UNCERTAINTY PRINCIPLES FOR A KINETIC EQUATION

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Dedicated with friendship and admiration to Alain Bensoussan issue on the occasion of his 85th birthday.

ABSTRACT. We study unique continuation properties for a kinetic equation. We establish sufficient conditions on the interaction potential and on the behavior of the solution at the initial and terminal times that ensure the solution is identically zero. Our strategy adapts that of Escauriaza et al. (2006) [13], combining the logarithmic convexity of certain quantities—which yields quadratic exponential decay at infinity of the solution—with a suitable Carleman inequality, which provides a lower bound for the L^2-norm of the solution in an appropriate annular domain.

1. Introduction

We consider the following transport equation (see [10, Chapter XXI]):

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + K[f](t,x,v) = 0, & (t,x,v) \in [0,1] \times \mathbb{R}^{2N}, \\ f(0,x,v) = f_0(x,v), & (x,v) \in \mathbb{R}^{2N}, \end{cases} \tag{1.1}$$

where $f = f(t,x,v)$ is the density distribution of particles at time $t \in [0,1]$ and position $x \in \mathbb{R}^N$, with $N \geqslant 1$, $v \in \mathbb{R}^N$ is the velocity, $f_0 : \mathbb{R}^{2N} \to \mathbb{R}$ is the initial distribution, and $K[f] : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a collision operator, which could be nonlinear.

Our goal is to investigate the *unique continuation properties* of solutions to (1.1): namely, we aim to establish sufficient conditions on the collision operator K and on the behavior of a solution f at two different times, $t = 0$ and $t = 1$, that ensure $f \equiv 0$ in $[0,1] \times \mathbb{R}^{2N}$.

1.1. Motivations and literature overview.

The main motivation comes from G. H. Hardy's uncertainty principle (see [24]; cf. also the generalization due to Morgan in [30]):[1]

$$\text{If } f(x) = O\left(e^{-|x|^2/\beta^2}\right) \text{ and } \hat{f}(\xi) = O\left(e^{-4|\xi|^2/\alpha^2}\right), \text{ with } 1/\alpha\beta > 1/4, \text{ then } f \equiv 0.$$

Moreover, if $1/\alpha\beta = 1/4$, then f is a constant multiple of $e^{-|x|^2/\beta^2}$. Throughout this paper, we use the notation

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} f(x) \, \mathrm{d}x$$

for the *Fourier transform* of f.

While the assumptions above are pointwise bounds, an L^2-Hardy uncertainty principle can be formulated as well (see [9, 34]):

$$\text{If } e^{|x|^2/\beta^2} f \text{ and } e^{4|\xi|^2/\alpha^2} \hat{f} \in L^2\left(\mathbb{R}^N\right), \text{ with } 1/\alpha\beta \geqslant 1/4, \text{ then } f \equiv 0.$$

This result can be interpreted as a sharp uniqueness result for the free solution of the Schrödinger equation (see [7, 13]),

$$\begin{cases} i\,\partial_t u + \Delta u = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ u(0,x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

where i denotes the imaginary unit, the unknown $\psi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{C}$ is the wave function of the particle, and $\psi_0 \in L^2(\mathbb{R}^N)$ is the initial datum. The statement is as follows:

$$\text{If } u(0,x) = O\left(e^{-|x|^2/\beta^2}\right) \text{ and } u(T,x) = O\left(e^{-|x|^2/\alpha^2}\right), \text{ with } T/\alpha\beta > 1/4, \text{ then } u \equiv 0.$$

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[1] We also refer to the textbooks [37, 25, 38, 33] for alternative proofs and further information.

Moreover, if $T/\alpha\beta = 1/4$, then u has as initial data a constant multiple of $e^{-\left(1/\beta^2+i/4T\right)|y|^2}$. Similarly, an L^2 statement holds:

$$\text{If } e^{|x|^2/\beta^2}u(0,x),\ e^{|\xi|^2/\alpha^2}u(x,T) \in L^2\left(\mathbb{R}^N\right),\ \text{with } T/\alpha\beta \geqslant 1/4,\ \text{then } u \equiv 0.$$

This research line was carried out in the fundamental papers [13, 12, 17, 8, 18, 15]. These results were generalized to semi-linear equations and to the covariant Schrödinger evolution in [16, 3, 6, 5]. Unique continuation results of this kind have been also established for the Kortweg–de Vries equation (see, e.g., [32, 14]), several nonlocal dispersive models (see [28]), the Navier–Stokes system (see, e.g., [11]), and discrete Schrödinger-type equations were studied in [4, 27, 20, 21, 1, 29, 22, 19]. We refer to [23] for a survey of this type of *dynamical versions* of Hardy's uncertainty principle.

In particular, we recall the following results from [13, Theorem 3] about a solution $u \in C([0,T];\, L^2\left(\mathbb{R}^N\right))$ of

$$\begin{cases} i\,\partial_t\psi(t,x) + \frac{1}{2}\Delta\psi(t,x) + V(t,x)\,\psi(x) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

If there exist positive constants α and β such that $T/\alpha\beta > 1/4$, and

$$\|e^{|x|^2/\beta^2}u(0,\cdot)\|_{L^2(\mathbb{R}^N)}, \quad \|e^{|x|^2/\alpha^2}u(T,\cdot)\|_{L^2(\mathbb{R}^N)} < \infty,$$

and the potential $V : \mathbb{R}_+\times\mathbb{R}^N \to \mathbb{C}$ is bounded and either $V(t,x) = V_1(x) + V_2(t,x)$, with V_1 real-valued and $\sup_{[0,T]}\|e^{T^2|x|^2/(\alpha t+\beta(T-t))^2}V_2(t)\|_{L^\infty(\mathbb{R}^N)} < +\infty$ or $\lim_{R\to+\infty}\|V\|_{L^1([0,T];\,L^\infty(\mathbb{R}^N\setminus B_R))} = 0$, then $u \equiv 0$.

1.2. **Main result.** In this note, we prove that result analogous to those for (1.2) also holds for (1.1).

Theorem 1.1 (Unique continuation for (1.1))**.** *Let us suppose that the collision operator K, which could be nonlinear, satisfies*

$$\|K[f]\|_{L^2([0,1]\times\mathbb{R}^{2N})} \leqslant C_K\|f\|_{L^2([0,1]\times\mathbb{R}^{2N})}, \tag{1.3}$$

$$K[vf](t,x,v) = vK[f](t,x,v), \quad \text{for all } (t,x,v) \in [0,1]\times\mathbb{R}^{2N}, \tag{1.4}$$

$$K[f](t,x,v) \leqslant 0, \quad \text{when } f(t,x,v) \geqslant 0, \quad \text{for all } (t,x,v) \in [0,1]\times\mathbb{R}^{2N}, \tag{1.5}$$

$$\int_{\mathbb{R}^{2N}} K[f]g\mathrm{d}x\mathrm{d}v \;=\; \int_{\mathbb{R}^{2N}} K[g]f\mathrm{d}x\mathrm{d}v \text{ for } f,g \in L^2([0,1]\times\mathbb{R}^{2N}), \tag{1.6}$$

$$\operatorname{supp}[K[f]] \;\subset [7/8,1]\times\{x\in\mathbb{R}^N \;\mid\; |x|\leqslant\rho\}\times\mathbb{R}^N \text{ for } f \in L^2([0,1]\times\mathbb{R}^{2N}) \tag{1.7}$$

where ρ is a fixed positive constant. Moreover, let us suppose that the there exist $R_0 > 0,\ \gamma > 12$ such that if the initial datum f_0 is not the zero function, it satisfies, for any $R > R_0$

$$\int_{\left(\gamma/2-2/3\right)R\leqslant|x|\leqslant\left(\gamma/2+1/3\right)R}\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R}|v|^2|f_0|^2\,\mathrm{d}v\,\mathrm{d}x \;\geqslant\; \mathfrak{c}_2e^{-\mathfrak{c}_1R^{2\beta}}, \tag{1.8}$$

for a positive constant β, $0<\beta<1$, and $\mathfrak{c}_1 > 0, \mathfrak{c}_2 \geqslant 0$ are constants depending on R_0, γ, β.

Let us assume that (1.1) has a strong non-negative solution f such that $f, |v|f \in C^1([0,1];\, H^1(\mathbb{R}^N, L^2(\mathbb{R}^N)))$. If

$$\||v|e^{\varphi}f(0,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} < \infty \quad \text{and} \quad \||v|e^{\varphi}f(1,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} < \infty,$$

where

$$\varphi(t,x,v) = \varphi_1(t,x) + \varphi_2(t,v), \tag{1.9}$$

$$\varphi_1(t,x) = \frac{M_1|x|^2}{(t+1)^2} + M_0|x|^2, \tag{1.10}$$

$$\varphi_2(t,v) = M_2|v|^2t^2, \tag{1.11}$$

with $M_2, M_1, M_0 > 0$ such that $M_2 + M_0 \geqslant 4M_1$, then

$$f(t,\cdot,\cdot) \equiv 0 \quad \text{for all } t \in [0,1].$$

Remark 1.2. An example of K is as follows. Let $\mathcal{M}(z) : \mathbb{R}_+ \to \mathbb{R}_+$, be a bounded function in $C_c(\mathbb{R}_+)$, we define

$$L[f](t,x,v) = \chi_{t\in[7/8,1]} \int_{|y|\leqslant\rho} \mathcal{M}(|x-y|) f(t,y,v)\mathrm{d}y, \text{ for } x \in \mathbb{R}^N, |x| \leqslant \rho,$$

and

$$L[f](x) = 0, \text{ for } x \in \mathbb{R}^N, |x| > \rho.$$

The proof of Theorem 1.1 is given in Section 3. Following the strategy of [13], it combines two main tools, which are contained in Section 2. The first one is the logarithmic convexity of certain quantities, which essentially yield quadratic exponential decay at infinity of f (see Lemma 2.2); the second one is a lower-bound for the L^2-norm of the solution in a suitable annular domain, for which we need a suitable Carleman inequality (see Lemmas 2.3–2.5).

1.3. **Further comments and extensions.** The current work is a test case of the method for the simple case of the linear transport equation (1.1). However, the method has the potential to be extended to various more complicated kinetic equations. In forthcoming works, we will analyze

$$\begin{cases} \partial_t W + k \cdot \nabla_x W = K[W], & (t,x,k) \in (0,+\infty) \times \mathbb{R}^N, \\ W(0,x,k) = W_0(x,k), & (x,k) \in \mathbb{R}^N, \end{cases} \tag{1.12}$$

where the collision operator K can be of classical type [2], nonlinear wave kinetic type [36] and nonlinear quantum kinetic type for $1 \leftrightarrow 2$ collisions [35] and $2 \leftrightarrow 2$ collisions [31]. Those cases are much harder, but highly relevant for applications.

2. Log-convexity and Carleman estimate

On $L^2(\mathbb{R}^N)$, whose inner product is denoted by $(\cdot,\cdot)$, let us consider the following abstract kinetic equation

$$\partial_t g \;+\; Tg \;=\; Lg \;+\; G, \tag{2.1}$$

where G is some source force, L is a collision operator satisfying

$$(Lh,k) \;=\; (h,Lk), \tag{2.2}$$

and T is a transport operator satisfying the following properties

$$(Tg,g) = 0, \tag{2.3}$$

$$(Th,k) = -(h,Tk), \tag{2.4}$$

$$Te^{\phi} = e^{\phi}T\phi, \tag{2.5}$$

$$T(hk) = (Th)k + hTk, \tag{2.6}$$

where ϕ, h, k are functions on $L^2(\mathbb{R}^N)$ such that the integrals and operators in (2.3), (2.4) and (2.5) are well-defined.

Let us define the function

$$W := \; e^{\phi} g, \tag{2.7}$$

where g is the solution of the equation (2.1) and $\phi(t,x)$ is some bounded function in $C^\infty([0,1]\times\mathbb{R}^N)$. We prove that, under suitable conditions, W is logarithmically convex.

Lemma 2.1 (Abstract log-convexity lemma)**.** *If there exist positive constants $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$, such that*

$$(\partial_t + T)^2\phi + (\partial_{tt} + \partial_t T)\phi \geqslant -\mathcal{C}_0, \qquad (t,x) \in [0,1] \times \mathbb{R}^N, \tag{2.8}$$

$$\|e^{\phi}Lf\|_{L^2} \leqslant \mathcal{C}_1\|f\|_{L^2}, \qquad \textit{for all } f \in L^2(\mathbb{R}^N), \tag{2.9}$$

$$\mathcal{C}_2 := \sup_{t\in[0,1]} \frac{\|e^{\phi}G(t)\|_{L^2}}{\|f(t)\|_{L^2}}, \tag{2.10}$$

then $W(t)$ is logarithmically convex in $[0,1]$ and there exists a constant $\mathcal{N} > 0$ such that

$$W(t) \leqslant e^{\mathcal{N}(\mathcal{C}_0+\mathcal{C}_1+\mathcal{C}_1^2+\mathcal{C}_2+\mathcal{C}_2^2)} W(0)^{1-t} W(1)^t. \tag{2.11}$$

Proof. We first observe that W satisfies the following equation

$$\partial_t W \;+\; TW \;-\; \tilde{G} \;=\; (\partial_t\phi \;+\; T\phi)W \;+\; LW, \tag{2.12}$$

where

$$\tilde{G} \;=\; Ge^{\phi}.$$

Define the function

$$M(t) \;=\; (W(t), W(t)), \tag{2.13}$$

and take the derivative of $\log M$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\log M) \;&=\; \frac{\dot{M}}{M} \;=\; \frac{2(\partial_t W, W)}{M} \;=\; \\ &= \frac{2(-TW + (\partial_t\phi + T\phi)W, W)}{M} \;+\; \frac{2(LW + \tilde{G}, W)}{M} \\ &= \frac{2((\partial_t\phi + T\phi)W, W)}{M} \;+\; \frac{2(LW + \tilde{G}, W)}{M}, \end{aligned} \tag{2.14}$$

where in the last line, we have used (2.3).
Let us consider the first term on the right hand side of (2.14) and set

$$P \;=\; 2((\partial_t\phi + T\phi)W, W). \tag{2.15}$$

Differentiating in time the above expression yields

$$\begin{aligned} \dot{P} \;&=2((\partial_{tt}\phi + \partial_t T\phi)W, W) \;+\; 4((\partial_t\phi + T\phi)W, \partial_t W) \\ &= 2((\partial_{tt}\phi + \partial_t T\phi)W, W) \;+\; 4((\partial_t\phi + T\phi)W, (\partial_t + T)W) \\ &\quad -\; 4((\partial_t\phi + T\phi)W, TW), \end{aligned} \tag{2.16}$$

where in the last line, we have employed the identity $\partial_t W = (\partial_t + T)W - TW$.

Now, by (2.4) and (2.6), the last term on the right hand side of (2.16) can be rewritten in the following way

$$\begin{aligned} ((\partial_t\phi + T\phi)W, TW) \;&= (-T((\partial_t\phi + T\phi)W), W) \\ &= \;-\,(((T\partial_t + T^2)\phi)W, W) \;-\; (((\partial_t + T)\phi)TW, W), \end{aligned} \tag{2.17}$$

which leads to

$$((\partial_t\phi + T\phi)W, TW) \;=\; -\frac{1}{2}(((T\partial_t + T^2)\phi)W, W). \tag{2.18}$$

Putting together the two identities (2.16) and (2.18) leads to

$$\begin{aligned} \dot{P} \;&= 2(((\partial_t + T)^2\phi)W, W) \;+\; 4((\partial_t\phi + T\phi)W, (\partial_t + T)W) \;+\; 2((\partial_{tt}\phi + \partial_t T\phi)W, W) \\ &\quad +\; 2((\partial_{tt}\phi + \partial_t T\phi)W, W), \end{aligned} \tag{2.19}$$

which, by the polarization identity, can be expressed as

$$\begin{aligned} \dot{P} \;&= 2(((\partial_t + T)^2\phi)W, W) \;+\; \|\partial_t W + TW + ((\partial_t + T)\phi)W\|_{L^2}^2 \\ &\quad -\; \|\partial_t W + TW - ((\partial_t + T)\phi)W\|_{L^2}^2 \;+\; 2((\partial_{tt}\phi + \partial_t T\phi)W, W). \end{aligned} \tag{2.20}$$

We, therefore, also have the following identities by multiplying both sides of (2.20) with M

$$\begin{aligned} \dot{P}M \;&= 2(((\partial_t + T)^2\phi)W, W)M \;+\; \|\partial_t W + TW + ((\partial_t + T)\phi)W\|_{L^2}^2\|W\|_{L^2}^2 \\ &\quad -\; \|\partial_t W + TW - ((\partial_t + T)\phi)W\|_{L^2}^2\|W\|_{L^2}^2 \;+\; 2((\partial_{tt}\phi + \partial_t T\phi)W, W)M, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \dot{M}P \;&= 4(((\partial_t + T)\phi)W, W)(\partial_t W, W) \\ &=(\partial_t W + ((\partial_t + T)\phi)W, W)^2 \;-\; (\partial_t W - ((\partial_t + T)\phi)W, W)^2 \\ &=(\partial_t W + TW + ((\partial_t + T)\phi)W, W)^2 \;-\; (\partial_t W + TW - ((\partial_t + T)\phi)W, W)^2, \end{aligned} \tag{2.22}$$

where the last line follows from (2.3).

Now, let us consider the time derivative of $\frac{P}{M}$, which is $\frac{\dot{P}M-\dot{M}P}{M^2}$, and can be computed by substracting (2.21) and (2.22) and then dividing by M^2, as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{M}\right) = {} & 2(((\partial_t+T)^2\phi)W,W)/M \;+\; \|\partial_t W + TW \\ & + ((\partial_t+T)\phi)W\|_{L^2}^2\|W\|_{L^2}^2/M^2 \\ & - \|\partial_t W + TW - ((\partial_t+T)\phi)W\|_{L^2}^2\|W\|_{L^2}^2/M^2 \\ & + (\partial_t W + TW - ((\partial_t+T)\phi)W,W)^2/M^2 \\ & - \;(\partial_t W + TW + ((\partial_t+T)\phi)W,W)^2/M^2 \;+\; 2((\partial_{tt}\phi + \partial_t T\phi)W,W)/M.\end{aligned} \tag{2.23}$$

By the Cauchy-Schwarz inequality

$$\|\dot{W} + TW + ((\partial_t+T)\phi)W\|_{L^2}^2\|W\|_{L^2}^2/M^2 \;\geqslant\; (\partial_t W + TW + ((\partial_t+T)\phi)W,W)^2/M^2,$$

and the positiveness of $(\partial_t W + TW - ((\partial_t+T)\phi)W,W)^2/M^2$, we deduce from (2.23) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{M}\right) \geqslant {} & 2(((\partial_t+T)^2\phi)W,W)/M - \|\partial_t W + TW \\ & - ((\partial_t+T)\phi)W\|_{L^2}^2\|W\|_{L^2}^2/M^2 \;+\; 2((\partial_{tt}\phi + \partial_t T\phi)W,W)/M,\end{aligned} \tag{2.24}$$

which, by (2.12), leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{M}\right) \geqslant {} & 2(((\partial_t+T)^2\phi)W,W)/\|W\|_{L^2}^2 \;-\|LW+\tilde{G}\|_{L^2}^2/\|W\|_{L^2}^2 \\ & +\; 2((\partial_{tt}\phi + \partial_t T\phi)W,W)/\|W\|_{L^2}^2.\end{aligned} \tag{2.25}$$

Using the hypothesis (2.8), (2.9) and (2.11), from (2.25), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{M}\right) \geqslant 2\mathcal{C}_0 - 2\mathcal{C}_1^2 - 2\mathcal{C}_2^2. \tag{2.26}$$

As a consequence of (2.14), we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\log M) = {} & \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{M}\right) \;+\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2(e^{\phi}Lf+\tilde{G},W)}{M}\right) \\ \geqslant {} & \;2\mathcal{C}_0 - 2\mathcal{C}_1^2 - 2\mathcal{C}_2^2 \;+\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2(e^{\phi}Lf+\tilde{G},W)}{\|W\|_{L^2}^2}\right),\end{aligned} \tag{2.27}$$

which, by putting the terms on the right-hand side to the left-hand side, implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\log M) \;-\; \frac{2(e^{\phi}Lf+\tilde{G},W)}{\|W\|_{L^2}^2} \;-\; (2\mathcal{C}_0 - 2\mathcal{C}_1^2 - 2\mathcal{C}_2^2)t\right) \geqslant 0 \quad \text{for all } t \in [0,1]. \tag{2.28}$$

The above inequality proves that $\frac{\mathrm{d}}{\mathrm{d}t}(\log M) \;-\; \frac{2(e^{\phi}Lf+\tilde{G},W)}{\|W\|_{L^2}^2} \;-\; (2\mathcal{C}_0 - 2\mathcal{C}_1^2 - 2\mathcal{C}_2^2)t$ is indeed an increasing function on the time interval $[0,1]$. In other words

$$\begin{aligned}& \frac{\mathrm{d}}{\mathrm{d}t}(\log M)(s_1) - \frac{2(e^{\phi}Lf+\tilde{G},W)}{\|W\|_{L^2}^2}(s_1) \;-\; (2\mathcal{C}_0 - 2\mathcal{C}_1^2 - 2\mathcal{C}_2^2)s_1 \\ \geqslant {} & \;\frac{\mathrm{d}}{\mathrm{d}t}(\log M)(s_2) \;-\; \frac{2(e^{\phi}Lf+\tilde{G},W)}{\|W\|_{L^2}^2}(s_2) \;-\; (2\mathcal{C}_0 - 2\mathcal{C}_1^2 - 2\mathcal{C}_2^2)s_2\end{aligned} \tag{2.29}$$

for $0 \leqslant s_2 \leqslant s_1 \leqslant 1$.

Using the fact that

$$\frac{2(e^{\phi}Lf+\tilde{G},W)}{\|W\|_{L^2}^2} \leqslant 2\mathcal{C}_1 + 2\mathcal{C}_2,$$

we get the conclusion of the lemma. □

As a consequence of Lemma 2.1, we deduce a log-convexity property for the solution of (1.1).

Lemma 2.2 (Log-convexity properties for (1.1))**.** *Let us suppose that (1.1) possess a strong solution $f \in C^1([0,1], H^1(\mathbb{R}^N, L^2(\mathbb{R}^N)))$.*

If $\|e^{\varphi} f(0,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$, $\|e^{\varphi} f(1,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} < \infty$, then $\|e^{\varphi} f(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$ is logarithmically convex in $[0,1]$ and there exist universal constants $\mathcal{N}, K_\infty$ depending only on K satisfying

$$\|e^{\varphi} f(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} < e^{\mathcal{N}(K_\infty^2+K_\infty)}\|e^{\varphi} f(0,\cdot,\cdot)\|^{1-t}_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}\|e^{\varphi} f(1,\cdot,\cdot)\|^{t}_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}, \quad t \in [0,1].$$

Proof. Let us define $u := e^{\varphi} f$, $T = -v\cdot\nabla_x$, and $L = K$. In order to apply the abstract result of Lemma 2.1, we need to check that conditions (2.8) and (2.9) hold.

Step 1. *Condition (2.8).* First, let us expand

$$(\partial_t + T)^2\varphi = \partial_{tt}\varphi + 2v\cdot\nabla_x\partial_t\varphi + v\cdot\nabla_x(v\cdot\nabla_x\varphi), \tag{2.30}$$

and compute explicitly (2.30) for φ to be the function defined in (1.9)

$$\begin{aligned}
(\partial_t + T)^2\varphi_2 &= 2M_2|v|^2,\\
\partial_t\varphi_1 &= -\frac{M_1 2|x|^2}{(t+1)^3}, \quad \nabla_x\partial_t\varphi_1 = -\frac{4M_1 x}{(t+1)^3},\\
2v\cdot\nabla_x\partial_t\varphi_1 &= -\frac{8M_1 x\cdot v}{(t+1)^3}, \quad \nabla_x\varphi_1 = \frac{2M_1 x}{(t+1)^2} + 2M_0 x,\\
v\cdot\nabla_x\varphi_1 &= \frac{2M_1 x\cdot v}{(t+1)^2} + 2M_0 x\cdot v, \quad v\cdot\nabla_x(v\cdot\nabla_x\varphi_1) = \frac{2M_1|v|^2}{(t+1)^2} + 2M_0|v|^2,\\
\partial_{tt}\varphi_1 &= \frac{M_1|x|^2 6}{(t+1)^4}, \quad \partial_{tt}\varphi_1 + \partial_t T\varphi_1 \geqslant \frac{6M_1|x|^2}{(t+1)^4} - \frac{4M_1 x\cdot v}{(t+1)^3},\\
\partial_{tt}\varphi_2 + \partial_t T\varphi_2 &\geqslant 2M_2|v|^2.
\end{aligned} \tag{2.31}$$

Combining the inequalities in (2.31) yields

$$\begin{aligned}
&(\partial_t + T)^2\varphi + (\partial_{tt} + \partial_t T)\varphi\\
&\geqslant 2(M_2+M_0)|v|^2 + \frac{2M_1|v|^2}{(t+1)^2} - \frac{12M_1 x\cdot v(t+1)}{(t+1)^4} + \frac{M_1|x|^2 4(t+1)^2}{(t+1)^6}.
\end{aligned} \tag{2.32}$$

Using the fact that $M_2 + M_0 \geqslant 4M_1$, we obtain

$$(\partial_t + T)^2\varphi + (\partial_{tt} + \partial_t T)\varphi \geqslant 0. \tag{2.33}$$

As a consequence, Condition (2.8) is satisfied.

Step 2. *Condition (2.9).* It is straightforward that for each t in the interval $[0,1]$, the following inequality holds true, for $C_K > 0$ depending on ρ

$$\|e^{\varphi} L u\|_{L^2(\mathbb{R}^{2N})} \leqslant C_K\|u\|_{L^2(\mathbb{R}^{2N})}, \tag{2.34}$$

which implies Condition (2.9). The conclusion of the proposition then follows from Lemma 2.1. □

As a second step, we turn to the proof of an L^2-Carleman inequality.

Lemma 2.3 (Carleman inequality)**.** *Let us suppose that the collision operator K satisfies* (1.3)-(1.7) *and that* (1.1) *possess a strong compactly supported solution $f \in C^1([0,1], H^1(\mathbb{R}^N, L^2(\mathbb{R}^N)))$.*

The following Carleman inequality holds

$$\int_0^1\int_{\mathbb{R}^N\times\mathbb{R}^N} e^{2\varphi} f^2(\partial_t + v\cdot\nabla_x)^2\varphi \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \leqslant \int_0^1\int_{\mathbb{R}^N\times\mathbb{R}^N} e^{2\varphi}|(\partial_t + v\cdot\nabla_x)f|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t, \tag{2.35}$$

where $\varphi \in C^\infty([0,1]\times\mathbb{R}^{2N})$ is a function satisfying

$$(\partial_t + v\cdot\nabla_x)^2\varphi \geqslant 0, \quad \textit{for all } (t,x,v) \in [0,1]\times\mathbb{R}^{2N},$$

on the support of f.

Proof. Letting

$$g := e^{\varphi} f,$$

we compute

$$
\begin{aligned}
e^{\varphi}(\partial_t f \ + \ v\cdot\nabla_x f) &= e^{\varphi}(\partial_t(e^{-\varphi}g) \ + \ v\cdot\nabla_x(e^{-\varphi}g)) \\
&= \ e^{\varphi}(-\partial_t\varphi e^{-\varphi}g \ + \ \partial_t g e^{-\varphi} \ - \ v\cdot\nabla_x\varphi e^{-\varphi}g + v\cdot\nabla_x g e^{-\varphi}) \\
&= \partial_t g \ + \ v\cdot\nabla_x g \ - \ (\partial_t\varphi \ + \ v\cdot\nabla_x\varphi)g.
\end{aligned} \tag{2.36}
$$

Letting

$$
\mathcal{A} = -v\cdot\nabla_x,
$$

$$
\mathcal{S} = \partial_t\varphi + v\cdot\nabla_x\varphi,
$$

we can write (2.36) as

$$
e^{\varphi}(\partial_t f + v\cdot\nabla_x f) = \partial_t g - \mathcal{A}g - \mathcal{S}g. \tag{2.37}
$$

Moreover, the operator $\mathcal{S}_t + [\mathcal{S},\mathcal{A}]$ can be computed explicitly as:

$$
\begin{aligned}
\mathcal{S}_t + [\mathcal{S},\mathcal{A}] = & \ \partial_{tt}\varphi \ + \ 2v\cdot\nabla_x\partial_t\varphi \ + \ v\cdot\nabla_x(v\cdot\nabla_x\varphi) \\
&= (\partial_t \ + \ v\cdot\nabla_x)^2\varphi.
\end{aligned} \tag{2.38}
$$

We now follow the standard method of the L^2-Carleman inequality (see [26]). The symmetric and skew-aymmetric part of the space-time operator $\partial_t - \mathcal{S} - \mathcal{A}$ are $-\mathcal{S}$ and $\partial_t - \mathcal{A}$ respectively and the commutator $[-\mathcal{S}, \partial_t - \mathcal{A}]$ is $\mathcal{S}_t + [\mathcal{S},\mathcal{A}]$. As a consequence, we can bound $\partial_t g - \mathcal{A}g - \mathcal{S}g$ as follows:

$$
\begin{aligned}
\|\partial_t g - \mathcal{A}g - \mathcal{S}g\|^2_{L^2([0,1]\times\mathbb{R}^{2N})} \ &= \|\partial_t g - \mathcal{A}g\|^2_{L^2([0,1]\times\mathbb{R}^{2N})} \ + \ \|\mathcal{S}g\|^2_{L^2([0,1]\times\mathbb{R}^{2N})} \\
& \quad - 2\int_0^1\int_{\mathbb{R}^{2N}} \mathcal{S}g(\partial_t g \ - \ \mathcal{A}g)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&\geqslant 2\int_0^1\int_{\mathbb{R}^{2N}} g[-\mathcal{S},\partial_t \ - \ \mathcal{A}]g\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&= 2\int_0^1\int_{\mathbb{R}^{2N}} (\mathcal{S}_t g \ + \ [\mathcal{S},\mathcal{A}]g)g\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&= 2\int_0^1\int_{\mathbb{R}^{2N}} g^2(\partial_t \ + \ v.\nabla_x)^2\varphi\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t,
\end{aligned} \tag{2.39}
$$

which yields (2.35). □

As a consequence of Lemma 2.3, we deduce the following lower-bound on the L^2-norm of vf in a suitable domain in $[0,1]\times\mathbb{R}^{2N}$.

Lemma 2.4 (Lower bound on the (localized) L^2-norm of vf)**.** *Let R, β, δ, γ be positive constants and $0<\beta<1$, $2>\delta>2\beta$, $\gamma>12$. Let us suppose that the collision operator K satisfies (1.3)-(1.7) and that equation (1.1) possesses a strong solution f such that $f, |v|f \in C^1([0,1], H^1(\mathbb{R}^N, L^2(\mathbb{R}^N)))$.*

Let us define

$$
\Omega_R(f) := \left(\int_0^1\int_{\frac{\gamma}{2}R-1\leqslant|v|\leqslant\gamma R+1}\int_{R\leqslant|x|\leqslant R+1} |fv|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.40}
$$

If

$$
\int_0^1\int_{\mathbb{R}^{2N}} |f|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \ \leqslant \ A^2, \tag{2.41}
$$

and there exist $R_0, C_1, C_2 > 0$ such that, for $R > R_0$,

$$
\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{|x|\leqslant R}\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R} |v|^2|f|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \ \geqslant \ C_2 e^{-C_1 R^{2\beta}}, \tag{2.42}
$$

then there exist positive constants c_1, c_2, R_* depending on δ and A such that for $R > R_*$:

$$
|\Omega_R(f)|^2 \ \geqslant \ c_2 e^{-c_1 R^{\delta}}. \tag{2.43}
$$

Proof. Step 1. *Construction of the weight functions.* Let us define

$$\phi(t,x) := \frac{|x|^2}{(R+1)^2} \;+\; 16t(1-t), \tag{2.44}$$

on $\mathbb{R}_+ \times \mathbb{R}^{2N}$, and the following cut-off function of f:

$$g(t,x,v) := \theta_R(x)\theta\left(\frac{|x|^2}{(R+1)^2} + 16t(1-t)\right)\rho_R(v)f(t,x,v), \tag{2.45}$$

where θ_R, ρ_R are functions in $C_c^\infty(\mathbb{R}^N)$ and θ is a function in $C^\infty(\mathbb{R})$, satisfying

- $\theta_R(x) = 1$ if $|x| \leqslant R$ and $\theta_R(x) = 0$ if $|x| \geqslant R+1$;
- $\theta(z) = 0$ if $|z| \leqslant 1$ and $\theta(z) = 1$ if $|z| \geqslant 3$;
- $\rho_R(v) = 1$ if $\frac{\gamma}{2}R \leqslant |v| \leqslant \gamma R$ and $\rho_R(v) = 0$ if $|v| > \gamma R + 1$ or $|v| < \frac{\gamma}{2}R - 1$.

We observe that $g(t,x,v)$ is compactly supported in $[0,1] \times \mathbb{R}^{2N}$ and $g(t,x,v) = f(t,x,v)$ if

$$(t,x,v) \in \mathcal{B} := \left\{0 \leqslant t \leqslant 1, \qquad |x| \leqslant R, \quad \frac{\gamma}{2}R \leqslant |v| \leqslant \gamma R \;:\; \frac{|x|^2}{(R+1)^2} + 16t(1-t) \geqslant 3\right\}. \tag{2.46}$$

We can see that

$$\mathcal{B}_R := \left[\frac{1}{4}, \frac{3}{4}\right] \times \{|x| \leqslant R\} \times \left\{\frac{\gamma}{2}R \leqslant |v| \leqslant \gamma R\right\} \subset \mathcal{B}. \tag{2.47}$$

Using (1.1), we deduce the following equation for g:

$$\begin{aligned}
&\partial_t g + v\cdot\nabla_x g + \theta_R(x)\theta\left(\frac{|x|^2}{(R+1)^2} + 16t(1-t)\right)\rho_R(v)Kf \\
&= g\theta_R\rho_R\theta' + g\theta\rho_R v\cdot\nabla_x\theta_R + \\
&\quad + \;(\partial_t f \;+\; v\cdot\nabla_x f)\theta_R\theta\rho_R \;+\; \theta_R(x)\theta\left(\frac{|x|^2}{(R+1)^2} + 16t(1-t)\right)\rho_R(v)Kf \\
&= f\theta_R\rho_R\theta' + f\theta\rho_R v\cdot\nabla_x\theta_R.
\end{aligned} \tag{2.48}$$

Step 2. *Estimates on the right-hand side of* (2.48). Let us consider the first term $f\theta_R\rho_R\theta'$ on the right-hand side of (2.48). Observe that $\theta'(z)$ is supported in $1 \leqslant |z| \leqslant 3$, which leads to $|\phi| \leqslant 3$ on the considered domain. We can estimate the L^2-norm with weight $e^{2\alpha\phi}$ of the first term on the right hand side of (2.48) as follows:

$$\begin{aligned}
&\left(\int_0^1\int_{\mathbb{R}^{2N}} |f\theta_R\rho_R\theta'|^2 e^{2\alpha\phi}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right)^{1/2} \\
&\leqslant Ce^{3\alpha}\left(\int_0^1\int_{\mathbb{R}^{2N}} |f|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right)^{1/2} \\
&\leqslant Ce^{3\alpha}A,
\end{aligned} \tag{2.49}$$

in which $e^{\alpha\phi}$ is bounded by $e^{3\alpha}$ on the domain of integration and C is some universal constant that varies from line to line.

Since $\nabla_x\theta_R$ is supported in the interval $R \leqslant |x| \leqslant R+1$, we can estimate the second term on the right hand side of (2.48) as follows

$$\begin{aligned}
&\left(\int_0^1\int_{\mathbb{R}^{2N}} |f\theta\rho_R v\cdot\nabla_x\theta_R|^2 e^{2\alpha\phi}\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right)^{1/2} \\
&\leqslant Ce^{17\alpha}\left(\int_0^1\int_{\frac{\gamma}{2}R-1\leqslant|v|\leqslant\gamma R+1}\int_{R\leqslant|x|\leqslant R+1} |fv|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right)^{1/2} \\
&= Ce^{17\alpha}\Omega_R(f),
\end{aligned} \tag{2.50}$$

where we have used the fact that

$$\phi(t,x,v) \;\leqslant\; 17$$

on the domain of integration.

Inequality (2.48), in combination with the 2 inequalities (2.49), (2.50), yields

$$\|e^{\alpha\phi}(\partial_t g + v\cdot\nabla_x g)\|_{L^2([0,1]\times\mathbb{R}^{2N})} \leqslant \left\| e^{\alpha\phi}\theta_R(x)\theta\left(\frac{|x|^2}{(R+1)^2}+16t(1-t)\right)\rho_R(v)K[f]\right\|_{L^2([0,1]\times\mathbb{R}^{2N})} + Ce^{17\alpha}\Omega_R(f) + Ce^{3\alpha}A. \tag{2.51}$$

Step 2. *Application of the Carleman inequality.* By Lemma 2.3, the left hand side of (2.51) can be bounded from below as follows:

$$\|e^{\alpha\phi}(\partial_t g + v\cdot\nabla_x g)\|_{L^2([0,1]\times\mathbb{R}^{2N})} \geqslant \alpha^{1/2}\|e^{\alpha\phi}g\sqrt{(\partial_t+v\cdot\nabla_x)^2\phi}\|_{L^2([0,1]\times\mathbb{R}^{2N})}. \tag{2.52}$$

We now estimate

$$(\partial_t+v\cdot\nabla_x)^2\phi = \frac{2|v|^2}{(R+1)^2} - 32. \tag{2.53}$$

For v in the support of g, $|v|>\frac{\gamma}{2}R-1$, which yields

$$(\partial_t+v\cdot\nabla_x)^2\phi \geqslant \frac{1}{(R+1)^2}\left(\frac{\gamma}{2}R-1\right)^2 - 32 > 1, \tag{2.54}$$

when $\gamma>12$ and R is chosen sufficiently large.
Putting together the three estimates (2.51), (2.52) and (2.54), we obtain

$$\alpha^{1/2}\|e^{\alpha\phi}g\|_{L^2([0,1]\times\mathbb{R}^{2N})} \leqslant C_K e^{3\alpha}\|f\|_{L^2([0,1]\times\mathbb{R}^{2N})} + Ce^{17\alpha}\Omega_R(f) + Ce^{3\alpha}A, \tag{2.55}$$

where C is some universal constant that varies from line to line.

Step 3. *Conclusion of the argument.* For $\alpha=R^\delta$ ($\delta>2\beta$) and R large enough, we obtain

$$R^{\delta/2}\|e^{\alpha\phi}g\|_{L^2([0,1]\times\mathbb{R}^{2N})} \leqslant Ce^{17\alpha}\Omega_R(f) + C_K e^{3\alpha}A + Ce^{3\alpha}A, \tag{2.56}$$

where C is some universal constant that varies from line to line.

Note that for $R>R_0$

$$\|e^{\alpha\phi}g\|_{L^2([0,1]\times\mathbb{R}^{2N})} \geqslant \|e^{\alpha\phi}f\|_{L^2\left([\frac13,\frac23]\times\{|x|\leqslant R\}\times\left\{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R\right\}\right)},$$

which, due to the fact that $\phi\geqslant\frac{32}{9}$ for $t\in\left[\frac13,\frac23\right]\subset\left[\frac14,\frac34\right]$, is bounded from below as

$$\|e^{\alpha\phi}f\|_{L^2\left([\frac13,\frac23]\times\{|x|\leqslant R\}\times\left\{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R\right\}\right)} \geqslant e^{\frac{32}{9}\alpha}\|f\|_{L^2\left([\frac13,\frac23]\times\{|x|\leqslant R\}\times\left\{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R\right\}\right)}.$$

Taking into account (2.42) and the choice $\alpha=R^\delta$, the above estimate becomes

$$\|e^{\alpha\phi}f\|_{L^2\left([\frac13,\frac23]\times\{|x|\leqslant R\}\times\left\{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R\right\}\right)} \geqslant Ce^{\frac{23}{9}R^\delta-C'R^{2\beta}} \geqslant Ce^{3R^\delta},$$

where the last inequality holds true for R sufficiently large and $\delta>2\beta$, C, C' are universal constants varying from lines to lines.

Using the above estimate, we then deduce from (2.56) that

$$R^{\delta/2}e^{3R^\delta} \leqslant Ce^{17R^\delta}\Omega_R(f) + (C_K+C)e^{3R^\delta}. \tag{2.57}$$

The conclusion of the lemma follows. □

We deduce another lower bound on a (localized) L^2-norm of vf.

Lemma 2.5 (Lower bound on the (localized) L^2-norm of vf)**.** *Let us suppose that the collision operator K satisfies* (1.3)-(1.7) *and that equation* (1.1) *possesses a strong non-negative solution f such that f, $|v|f\in C^1([0,1];\,H^1(\mathbb{R}^N,L^2(\mathbb{R}^N)))$.*

Let us adopt the notations γ,β from Lemma 2.4. Then following inequality holds true:

$$\int_{\frac13}^{\frac23}\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R}\int_{|x|\leqslant R}|vf|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \geqslant \int_{\frac13}^{\frac23}\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R}\int_{|x|\leqslant R}|vf_0(x-vt,v)|^2\,\mathrm{d}x\,\mathrm{d}v\mathrm{d}t. \tag{2.58}$$

Moreover, if f_0 satisfies (1.8)*, then there exist constants $\mathfrak{C}_1,\mathfrak{C}_2>0$ such that the following inequality holds true:*

$$\int_{\frac13}^{\frac23}\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R}\int_{|x|\leqslant R}|vf|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \geqslant \mathfrak{C}_1 e^{-\mathfrak{C}_2 R^{2\beta}}. \tag{2.59}$$

Proof. We have

$$f(t,x,v) \;=\; -\int_0^t K[f](t-s,x-vs,v)ds \;+\; f_0(x-vt,v). \tag{2.60}$$

Since $K \leqslant 0$ (owing to (1.5)), from (2.60) we deduce

$$f(t,x,v) \geqslant f_0(x-vt,v). \tag{2.61}$$

Integrating the inequality (2.61) in x and v on the domain $\{\frac{\gamma}{2}R \leqslant |v| \leqslant \gamma R\} \times \{|x| \leqslant R\}$ yields

$$\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R}\int_{|x|\leqslant R} |vf|^2 \,\mathrm{d}x\,\mathrm{d}v \geqslant \int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R}\int_{|x|\leqslant R} |vf_0(x-vt,v)|^2 \,\mathrm{d}x\,\mathrm{d}v.$$

This proves (2.58). Moreover, if f_0 satisfies (1.8), we can further estimate the right-hand side and deduce that (2.59) holds. □

3. Proof of Theorem 1.1

By putting together the lemmas contained in Section 2, we can prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that $f \neq 0$, we can assume after a possible multiplication by a constant, dilation, translation that the conditions of Lemma 2.4 are satisfied. Fix positive constants $R > 0$, $0 < \beta < 1$, $\gamma > 8$. We start by noting that $|v|f$ satisfies the following equation:

$$\partial_t(|v|f)(t,x,v) + v\cdot\nabla_x(|v|f)(t,x,v) + K[|v|f](t,x,v) = 0, \qquad (t,x,v) \in [0,1]\times\mathbb{R}^{2N}. \tag{3.1}$$

<u>Step 1.</u> *Log-convexity.* Applying Lemma 2.2 to (3.1), we deduce

$$\sup_{t\in[0,1]} \|e^{\varphi}|v|f(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^N\times\mathbb{R}^N)} \leqslant C, \tag{3.2}$$

where C depends on the norms $\|e^{\varphi}|v|f(0,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$, $\|e^{\varphi}|v|f(1,\cdot,\cdot)\|_{L^2(\mathbb{R}^N\times\mathbb{R}^N)}$. This mplies

$$\begin{aligned}
&\int_{R\leqslant|x|\leqslant R+1}\int_{\frac{\gamma}{2}R-1\leqslant|v|\leqslant\gamma R+1} |v|^2|f|^2 \exp\left(\frac{M_1|x|^2}{(t+1)^2} \;+\; M_2|v|^2(t+1)^2 \;+\; M_0|v|^2\right)\mathrm{d}x\,\mathrm{d}v\\
&\leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v|^2|f|^2 \exp\left(\frac{M_1|x|^2}{(t+1)^2} \;+\; M_2|v|^2(t+1)^2 \;+\; M_0|v|^2\right)\mathrm{d}x\,\mathrm{d}v\\
&\leqslant C.
\end{aligned} \tag{3.3}$$

Using the fact that

$$\frac{|x|^2}{(t+1)^2} \geqslant \frac{R^2}{4},$$

on the domain of integration of (3.3), we deduce that

$$\int_0^1\int_{R\leqslant|x|\leqslant R+1}\int_{\frac{\gamma}{2}R-1\leqslant|v|\leqslant\gamma R+1} |v|^2|f|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant Ce^{-\frac{M_1R^2}{4}}, \tag{3.4}$$

where C is a universal constant that varies from line to line.

<u>Step 2.</u> *Lower-bound.* Owing to Lemma 2.5 (since condition (1.8) holds), we have the following estimate:

$$\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{|x|\leqslant R}\int_{\frac{\gamma}{2}R\leqslant|v|\leqslant\gamma R} |vf|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \;\geqslant\; c_2e^{-c_1R^{2\beta}}.$$

Furthermore, from Lemma 2.4 and the estimate above, we deduce

$$\int_0^1\int_{R\leqslant|x|\leqslant R+1}\int_{\frac{\gamma}{2}R-1\leqslant|v|\leqslant\gamma R+1} |vf|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \geqslant c_2'e^{-c_1'R^{\delta}}, \tag{3.5}$$

for $2\beta < \delta < 2$, $\delta > 1$.

<u>Step 3.</u> *Conclusion of the proof.* Combining the two inequalities (3.4) and (3.5) yields

$$c_2'e^{-c_1'R^{\delta}} \;\leqslant\; Ce^{-\frac{M_1R^2}{4}},$$

which leads to a contradiction as R tends to infinity since $\delta < 2$. □

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