{\phi}^*(-k,t/\varepsilon)\Big)O_{col}^{M_N}1_{(-R,R)}dk\right|.\end{aligned}$$

Thus, given that $\hat{K}_{M_N}$ is bounded, we have $O_{col}^{M_N}1_{(-R,R)} \lesssim R + 1_{(-R,R)}$. This leads to the estimate

$$\left|\frac{d}{dt}\langle 1_{(-R,R)}, E^\varepsilon(t)\rangle\right| \lesssim R + \langle 1_{(-R,R)}, E^\varepsilon(t)\rangle.$$

By Gronwall's inequality, we obtain

$$\langle 1_{[-R,R]}, E^{\varepsilon}(t)\rangle \lesssim \left(R + \langle 1_{(-R,R)}, E^{\varepsilon}(0)\rangle\right) e^{C_8 t}$$

for a constant $C_8 > 0$ and for each $t \in [0, T]$. By Assumption (μ4), the conclusion follows.

6.3. Proof of Theorem 5

We recall the definition of the Wigner distribution in the Fourier space

$$\langle S, W^{\varepsilon}(t)\rangle := \frac{\varepsilon}{2} \int\limits_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}} \hat{\phi}(k+\varepsilon\xi/2)\hat{\phi}^*(k-\varepsilon\xi/2)\hat{S}^*(\xi,k)d\xi dk,$$

where $\mathbb{T}_{2/\varepsilon}$ is the torus of size $2/\varepsilon$ and $\hat{S}(\xi,k) = \int_{\mathbb{R}} e^{-2\pi i x\xi} S(x,k)dx$. Using the definition, we compute:

$$\frac{d}{dt}\langle S, W^{\varepsilon}(t)\rangle = \frac{\varepsilon}{2} \int\limits_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}} \varepsilon^{-1}\mathbb{E}_{\varepsilon}\left[O^H(\hat{\phi}(k+\varepsilon\xi/2, t/\varepsilon)\hat{\phi}^*(k-\varepsilon\xi/2, t/\varepsilon))\right]\hat{S}^*(\xi,k)d\xi dk$$
$$+ \frac{\varepsilon}{2}\sum_{n',n''}\mathbb{E}_{\varepsilon}\left[O^{M_N}(\phi_{n'}(t/\varepsilon)\phi^*_{n''}(t/\varepsilon))\right]\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2}, n'-n''\right).$$

First, we deal with the Hamiltonian operator. By (9), we have:

$$O^H(\hat{\phi}(k+\varepsilon\xi/2)\hat{\phi}^*(k-\varepsilon\xi/2)) = -i(\omega(k+\varepsilon\xi/2)-\omega(k-\varepsilon\xi/2))\hat{\phi}(k+\varepsilon\xi/2)\hat{\phi}^*(k-\varepsilon\xi/2). \tag{64}$$

We observe that $\varepsilon^{-1}(\omega(k+\varepsilon\xi/2)-\omega(k-\varepsilon\xi/2))$ can be estimated by $\omega'(k)\xi$. We prove that

$$\begin{aligned}&\frac{\varepsilon}{2}\int\limits_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}} \mathbb{E}_{\varepsilon}\left[\hat{\phi}(k+\varepsilon\xi/2, t/\varepsilon)\hat{\phi}^*(k-\varepsilon\xi/2, t/\varepsilon)\right] \\ &\qquad \times (\varepsilon(\omega(k+\varepsilon\xi/2)-\omega(k-\varepsilon\xi/2)) - \omega'(k)\xi)\hat{S}^*(\xi,k)d\xi dk\end{aligned} \tag{65}$$

converges to 0 as $\varepsilon \to 0$. We consider the no-pinning case and we split the domain of integration into three regions: $|k| \geq R > \varepsilon|\xi|$, $R \leq \varepsilon|\xi|$, and $|k| < R$ for sufficiently small values of ε and R.

In the case $|k| \geq R > \varepsilon|\xi|$, the function ω is smooth on the interval between $k \pm \varepsilon\xi/2$ (consider it on $(0,1)$ or $(-1,0)$ as k may be close to $\pm 1/2$). Thus, we obtain

$$|\varepsilon^{-1}(\omega(k+\varepsilon\xi/2)-\omega(k-\varepsilon\xi/2)) - \omega'(k)\xi| = \frac{1}{2}|\omega''(k')|\varepsilon\xi^2 \tag{66}$$

where k' is between $k \pm \varepsilon\xi/2$. We have an estimate on the second-order derivative:

$$\omega''(k') = \frac{\hat{\sigma}''(k')\omega(k') - \hat{\sigma}'(k')\omega'(k')}{2\hat{\sigma}(k')} = \frac{\hat{\sigma}''(k')\omega(k') - (\hat{\sigma}'(k'))^2/(2\omega(k'))}{2\hat{\sigma}(k')}.$$

We observe that

$$|\omega''(k')| \approx |k'|^{-1} \approx |k|^{-1} \lesssim R^{-1}$$

because $\hat{\sigma}''(k') \approx 1$, $\omega(k') \approx |k'|$, $\hat{\sigma}'(k') \approx |k'|$, and $\hat{\sigma}(k') \approx |k'|^2$. The integral (65) over this domain is bounded by

$$\varepsilon \int\limits_{|k|>R} \mathbb{E}_\varepsilon \left[|\hat{\phi}(k)|^2 \right] dk \sup_k \left| \int\limits_{-R\varepsilon^{-1}}^{R\varepsilon^{-1}} \hat{S}^*(\xi, k) R^{-1} \varepsilon \xi^2 d\xi \right|.$$

As S is in the Schwartz space, we can bound $\int_{\mathbb{R}} |\hat{S}(\xi, k)||\xi|^2 d\xi$ by a constant. Therefore, the integral (65) over this domain tends to 0 as $\varepsilon \to 0$ for each fixed R.

Considering the case $R \le \varepsilon|\xi|$, we use the estimate

$$|\varepsilon^{-1}(\omega(k+\varepsilon\xi/2) - \omega(k-\varepsilon\xi/2)) - \omega'(k)\xi)| \lesssim R^{-1}|\xi|.$$

The integral (65) over this domain is bounded by

$$\varepsilon \int\limits_{\mathbb{T}} \mathbb{E}_\varepsilon \left[|\hat{\phi}(k)|^2 \right] dk \sup_k \int\limits_{R \le \varepsilon|\xi| \le 2} \hat{S}^*(\xi, k) R^{-1} |\xi| d\xi.$$

We use $|\xi|^{-1} \le R^{-1}\varepsilon$ and again we bound $\int_{\mathbb{R}} |\hat{S}(\xi, k)||\xi|^2 d\xi$. Thus, the integral (65) also tends to 0 as $\varepsilon \to 0$ in this domain.

For the case $|k| < R$, we use (33). We need to bound

$$\int\limits_{\mathbb{T}_{2/\varepsilon}} \hat{S}^*(\xi, k)(\varepsilon^{-1}(\omega(k+\varepsilon\xi/2) - \omega(k-\varepsilon\xi/2)) - \omega'(k)\xi) d\xi. \tag{67}$$

When $|k| > \varepsilon|\xi|$, we apply (66) with the estimate $|\omega''(k')|\varepsilon|\xi| \lesssim 1$ and use a bound for $\int_{\mathbb{R}} |\hat{S}||\xi| d\xi$. When $|k| \le \varepsilon|\xi|$ we have

$$|\varepsilon^{-1}(\omega(k+\varepsilon\xi/2) - \omega(k-\varepsilon\xi/2)) - \omega'(k)\xi)| \le \varepsilon^{-1}(\varepsilon|\xi|/2 + |k| + \varepsilon|\xi|/2 - |k|) + |\omega'(k)||\xi| \lesssim |\xi|.$$

We again use a bound for $\int_{\mathbb{R}} |\hat{S}||\xi| d\xi$. Thus, (67) is bounded.

In the pinning case, since ω is smooth, we use (66) and the fact that ω'' is bounded, allowing us to easily obtain the limit 0 for (65) as $\varepsilon \to 0$. Therefore, the Hamiltonian operator yields

$$\frac{\varepsilon}{2} \int\limits_{\mathbb{T}_{2/\varepsilon} \times \mathbb{T}} \mathbb{E}_\varepsilon \left[\hat{\phi}(k+\varepsilon\xi/2, t/\varepsilon) \hat{\phi}^*(k-\varepsilon\xi/2, t/\varepsilon) \right] (-i\xi)\omega'(k) \hat{S}^*(\xi, k) d\xi dk.$$

Since

$$\frac{\partial \hat{S}}{\partial x}(\xi, k) = (-2\pi i \xi) \hat{S}(\xi, k),$$

the term (64) at the limit $\varepsilon \to 0$ is

$$\frac{1}{2\pi}\left\langle \omega'(k)\frac{\partial S}{\partial x}, W^\varepsilon(t)\right\rangle.$$

The perturbation is computed using the techniques developed in Section 5.1. We write

$$\begin{aligned}
&\sum_{n',n''}\mathbb{E}_\varepsilon\left[\lambda_n^{M_N}\phi_{n'}\lambda_n^{M_N}\phi^*_{n''}\right]S^*\left(\frac{\varepsilon(n'+n'')}{2}, n'-n''\right)\\
&\quad=\sum_{n',d,d'}\mathbb{E}_\varepsilon\left[\alpha_{n'}\alpha_{n'+d}\right]K_{M_N}(d,d')\tilde{S}^*\left(\frac{\varepsilon(2n'+d')}{2}, d'\right)\\
&\quad=\sum_{n',n'',d'}\mathbb{E}_\varepsilon\left[\alpha_{n'}\alpha_{n''}\right]K_{M_N}(n''-n',d')\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2}, d'\right)+a(\varepsilon)\\
&\quad=\frac{1}{2}\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi^*_{n''}+\phi^*_{n'}\phi_{n''}\right]\int_{\mathbb{T}^2}e^{2\pi i k(n''-n')}\hat{K}_{M_N}(k,k')S^*\left(\frac{\varepsilon(n'+n'')}{2}, k'\right)dkdk'\\
&\quad-\frac{1}{2}\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi_{n''}+\phi^*_{n'}\phi^*_{n''}\right]\int_{\mathbb{T}^2}e^{2\pi i k(n''-n')}\hat{K}_{M_N}(k,k')S^*\left(\frac{\varepsilon(n'+n'')}{2}, k'\right)dkdk'+a(\varepsilon)
\end{aligned}$$

where $a(\varepsilon)$ satisfies $\varepsilon a(\varepsilon) \to 0$ as $\varepsilon \to 0$. We change $\phi^*_{n'}\phi_{n''}$ into $\phi_{n'}\phi^*_{n''}$. This is possible because $\hat{K}_{M_N}(k,k')$ is even with respect to k. We also obtain that

$$\begin{aligned}
&\sum_{n',n''}\mathbb{E}_\varepsilon\left[(O^{M_N}\phi_{n'})\phi^*_{n''}+(O^{M_N}\phi^*_{n''})\phi_{n'}\right]\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2}, n'-n''\right)\\
&=\sum_{n',n''}\mathbb{E}_\varepsilon\Bigg[(-\sum_d K_{M_N}(-d,0)(\phi_{n'+d}-\phi^*_{n'+d}))\phi^*_{n''}+(\sum_d K_{M_N}(d,0)(\phi_{n''-d}-\phi^*_{n''-d}))\phi_{n'}\Bigg)\\
&\quad\times\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2}, n'-n''\right)\\
&=-2\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi^*_{n''}-\frac{1}{2}(\phi_{n'}\phi_{n''}+\phi^*_{n'}\phi^*_{n''})\right]\sum_d K_{M_N}(d,0)\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2}, n'-n''+d\right)\\
&\quad+a(\varepsilon)\\
&=-2\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi^*_{n''}-\frac{1}{2}(\phi_{n'}\phi_{n''}+\phi^*_{n'}\phi^*_{n''})\right]\int_{\mathbb{T}^2}e^{2\pi i k(n''-n')}\hat{K}(k,k')S^*\left(\frac{\varepsilon(n'+n'')}{2}, k\right)dkdk'\\
&\quad+a(\varepsilon).
\end{aligned}$$

Therefore, the perturbation gives

$$\frac{\varepsilon}{2}\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi^*_{n''}-\frac{1}{2}(\phi_{n'}\phi_{n''}+\phi^*_{n'}\phi^*_{n''})\right]\int_{\mathbb{T}} e^{2\pi ik(n''-n')}O^{M_N}_{col}S^*\left(\frac{\varepsilon(n'+n'')}{2},k\right)dk+\varepsilon a(\varepsilon).$$

To reach the conclusion, we show that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_{\mathbb{T}}\mathbb{E}_\varepsilon\left[\phi_n(t/\varepsilon)\phi_{n''}(t/\varepsilon)\right]\int_{\mathbb{T}} e^{2\pi ik(n''-n')}O^{M_N}_{col}S^*\left(\frac{\varepsilon(n'+n'')}{2},k\right)dk=0.$$

We have

$$\begin{aligned}
&\frac{d}{dt}\frac{\varepsilon}{2}\int_{\mathbb{T}}\mathbb{E}_\varepsilon\left[\phi_n(t/\varepsilon)\phi_{n''}(t/\varepsilon)\right]\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2},n'-n''\right)\\
&=\frac{\varepsilon}{2}\int_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}}\varepsilon^{-1}\mathbb{E}_\varepsilon\left[O^H(\hat{\phi}(k+\varepsilon\xi/2,t/\varepsilon))\hat{\phi}(k-\varepsilon\xi/2,t/\varepsilon)\right]\hat{S}^*(\xi,k)\,d\xi\,dk\\
&\quad+\frac{\varepsilon}{2}\sum_{n',n''}\mathbb{E}_\varepsilon\left[O^{M_N}(\phi_{n'}(t/\varepsilon)\phi_{n''}(t/\varepsilon))\right]\tilde{S}^*\left(\frac{\varepsilon(n'+n'')}{2},n'-n''\right)\\
&=\frac{-i\varepsilon}{2}\int_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}}\mathbb{E}_\varepsilon\left[\hat{\phi}(k+\varepsilon\xi/2,t/\varepsilon)\hat{\phi}(k-\varepsilon\xi/2,t/\varepsilon)\right]\varepsilon^{-1}\left(\omega(k+\varepsilon\xi/2)+\omega(k-\varepsilon\xi/2)\right)\\
&\quad\times\hat{S}^*(\xi,k)\,d\xi\,dk\\
&\quad-\frac{1}{2}\left(\frac{d}{dt}\langle S,W^\varepsilon(t)\rangle+\frac{d}{dt}\langle S,W^\varepsilon(t)^*\rangle\right)\\
&\quad+\frac{1}{2\pi}\left(\langle\omega'\frac{\partial S}{\partial x},W^\varepsilon(t)\rangle+\langle\omega'\frac{\partial S}{\partial x},W^\varepsilon(t)^*\rangle\right)\\
&\quad+\frac{\varepsilon}{4}\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi_{n''}+\phi^*_{n'}\phi^*_{n''}\right]\int_{\mathbb{T}^2}e^{2\pi ik(n''-n')}\hat{K}_{M_N}(k,k')S^*\left(\frac{\varepsilon(n'+n'')}{2},k'\right)dk\,dk'\\
&\quad-\frac{\varepsilon}{4}\sum_{n',n''}\mathbb{E}_\varepsilon\left[\phi_{n'}\phi_{n''}-\phi^*_{n'}\phi^*_{n''}\right]\int_{\mathbb{T}^2}e^{2\pi ik(n''-n')}\hat{K}_{M_N}(k,k')S^*\left(\frac{\varepsilon(n'+n'')}{2},k\right)dk\,dk'.
\end{aligned}$$

We find that

$$\begin{aligned}
&\int_0^t\frac{\varepsilon}{2}\int_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}}\mathbb{E}_\varepsilon\left[\hat{\phi}(k+\varepsilon\xi/2,s/\varepsilon)\hat{\phi}(k-\varepsilon\xi/2,s/\varepsilon)\right](\omega(k+\varepsilon\xi/2)+\omega(k-\varepsilon\xi/2))\\
&\quad\times\hat{S}^*(\xi,k)d\xi dk ds
\end{aligned}$$

tends to 0 as $\varepsilon\to 0$ for each $t\in[0,T]$. We replace $\omega(k+\varepsilon\xi/2)+\omega(k-\varepsilon\xi/2)$ with $2\omega(k)$. This approximation is valid because, if $|k|>\varepsilon|\xi|$, then

$$\omega(k+\varepsilon\xi/2)+\omega(k-\varepsilon\xi/2)-2\omega(k)=\frac{1}{2}(\omega''(k_+)+\omega''(k_-))\varepsilon^2\xi^2.$$

In the no-pinning case $\omega''(k_\pm)\varepsilon|\xi|\lesssim 1$, and in the pinning case $\omega''(k_\pm)\lesssim 1$. Thus, in both scenarios, there is always at least one free ε. If $|k|\le\varepsilon|\xi|$, then

$$|\omega(k+\varepsilon\xi/2)+\omega(k-\varepsilon\xi/2)-2\omega(k)|\lesssim\varepsilon|\xi|/2+|k|+\varepsilon|\xi|/2-|k|+2|k|\lesssim\varepsilon|\xi|.$$

This also provides a free ε.

We can replace $\omega(k+\varepsilon\xi/2)+\omega(k-\varepsilon\xi/2)$ by $2\omega(k)$, leading to the expression:

$$\int_0^t\frac{\varepsilon}{2}\int_{\mathbb{T}_{2/\varepsilon}\times\mathbb{T}}\mathbb{E}_\varepsilon\left[\hat{\phi}\left(k+\frac{\varepsilon\xi}{2},\frac{s}{\varepsilon}\right)\hat{\phi}\left(k-\frac{\varepsilon\xi}{2},\frac{s}{\varepsilon}\right)\right]2\omega(k)\hat{S}^*(\xi,k)\,d\xi\,dk\,ds,$$

which tends to 0 as $\varepsilon\to 0$ for $t\in[0,T]$. Replacing S by $\frac{O_{\rm col}^{M_N}S(k)}{\omega(k)}$, we obtain the desired result.

6.4. Proof of Proposition 9

In the 1-dimensional case, for any N, we have

$$\begin{aligned}\hat{K}_{M_N}(k,k')&=4\sum_{1\le|d|,|d'|\le 2N}K_{M_N}(d,d')\sin^2(\pi dk)\sin^2(\pi d'k')\\&=16\sum_{1\le d,d'\le 2N}K_{M_N}(d,d')\sin^2(\pi k)\sin^2(\pi k')\left(\sum_{d_1=0}^{d-1}U_{2d_1}(\cos(\pi k))\right)\\&\quad\times\left(\sum_{d_2=0}^{d'-1}U_{2d_2}(\cos(\pi k'))\right)\end{aligned}$$

where $U_d(x)$ is the Chebyshev polynomial defined by

$$\sin((d+1)\theta)=\sin(\theta)U_d(\cos(\theta)),\quad d\ge 0.$$

Noting that

$$\begin{aligned}\sin^2(\theta)\left(\sum_{d_1=0}^{d-1}U_{2d_1}(\cos(\theta))\right)&=\sum_{d_1=0}^{d-1}\sin(\theta)\sin((2d_1-1)\theta)\\&=\sum_{d_1=0}^{d-1}\frac{\cos(2d_1\theta)-\cos(2(d_1+1)\theta)}{2}\\&=\frac{1-\cos(2d\theta)}{2}=\sin^2(d\theta)\end{aligned}$$

we see that the kernel $\hat{K}_{M_N}$ is of the form

$$16\sin^2(\pi k)\sin^2(\pi k')\sum_{d_1=0}^{2N-1}\sum_{d_2=0}^{2N-1} L_{M_N}(d_1,d_2)U_{2d_1}(\cos(\pi k))U_{2d_2}(\cos(\pi k')),$$

where L_{M_N} is defined by

$$L_{M_N}(d_1,d_2)=\sum_{d=d_1+1}^{2N}\sum_{d'=d_2+1}^{2N} K_{M_N}(d,d').$$

According to (12) to (23) and Assumption $(M2)$ (pick $M_N(d)=0, d\le N/2$ for simplicity), $L_{M_N}(d_1,d_2)$ can be rewritten as the sum

$$\sum_{N/2<d\le d'\le 2N} c_{d,d'}(d_1,d_2)M_N(d)M_N(d').$$

We define the polynomial $P_{i,j}(x,y)$ by

$$P_{i,j}(x,y)=\sum_{d_1,d_2} c_{i+\lfloor N/2\rfloor, j+\lfloor N/2\rfloor}(d_1,d_2)U_{2d_1}(x)U_{2d_2}(y).$$

Therefore, for

$$\mathcal{K}(k,k')=\sin^2(\pi k)\sin^2(\pi k')v(\cos(\pi k),\cos(\pi,k')),$$

where $v=C_iC_jP_{i,j}(x,y)$, we choose

$$M_N\left(i+\left\lfloor\frac{N}{2}\right\rfloor\right)=\frac{C_i}{4}.$$

With this choice, we obtain

$$\hat{K}_{M_N}=\mathcal{K}.$$

Appendix A. Example for small distances

Let us consider $\mathbb{d}=1, N=3$ as the simplest example. We compute the polynomials that define the space of polynomials V in Proposition 9. We make $M_3(2), M_3(3)$ as free parameters, while $M_3(1)=0$. We compute $K_{M_3}(d,d')$ for $1\le d\le d'\le 6$ in Table 1.

Then, we compute $L_{M_3}(d_1,d_2)$ for $0\le d_1\le d_2\le 5$ (see Table 2).

We define

$$\begin{aligned}P(x,y) &:= \frac{3}{4}(U_2(x)+1)(U_2(y)+1)\\ &\quad+\frac{1}{4}\left(U_4(x)+U_4(y)+U_4(x)U_2(y)+U_2(x)U_4(y)-U_4(x)U_4(y)\right)\end{aligned}$$

Table 1
K_{M_N} for $N = 3$.

d' \ d	1	2	3	4	5	6
1	$\frac{1}{2}M_3(2)M_3(3)$					
2	$-\frac{1}{2}M_3(2)M_3(3)$	0				
3	$-\frac{1}{2}M_3(2)M_3(3)$	0	0			
4	0	$\frac{1}{2}M_3(2)^2$	0	$-\frac{1}{4}M_3(2)^2$		
5	0	$\frac{1}{2}M_3(2)M_3(3)$	$\frac{1}{2}M_3(2)M_3(3)$	0	$-\frac{1}{2}M_3(2)M_3(3)$	
6	0	0	$\frac{1}{2}M_3(3)^2$	0	0	$-\frac{1}{4}M_3(3)^2$

Table 2
L_{M_N} for $N = 3$.

d_2 \ d_1	0	1
0	$\frac{3}{4}\left(M_3(2)^2 + M_3(3)^2\right)$	
1	$\frac{3}{4}\left(M_3(2)^2 + M_3(3)^2\right) + \frac{1}{2}M_3(2)M_3(3)$	$\frac{3}{4}\left(M_3(2)^2 + M_3(3)^2\right) + \frac{3}{2}M_3(2)M_3(3)$
2	$\frac{1}{4}M_3(2)^2 + \frac{3}{4}M_3(3)^2 + \frac{1}{2}M_3(2)M_3(3)$	$\frac{1}{4}M_3(2)^2 + \frac{3}{4}M_3(3)^2 + M_3(2)M_3(3)$
3	$\frac{1}{4}\left(M_3(2)^2 + M_3(3)^2\right) + \frac{1}{2}M_3(2)M_3(3)$	$\frac{1}{4}\left(M_3(2)^2 + M_3(3)^2\right) + \frac{1}{2}M_3(2)M_3(3)$
4	$\frac{1}{4}M_3(3)^2 + \frac{1}{2}M_3(2)M_3(3)$	$\frac{1}{4}M_3(3)^2 + \frac{1}{2}M_3(2)M_3(3)$
5	$\frac{1}{4}M_3(3)^2$	$\frac{1}{4}M_3(3)^2$

d_2 \ d_1	2	3
2	$-\frac{1}{4}M_3(2)^2 + \frac{3}{4}M_3(3)^2 + \frac{1}{2}M_3(2)M_3(3)$	
3	$\frac{1}{4}\left(-M_3(2)^2 + M_3(3)^2\right)$	$-\frac{1}{4}\left(M_3(2)^2 + M_3(3)^2\right) - \frac{1}{2}M_3(2)M_3(3)$
4	$\frac{1}{4}M_3(3)^2$	$-\frac{1}{4}M_3(3)^2 - \frac{1}{2}M_3(2)M_3(3)$
5	$\frac{1}{4}M_3(3)^2$	$-\frac{1}{4}M_3(3)^2$

d_2 \ d_1	4	5
4	$-\frac{1}{4}M_3(3)^2 - \frac{1}{2}M_3(2)M_3(3)$	
5	$-\frac{1}{4}M_3(3)^2$	$-\frac{1}{4}M_3(3)^2$

$$
\begin{aligned}
&+\frac{1}{4}\left(U_6(x) + U_6(y) + U_6(x)U_2(y) + U_2(x)U_6(y) - U_6(x)U_4(y)\right.\\
&\left.-U_4(x)U_6(y) - U_6(x)U_6(y)\right)
\end{aligned}
$$

$$
\begin{aligned}
Q(x,y) :=\ & \frac{3}{2}U_2(x)U_2(y) + U_4(x)U_2(y) + U_2(x)U_4(y)\\
&+\frac{1}{2}\left(U_2(x) + U_2(y) + U_4(x) + U_4(y) + U_6(x) + U_6(y) + U_8(x) + U_8(y)\right)\\
&+\frac{1}{2}\left(U_6(x)U_2(y) + U_2(x)U_6(y) + U_8(x)U_2(y) + U_2(x)U_8(y) + U_4(x)U_4(y)\right)\\
&-\frac{1}{2}\left(U_6(x)U_6(y) + U_8(x)U_6(y) + U_6(x)U_8(y) + U_8(x)U_8(y)\right)
\end{aligned}
$$

$$
\begin{aligned}
R(x,y) := {} & \frac{3}{4}\left(1 + U_2(x) + U_2(y) + U_2(x)U_2(y) + U_4(x) + U_4(y) + U_4(x)U_2(y)\right. \\
& \left. + U_2(x)U_4(y) + U_4(x)U_4(y)\right) \\
& + \frac{1}{4}\left(U_6(x) + U_6(y) + U_6(x)U_2(y) + U_2(x)U_6(y) + U_6(x)U_4(y)\right. \\
& \left. + U_4(x)U_6(y) + U_6(x)U_6(y)\right) \\
& + \frac{1}{4}\left(U_8(x) + U_8(y) + U_8(x)U_2(y) + U_2(x)U_8(y) + U_8(x)U_4(y) + U_4(x)U_8(y)\right) \\
& - \frac{1}{4}\left(U_8(x)U_6(y) + U_6(x)U_8(y) + U_6(x)U_6(y) + U_8(x)U_8(y)\right) \\
& + \frac{1}{4}U_{10}(x)\left(1 + U_2(y) + U_4(y) - U_6(y) - U_8(y) - U_{10}(y)\right) \\
& + \frac{1}{4}U_{10}(y)\left(1 + U_2(x) + U_4(x) - U_6(x) - U_8(x) - U_{10}(x)\right)
\end{aligned}
$$

We have the set of polynomials

$$
V = \{C_1^2 P(x,y) + C_1 C_2 Q(x,y) + C_2^2 R(x,y) \mid C_1, C_2 \in \mathbb{R}\}.
$$

Proposition 9 states that we can always find a perturbation to get the kernel in the form of

$$
\sin^2(\pi k)\sin^2(\pi k')v(\cos(\pi k), \cos(\pi k')) \quad \text{for } v \in V.
$$

Data availability

No data was used for the research described in the article.

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