Mathematische Zeitschrift (2025) 310:19
https://doi.org/10.1007/s00209-025-03734-6

Mathematische Zeitschrift

On the wave turbulence theory: ergodicity for the elastic beam wave equation

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Received: 25 August 2024 / Accepted: 8 March 2025

Abstract

We analyse a 3-wave kinetic equation, derived from the elastic beam wave equation on the lattice. The ergodicity condition states that two distinct wavevectors are supposed to be connected by a finite number of collisions. In this work, we prove that the ergodicity condition is violated and the equation domain is broken into disconnected domains, called no-collision and collisional invariant regions. If one starts with a general initial condition, whose energy is finite, then in the long-time limit, the solutions of the 3-wave kinetic equation remain unchanged on the no-collision region and relax to local equilibria on the disjoint collisional invariant regions. To our best knowledge, this is the first time that the violation of the ergodicity condition is observed and proved for a kinetic equation.

Keyword Wave turbulence · Convergence to equilibrium · Ergodicity condition

Contents

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Published online: 05 April 2025

1 Introduction

Having the origin in the works of Benney-Saffman-Newell [6, 7], Hasselmann [25, 26], Peierls [45, 46] and Zakharov [55], wave kinetic equations have been shown to play important roles in a vast range of physical examples and this is why a huge and still growing number of situations have used WT theory: inertial waves due to rotation; Alfvén wave turbulence in the solar wind; waves in plasmas of fusion devices; and many others, as discussed in the books of Nazarenko [38], Zakharov et al. [55] and the review papers of Newell and Rumpf [39, 40].

We consider the quadratic elastic beam wave equation (Bretherton-type equation) (see Benney-Newell [5], Bretherton [8] and Love [34])

$$\begin{aligned} &\frac{\partial^2\psi}{\partial T^2}(x,T)+(\Delta+c)^2\psi(x,T)+\lambda\psi^2(x,T)=0,\\ &\psi(x,0)=\psi_0(x),\quad \frac{\partial\psi}{\partial T}(x,0)=\psi_1(x), \end{aligned}\tag{1}$$

for x being on $\mathbb{Z}^3$, $T\in\mathbb{R}_+$, $c\in\mathbb{R}$ is some real constant, λ is a small constant describing the smallness of the nonlinearity. Equations of type (1) have been widely studied in control theory, and have been shown to have a Schrödinger structure (see, for instance, Burq [9], Fu-Zhang-Zuazua [19], Haraux [24], Lebeau [28], Lions [33], and Zuazua-Lions [56].) The analysis of (1) is also an interesting mathematical question of current interest (see, for instance, Hebey-Pausader [27], Levandosky-Strauss [32], Pausader [43] Pausader-Strauss [44].)

We obtain the 3-wave kinetic equation

$$\begin{aligned} &\partial_t f(k,t)=Q_c[f](k),\qquad f(k,0)=f_0(k),\quad \forall k\in\mathbb{T}^3,\\ &Q_c[f](k)=\int_{\mathbb{T}^6}K(\omega,\omega_1,\omega_2)\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[f_1f_2-ff_1\\ &\qquad -ff_2]\mathrm{d}k_1\mathrm{d}k_2\\ &\qquad -2\int_{\mathbb{T}^6}K(\omega,\omega_1,\omega_2)\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_2f-ff_1\\ &\qquad -f_1f_2]\mathrm{d}k_1\mathrm{d}k_2, \end{aligned}\tag{2}$$

where $K(\omega,\omega_1,\omega_2)=[\sqrt{8}\omega(k)\omega(k_1)\omega(k_2)]^{-1}$, with

$$\omega(k)=\omega_0+\sum_{j=1}^{3}2\Big(1-\cos(2\pi k^j)\Big),$$

and $\mathbb{T}^d$ is the periodic torus $[0,1]^d$.

One of the main challenges in understanding the behaviors of solutions to the 3-wave kinetic equations is the so-called *ergodicity*, which is quite typical for 3-wave processes. Ergodicity has played a very important role and has a long history in physics [3, 29–31] and we refer to the lecture notes [51] [Section 17] for a more detailed discussion. To define ergodicity, we will need the concept of the connectivity between two wave vectors k and k', which we briefly discuss here, leaving the precise definition for later. Given a wave vector k, a wave vector k' is understood to be connected to k in a collision if either $\omega(k')=\omega(k)+\omega(k'-k)$, $\omega(k)=\omega(k')+\omega(k-k')$, or $\omega(k+k')=\omega(k)+\omega(k')$.

Ergodicity Condition (E): *For every* $k,k'\in\mathbb{T}^3\backslash\{0\}$, *there is a finite sequence of collisions such that* k *is connected to* k'.

When the Ergodicity Condition (E) is violated, the system is partitioned into smaller subsystems which are dynamically disconnected and each subsystem thermalizes by itself.

It was shown that (see [51]) under the Ergodicity Condition (E), the only stationary solutions of the spatially homogeneous Boltzmann equations (2) take the forms

$$\frac{1}{\beta\omega(k)},$$

in which β can be computed via the conservation laws.

The aim of this work is to develop a rigorous analysis and prove that the Ergodicity Condition (E) is violated for the equation (2). We will show that the domain of integration is broken into disconnected domains. Those subregions are then proved to be dynamically disconnected. There is one region, in which if one starts with any initial condition, the solutions remain unchanged as time evolves. In general, the equilibration temperature will differ from region to region. We call it the “no-collision region”. The rest of the domain is divided into disconnected regions, each has their own local equilibria. If one starts with any initial condition, whose energy is finite on one subdomain, the solutions will relax to the local equilibria of this subregion, as time evolves, and as thus each subsystem thermalizes by itself. Those subregions are named “collisional invariant regions”, due to the fact that we can rigorously establish unique local collisional invariants on each of them, using the conservation of energy. This confirms Spohn’s prediction and enlightening physical intuitions [51] on the behavior of 3-wave systems. To our best knowledge, this is the first example in which the important ergodicity condition is violated for a kinetic equation. We also remark that the 3-wave kinetic equation considered in this work describes the translation invariant system and the results proven (decomposition of the frequency space $\mathbb{T}^d$ into disjoint equivalence classes under connectedness via collisions, that are invariant under the flow) do not hold for the spatially inhomogenous version of the equation.

In addition to 3-wave kinetic equations, 4-wave kinetic equations have also played an important role in wave turbulence and have been first studied in the work of Escobedo and Velazquez in [16, 17] as well as several other works [2, 11, 13, 14, 21, 37, 52, 53].

2 From the Bretheton equation to the 3-wave kinetic equation

For the sake of completeness, in this section, we recall the formal derivation of the 3-wave kinetic equation from the Bretheton equation for the general dimension $d > 2$. We follow the same strategy of [36, 51] to put the equation on a lattice

$$\Lambda = \Lambda(D) = \{1, \dots, 2D\}^d, \tag{3}$$

for some constant $D \in \mathbb{N}$.

The discretized equation is now

$$\begin{aligned} \partial_{TT}\psi(x,T) &= -\sum_{y\in\Lambda} O_1(x-y)\psi(y,T) - \lambda(\psi(x,T))^2, \\ \psi(x,0) &= \psi_0(x),\ \partial_T\psi(x,0) = \psi_1(x), \forall (x,T) \in \Lambda \times \mathbb{R}_+, \end{aligned} \tag{4}$$

where $O_1(x-y)$ is a finite difference operator that we will express below in the Fourier space. We remark that a similar beam dynamics of non-acoustic chains has also been considered in

[4][Section 7]. To obtain the lattice dynamics, we introduce the Fourier transform

$$\hat{\psi}(k) = \sum_{x\in\Lambda} \psi(x) e^{-2\pi \mathbf{i} k\cdot x}, \quad k \in \Lambda^* = \Lambda^*(D) = \left\{0, \ldots, \frac{2D}{2D+1}\right\}^d, \tag{5}$$

which is a subset of the d-dimensional torus $[0, 1]^d$. We also define the mesh size to be

$$h^d = \left(\frac{1}{2D+1}\right)^d. \tag{6}$$

At the end of this standard procedure, (4) can be rewritten in the Fourier space as a system of ODEs

$$\begin{aligned} \partial_{TT}\hat{\psi}(k, T) &= -\omega(k)^2 \hat{\psi}(k, T) \\ &\quad - \lambda \sum_{k_1,k_2\in\Lambda^*} \hat{\psi}(k_1, T)\delta(k - k_1 - k_2)\hat{\psi}(k_2, T), \\ \hat{\psi}(k, 0) &= \hat{\psi}_0(k), \qquad \partial_T \hat{\psi}(k, 0) = \hat{\psi}_1(k), \end{aligned} \tag{7}$$

where the dispersion relation takes the discretized form

$$\omega_k = \omega(k) = \sin^2(2\pi h k^1) + \cdots + \sin^2(2\pi k^d) + c, \tag{8}$$

with $k = (k^1, \cdots, k^d)$.

We define the inverse Fourier transform to be

$$f(x) = \sum_{k\in\Lambda_*} \hat{f}(k) e^{2\pi \mathbf{i} k\cdot x}. \tag{9}$$

We also use the following notations

$$\int_\Lambda \mathrm{d}x = h^d \sum_{x\in\Lambda}, \qquad \langle f, g\rangle = h^d \sum_{x\in\Lambda} f(x)^* g(x), \tag{10}$$

where if $z \in \mathbb{C}$, then $\bar{z}$ is the complex conjugate, as well as the Japanese bracket

$$\langle x\rangle = \sqrt{1 + |x|^2}, \quad \forall x \in \mathbb{R}^d. \tag{11}$$

And

$$\sum_{k\in\Lambda^*} = \int_{\Lambda^*} \mathrm{d}k. \tag{12}$$

Moreover, for any $N \in \mathbb{N}\backslash\{0\}$, following precisely [36] [equation (2.9)], we define the delta function δ_N on $(\mathbb{Z}/N)^d$ as

$$\delta_N(k) = |N|^d \mathbf{1}(k \bmod 1 = 0), \qquad \forall k \in (\mathbb{Z}/N)^d. \tag{13}$$

In our computations, we omit the sub-index N and simply write

$$\delta(k) = |N|^d \mathbf{1}(k \bmod 1 = 0), \qquad \forall k \in (\mathbb{Z}/N)^d. \tag{14}$$

Remark 1 Note that, the above definition of the discrete delta function follows the classical definition of Lukkarinen-Spohn [36] [equation (2.9)], commonly used in the derivation of wave kinetic equations. The factor $|N|^d$ is needed as it guarantees the convergence of the discrete delta function to the continuum delta function in the limit of N going to ∞.

Equation (7) can now be expressed as a coupling system

$$\begin{aligned}&\frac{\partial}{\partial T}q(k,T)=p(k,T),\\&\frac{\partial}{\partial T}p(k,T)=-\omega^2(k)q(k,T)\\&\qquad\qquad -\lambda\int_{(\Lambda^*)^2}\mathrm{d}k_1\mathrm{d}k_2\delta(k-k_1-k_2)q(k_1,T)q(k_2,T),\\&\qquad q(k,0)=\hat{\psi}_0(k),\quad p(k,0)=\hat{\psi}_1(k),\qquad \forall(k,T)\in\Lambda^*\times\mathbb{R}_+,\end{aligned}\tag{15}$$

which, under Spohn's transformation (see [51])

$$a(k,T)=\frac{1}{\sqrt{2}}\Big[\omega(k)^{\frac12}q(k,T)+\frac{i}{\omega(k)^{\frac12}}p(k,T)\Big],\tag{16}$$

leads to the following system of ordinary differential equations

$$\begin{aligned}\frac{\partial}{\partial T}a(k,T)&=i\omega(k)a(k,T)-i\lambda\int_{(\Lambda^*)^2}\mathrm{d}k_1\mathrm{d}k_2\delta(k-k_1-k_2)[8\omega(k)^2 1\omega(k_1)^2\omega(k_2)^2]^{-\frac12}\\&\qquad\times\Big[a(k_1,T)+a^*(-k_1,T)\Big]\Big[a(k_2,T)+a^*(-k_2,T)\Big],\\a(k,0)&=a_0(k)=\frac{1}{\sqrt{2}}\Big[\omega(k)q(k,0)\;+\;\frac{i}{\omega(k)}p(k,0)\Big],\forall(k,T)\in\Lambda^*\times\mathbb{R}_+.\end{aligned}\tag{17}$$

In order to absorb the quantity $i\omega(k)\hat{a}(k,\sigma,T)$ on the right hand side of the above system, we set

$$\alpha(k,T)=a(k,T)e^{-i\omega(k)T}.\tag{18}$$

The following system can be now derived for $\alpha_T(k)$

$$\begin{aligned}\frac{\partial}{\partial T}\alpha(k,T)&=-i\sigma\lambda\sum_{k_1,k_2\in\Lambda^*}\delta(k-k_1-k_2)[8\omega(k)^2\omega(k_1)^2\omega(k_2)^2]^{-\frac12}\times\\&\qquad\times\Big[\alpha(k_1,T)+\alpha^*(-k_1,T)\Big]\Big[\alpha(k_2,T)+\alpha^*(-k_2,T)\Big]e^{-iT(-\omega(k_1)-\omega(k_2)+\omega(k))}.\end{aligned}\tag{19}$$

Consider the two-point correlation function

$$f_{\lambda,D}(k,T)=\langle\alpha_T(k,-1)\alpha_T(k,1)\rangle.\tag{20}$$

In the limit of $D\to\infty,\lambda\to 0$ and $T=\lambda^{-2}t=\mathcal{O}(\lambda^{-2})$, the two-point correlation function $f_{\lambda,D}(k,T)$ has the limit

$$\lim_{\lambda\to 0,D\to\infty}f_{\lambda,D}(k,\lambda^{-2}t)=f(k,t)$$

which solves the 3-wave equation (2), by the standard formal derivation of [51].

Remark 2 As a consequence of the definition (13)–(14), the delta function $\delta(k-k_1-k_2)$ in the collision operator of (2) means that there exists a vector $z\in\mathbb{Z}^d$ such that $k=k_1+k_2+z$.

3 Main results

Let us first normalize the dispersion ω as

$$\omega(k) = \omega_0 + \sum_{j=1}^{3} 2\Big(1 - \cos(2\pi k^j)\Big), \tag{21}$$

where $2 < \omega_0 < 3$, and $k = (k^1, k^2, k^3)$. This will result in an addition factor 4 comparison to the dispersion relation defined in (8), leading to a factor of 4 to the kernel $K(\omega, \omega_1, \omega_2)$. In our proof, we suppose $K(\omega, \omega_1, \omega_2)$ is $[\omega(k)\omega_1(k)\omega_1(k)]^{-1}$ for the sake of simplicity.

For $\infty > m \geq 1$, let $\mathcal{S}$ be a Lebesgue measurable subset of $\mathbb{T}^3$ such that its measure is strictly positive, we introduce the function space $L^m(\mathcal{S})$, defined by the norm

$$\|f\|_{L^m(\mathcal{S})} := \left(\int_{\mathcal{S}} |f(p)|^m \mathrm{d}p\right)^{\frac{1}{m}}. \tag{22}$$

In addition, we also need the space $L^\infty(\mathcal{S})$, defined by the norm

$$\|f\|_{L^\infty(\mathcal{S})} := \mathrm{esssup}_{p\in\mathcal{S}} |f(p)|. \tag{23}$$

We denote by $C^m(\mathcal{S})$, $m = 0, 1, 2, \ldots$, the restrictions of all continuous and m-time differentiable functions on $\mathbb{T}^3$ onto $\mathcal{S}$. The space $C^0(\mathcal{S}) = C(\mathcal{S})$ is endowed with the usual sup-norm (23). In addition, for any normed space $(Y, \|\cdot\|_Y)$, we define

$$C([0,T), Y) := \Big\{F : [0,T) \to Y \;\Big|\, F \text{ is continuous from } [0,T) \text{ to } Y\Big\} \tag{24}$$

and

$$C^1((0,T), Y) := \Big\{F : (0,T) \to Y \;\Big|\, F \text{ is continuous and differentiable from } (0,T) \text{ to } Y\Big\}, \tag{25}$$

for any $T \in (0, \infty]$. The above definitions can also be extended to the spaces $C([0, T], Y)$, $C^1((0, T], Y)$ for any $T \in (0, \infty)$.

Let us state our main theorem.

Theorem 3 *Under the assumption that there exists a positive, classical solution f in $C([0,\infty), C^1(\mathbb{T}^3)) \cap C^1((0,\infty), C^1(\mathbb{T}^3))$ of (2), with the initial condition $f_0 \in C(\mathbb{T}^3)$, $f_0(k) \geq 0$ for all $k \in \mathbb{T}^3$.*

There exist subsets $\mathfrak{V}, \mathfrak{I} \subset \mathbb{T}^3$ such that the torus $\mathbb{T}^3$ can be decomposed into disjoint subsets as follows

$$\mathbb{T}^3 = \mathfrak{I} \cup \bigcup_{x\in\mathfrak{V}} \mathcal{S}(x), \tag{26}$$

where $\mathcal{S}(x) \cap \mathcal{S}(y) = \emptyset$ and $\mathcal{S}(x) \cap \mathfrak{I} = \emptyset$ for $x, y \in \mathfrak{V}$. The set $\mathfrak{I}$ is not empty and is called the "no-collision region". The set $\mathcal{S}(x)$ is called the "collisional-invariant region". The solution f behaves differently on each sub-region.

(I) On $\mathfrak{I}$ the solution stays the same for all time

$$f(t,k) = f_0(k), \qquad \forall t \geq 0, \qquad \forall k \in \mathfrak{I}.$$

(II) *Let x be in $\mathfrak{V}$, suppose that the Lebesgue measure $\mathscr{L}(\mathcal{S}(x))$ of $\mathcal{S}(x)$ is strictly positive, let $E_x \in \mathbb{R}_+$ be a constant and assume further that it is indeed the local energy of the initial condition on $\mathcal{S}(x)$*

$$\int_{\mathcal{S}(x)} f_0(k)\omega(k)\mathrm{d}k = E_x.$$

Suppose that

$$\frac{1}{a_x}\int_{\mathcal{S}(x)} \mathrm{d}k = \frac{\mathscr{L}(\mathcal{S}(x))}{a_x} = E_x, \tag{27}$$

with $a_x \in \mathbb{R}_+$; the local equilibrium on the collision invariant region $\mathcal{S}(x)$ can be uniquely determined as

$$\frac{1}{a_x\omega(k)}. \tag{28}$$

Then, the following limits always holds true

$$\lim_{t\to\infty}\left\| f(t,k) - \frac{1}{a_x\omega(k)}\right\|_{L^1(\mathcal{S}(x))} = 0. \tag{29}$$

and

$$\lim_{t\to\infty}\left|\int_{\mathcal{S}(x)} \ln[f]\mathrm{d}k - \int_{\mathcal{S}(x)} \ln\left[\frac{1}{a_x\omega(k)}\right]\mathrm{d}k\right| = 0. \tag{30}$$

If, in addition, there is a positive constant $M^* > 0$ such that $f(t,k) < M^*$ for all $t \in [0,\infty)$ and for all $k \in \mathcal{S}(x)$, then

$$\lim_{t\to\infty}\left\| f(t,\cdot) - \frac{1}{a_x\omega(k)}\right\|_{L^p(\mathcal{S}(x))} = 0, \qquad \forall p \in [1,\infty). \tag{31}$$

If we assume further that $f_0(k) > 0$ for all $k \in \mathcal{S}(x)$, there exists a constant M_* such that $f(t,k) > M_*$ for all $t \in [0,\infty)$ and for all $k \in \mathcal{S}(x)$.

Remark 4 In the above theorem, we assume the well-posedness of the equation. As this piece of analysis is quite subtle and long, we reserve it for a separate paper.

Remark 5 Notice that, according to our result, the torus $\mathbb{T}^3$ can be decomposed into disjoint subsets as follows

$$\mathbb{T}^3 = \mathfrak{I} \cup \bigcup_{x\in\mathfrak{V}} \mathcal{S}(x), \tag{32}$$

where $\mathcal{S}(x) \cap \mathcal{S}(y) = \emptyset$ and $\mathcal{S}(x) \cap \mathfrak{I} = \emptyset$ for $x, y \in \mathfrak{V}$. However, those disjoint subsets might be topologically disconnected sets.

Remark 6 Since our solutions are assumed to be are regular and non-measured, in (II) of the above theorem, the condition that $\mathscr{L}(\mathcal{S}(x)) > 0$ is essential. When $\mathscr{L}(\mathcal{S}(x)) = 0$, it follows that $\frac{1}{a_x}\int_{\mathcal{S}(x)} \mathrm{d}k = \int_{\mathcal{S}(x)} f_0(k)\omega(k)\mathrm{d}k = 0$, and those cases are negligible due to the assumption on our solutions. The case when $\mathscr{L}(\mathcal{S}(x)) = 0$ is more interesting when the solutions are measures and this problem is being investigated and will be reported in an upcoming work.

The above two theorems assert that those subregions are all non-empty. In the no-collision region $\mathfrak{I}$, any wavevector $k \in \mathfrak{I}$ is totally disconnected to other wavevectors, and thus the solutions on $\mathfrak{I}$ do not change as time evolves. In each of the collisional invariant regions $\mathcal{S}(x)$, as time goes to infinity, the solutions converge in the $L^1(\mathcal{S}(x))$-norm to $\frac{1}{a_x\omega(k)}$. In the classical case, to obtain the convergence, we need more regularity on the solutions: we assume that the solutions are in $C([0,\infty), C^1(\mathbb{T}^3)) \cap C^1((0,\infty), C^1(\mathbb{T}^3))$.

Let us also mention that this asymptotic behavior of the solutions to this 3-wave equations is very different from what is observed in spatially homogeneous and isotropic capillary or acoustic kinetic wave equations. It is showed in [50] that if one looks for a solution whose energy is a constant for all time to one of these isotropic capillary/acoustic kinetic wave equations, then this solution can exist only up to a finite time, after this time, some energy is lost to infinity. In other words, the solution exhibits the so-called energy cascade phenomenon.

4 The analysis of the 3-wave kinetic equation

In our proof, as discussed above, we suppose $K(\omega, \omega_1, \omega_2)$ is $[\omega(k)\omega_1(k)\omega_1(k)]^{-1}$ for the sake of simplicity. We assume, when needed that the Lebesgue measure of each collisional region $\mathscr{L}(\mathcal{S}(x))$ is strictly positive due to Remark 6.

4.1 No-collision, collisional regions and the 3-wave kinetic operator on these local disjoint sets

In this section x, y, z are now used for frequency vectors, in contrast to the previous discussion in equation ((1)) and the previous sections. Therefore, that the notations will now be unlinked from what they were in prior sections.

4.1.1 Collisional invariant regions

For a vector $x = (x^1, x^2, x^3) \in \mathbb{T}^3$, we say that the wave vector x is connected to the wave vector $y = (y^1, y^2, y^3) \in \mathbb{T}^3$ by a **forward collision** if and only if

$$\mathfrak{F}_x^f(y) := \sum_{j=1}^{3} 2[\cos(2\pi(y_j - x_j)) + \cos(2\pi x_j) - \cos(2\pi y_j)] - 6 - \omega_0 = 0. \quad (33)$$

In a forward collision, a particle with wave vector $y - x$ merges with a particle with wave vector x, resulting in a new particle with wave vector y. Following Remark 2, we could see that $y - x$ does not need to belong to $\mathbb{T}^d$. Indeed, there exists a vector $z \in \mathbb{Z}^d$ such that $y - x - z \in \mathbb{T}^d$. In this collision, the conservation of energy $\omega(y) = \omega(x) + \omega(y - x)$, describing by equation (33), needs to be satisfied. Therefore, given a particle with wave vector x, there maybe no wave vector y such that the conservation of energy is guaranteed. In other words, there may be no y such that x is connected to y by a forward collision.

On the other hand, we say that the wave vector x is connected to the wave vector $y = (y^1, y^2, y^3) \in \mathbb{T}^3$ by a **backward collision** if and only if

$$\mathfrak{F}_x^b(y) := \sum_{j=1}^{3} 2[\cos(2\pi y_j) + \cos(2\pi(x_j - y_j)) - \cos(2\pi x_j)] - 6 - \omega_0 = 0. \quad (34)$$

Different from forward collisions, in a backward collision, a particle with wave vector x is broken into two particles, one with wave vector y, and the other one with wave vector $x-y$. Again, in a backward collision, the conservation of energy $\omega(x)=\omega(y)+\omega(x-y)$ needs to be satisfied; and therefore, for a given wave vector x, it could happen that one cannot break x into y and $x-y$, such that the energy conservation (34) is satisfied. Again, following Remark 2, we could see that $x-y$ does not need to belong to $\mathbb{T}^d$. Indeed, there exists a vector $z\in\mathbb{Z}^d$ such that $x-y-z\in\mathbb{T}^d$.

Finally, we say that the wave vector x is connected to the wave vector y or the wave vector y is connected to the wave vector x by a **central collision** if and only if

$$\mathfrak{F}^c_x(y)=\mathfrak{F}^c_y(x):=\sum_{j=1}^3 2[\cos(2\pi y_j)+\cos(2\pi(x_j))-\cos(2\pi(x_j+y_j))]-6-\omega_0=0. \tag{35}$$

Similarly to the above types of collisions, in a central collision, we require that $\omega(x)+\omega(y)=\omega(x+y)$ and this conservation of energy is not always satisfied. Following Remark 2, we could see that $y+x$ does not need to belong to $\mathbb{T}^d$. Indeed, there exists a vector $z\in\mathbb{Z}^d$ such that $y+x-z\in\mathbb{T}^d$.

Note that if y is connected to x by a forward collision, then x is connected to y by a backward collision. Moreover, if y is connected to x by a central collision, then x is connected to y by a central collision and $x+y$ is connected to both x and y by backward collisions. We simply say that x and y are connected by one collision; or x is connected to y and y is connected to x by one collision.

If a wave vector k is not connected to any other wave vectors in forward collisions, the second term in the collision operator $Q_c[f](k)$

$$\int_{\mathbb{T}^6}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_2f-ff_1-f_1f_2]\mathrm{d}k_1\mathrm{d}k_2$$

vanishes, no matter how we choose the function f.

If a wave vector k is not connected to any other wave vectors in backward collisions, the first term in the collision operator $Q_c[f](k)$

$$\int_{\mathbb{T}^6}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[f_1f_2-ff_1-ff_2]\mathrm{d}k_1\mathrm{d}k_2$$

vanishes.

We define the set of all wave vectors k such that k is not connected to any other wave vectors to be **the no-collision region** $\mathfrak{I}$. It is clear that $\mathfrak{F}^f_0(y)=\mathfrak{F}^c_0(y)=-\omega_0<0$ and

$$\mathfrak{F}^b_0(y)=\sum_{j=1}^3 2[2\cos(2\pi y_j)-1]-6-\omega_0=\sum_{j=1}^3 2[2\cos(2\pi y_j)-2]-\omega_0\le-\omega_0<0,$$

for all wave vectors y. As a consequence, the origin belongs to $\mathfrak{I}$. Since $\mathfrak{F}^f_0(y),\mathfrak{F}^b_0(y),\mathfrak{F}^c_0(y)\le-\omega_0<0$, there exists a ball $B(0,R):=\{x\in\mathbb{R}^3 \mid |x|<R\}$, $(R>0)$, such that $\mathfrak{F}^f_x(y),\mathfrak{F}^b_x(y),\mathfrak{F}^c_x(y)<0$, for all $y\in\mathbb{T}^3$ and for all $x\in B(0,R)$. The ball $B(0,R)$ is therefore a subset of the no-collision region $\mathfrak{I}$.

The condition $2<\omega_0<3$ implies that the set $\mathbb{T}^3\backslash\mathfrak{I}$ is then not empty. For a vector $x\in\mathbb{T}^3\backslash\mathfrak{I}$, we define $\mathcal{S}^1(x)$ to be the **one-collision connection set of** x, containing all wave vectors $y\in\mathbb{T}^3$ such that y is connected to x by a collision. By a recursive manner, we also define $\mathcal{S}^n(x) = \mathcal{S}^1(\mathcal{S}^{n-1}(x))$, the **$n$-collision connection set of** x, for $n \geq 2, n \in \mathbb{N}$. This set consists of all wave vectors connecting to x by at most n collisions. The union

$$\mathcal{S}(x) = \bigcup_{1 \leq n < \infty} \mathcal{S}^n(x) \tag{36}$$

contains all wave vectors y connecting to x by a finite number of collisions. We then call $\mathcal{S}(x)$ a **finite collision connection set of** x or a **collision invariant region**.

Note that if $k \in \mathcal{S}(x)$ and k is connected to $k + k' \in \mathcal{S}(x)$ by a forward collision, then $k + k'$ is also connected with k' by a backward collision, and hence $k' \in \mathcal{S}(x)$.

Proposition 7 *(The effect of the collision operator on the no-collision region) Any smooth solution $f(t,k)$ of (2), is time invariant on the no-collision region $\mathfrak{I}$. In other words, $f(t,k) = f_0(k)$ for all $k \in \mathfrak{I}$.*

Proof Since $k \in \mathfrak{I}$, the wave vector k is not connected to any other wave vectors in any collisions, the collision operator $Q_c[f](k)$ vanishes, which implies $\partial_t f(t,k) = 0$ for all $k \in \mathfrak{I}$. Therefore, $f(t,k) = f_0(k)$ for all $k \in \mathfrak{I}$. □

Proposition 8 *(Decomposition into collisional invariant regions) Let x, y be two wave vectors in $\mathbb{T}^3 \backslash \mathfrak{I}$, then either $\mathcal{S}(x) = \mathcal{S}(y)$ or $\mathcal{S}(x) \cap \mathcal{S}(y) = \emptyset$. In other words, either x and y are connected by a finite number of collisions ($\exists m > 0$ such that $x \in \mathcal{S}^m(y)$) or they are totally disconnected ($\nexists m > 0$ such that $x \in \mathcal{S}^m(y)$).*

As a consequence, there exists a subset $\mathfrak{V}$ of $\mathbb{T}^3 \backslash \mathfrak{I}$ such that the torus $\mathbb{T}^3$ can be decomposed into disjoint collisional invariant regions, as follows

$$\mathbb{T}^3 \backslash \mathfrak{I} = \bigcup_{x \in \mathfrak{V}} \mathcal{S}(x), \tag{37}$$

and $\mathcal{S}(x) \cap \mathcal{S}(y) = \emptyset$ for $x, y \in \mathfrak{V}$.

Proof Let x, y be two wave vectors in $\mathbb{T}^3 \backslash \mathfrak{I}$ and suppose that $\mathcal{S}(x) \cap \mathcal{S}(y) \neq \emptyset$, we can therefore choose a wave vector z belonging to both sets $\mathcal{S}(x)$ and $\mathcal{S}(y)$, that means z is connected to both wave vectors x and y by finite numbers of collisions. It follows that $z \in \mathcal{S}^n(x)$ and $z \in \mathcal{S}^m(y)$, for some positive integers n and m. Since $z \in \mathcal{S}^n(x)$, it is clear that $\mathcal{S}(z) \subset \mathcal{S}^{n+1}(x)$, and in general $\mathcal{S}^p(z) \subset \mathcal{S}^{n+p}(x)$ for all $p \in \mathbb{N}$. As a result, $\mathcal{S}(z) \subset \mathcal{S}(x)$. By a similar argument, it also follows that $\mathcal{S}(z) \subset \mathcal{S}(y)$. Now, let ϑ be an wave vector of $\mathcal{S}(y) \backslash \mathcal{S}(z)$. Being a wave vector of $\mathcal{S}(y)$, ϑ is connected to y by a finite number $p \in \mathbb{N}$ of collisions. Since z is connected to y by m collisions, ϑ is connected to z by at most $p + m$ collisions. In other words, $\vartheta \in \mathcal{S}^{p+m}(z)$; and hence, $\vartheta \in \mathcal{S}(z)$, contradicting the fact that $\vartheta \in \mathcal{S}(y) \backslash \mathcal{S}(z)$. This contradiction leads to $\mathcal{S}(y) \subset \mathcal{S}(z)$; however, as shown above $\mathcal{S}(z) \subset \mathcal{S}(y)$, it then follows $\mathcal{S}(y) = \mathcal{S}(z)$. The same argument can also be used to prove $\mathcal{S}(x) = \mathcal{S}(z)$. We finally get $\mathcal{S}(y) = \mathcal{S}(x)$.

The existence of $\mathfrak{V}$ and the decomposition (37) then follows straightforwardly. □

Remark 9 The decomposition of the domain $\mathbb{T}^3$ in to several collisional invariant and no-collision regions is a very special and interesting feature of the specific form of the dispersion relation (21).

In the previous works, several other dispersion relations have been considered in many other contexts $\omega(k) = |k|$ for very low temperature bosons (see [1, 15]), $\omega(k) = |k|^\gamma$, $(1 < \gamma \leq 2)$ for capillary waves (see [41]), $\omega(k) = \sqrt{c_1|k|^2 + c_2|k|^4}$, $(0 < c_1, 0 \leq c_2)$ for bosons (see [47, 49]) and the space of the frequency k is $\mathbb{R}^d$. In all of these cases, the division of the domain of wavenumbers into disjoint regions has never been observed due to the fact that the frequency space is $\mathbb{R}^d$ instead of $\mathbb{T}^d$. On the other hand, important results on 4-wave kinetic equations set the torus $\mathbb{T}^d$ have been recently obtained in [18, 22, 35].

Notice that in [20], the dispersion relation $\omega(k) = \sqrt{c_1 + c_2|k|^2}$, $(0 < c_1, c_2)$ for stratified flows in the ocean, has been considered. However, the resonance is broadened and the extended resonance manifold is then studied

$$k = k_1 + k_2, \qquad |\omega(k) - \omega(k_1) - \omega(k_2)| \le \theta, \qquad k, k_1, k_2 \in \mathbb{R}^2,$$

for $\theta > 0$, in stead of the exact resonance one

$$k = k_1 + k_2, \qquad \omega(k) = \omega(k_1) + \omega(k_2), \qquad k, k_1, k_2 \in \mathbb{R}^3,$$

due to the fact that the exact resonance configuration is no longer correct (see [48]). Of course, in all resonance broadening cases, the decomposition of the full domain into local no-collision and collisional invariant regions does not exist.

Proposition 10 *The set $\mathcal{S}^n(x)$ is a closed subset of $\mathbb{T}^3$ for all $n \in \mathbb{N}\backslash\{0\}$.*

Proof We first observe that the set $\mathcal{S}^1(x)$ contains all wave vectors y such that x is connected to y by either a forward, a backward or a central collision. By definition, the set of all y such that x is connected to y by a forward collision is

$$\mathcal{S}^1_f(x) = \left[\mathfrak{F}^f_x\right]^{-1}(\{0\}). \tag{38}$$

Similarly, the sets of all y such that x is connected to y by backward and central collisions are

$$\mathcal{S}^1_b(x) = \left[\mathfrak{F}^b_x\right]^{-1}(\{0\}), \tag{39}$$

and

$$\mathcal{S}^1_c(x) = \left[\mathfrak{F}^c_x\right]^{-1}(\{0\}). \tag{40}$$

By the continuity of $\mathfrak{F}^f_x$, $\mathfrak{F}^b_x$ and $\mathfrak{F}^c_x$, the sets $\mathcal{S}^1_f(x)$, $\mathcal{S}^1_b(x)$ and $\mathcal{S}^1_c(x)$ are all closed. Since $\mathcal{S}^1(x) = \mathcal{S}^1_f(x) \cup \mathcal{S}^1_b(x) \cup \mathcal{S}^1_c(x)$, it is also a closed set.

We now follow an induction argument in n. When $n = 1$, it is clear from the above argument that $\mathcal{S}^1(x)$ is closed. Suppose that $\mathcal{S}^k(x)$ is closed, we will show that $\mathcal{S}^{k+1}(x)$ is also closed for all $k \ge 1$. To this end, let us suppose that $\{x_m\}_{m=1}^\infty$ is a sequence in $\mathcal{S}^{k+1}(x)$ and $\lim_{m\to\infty} x_m = x_*$. By the definition of the set $\mathcal{S}^{k+1}(x)$, there exists a sequence $\{y_m\}_{m=1}^\infty$ such that $y_m \in \mathcal{S}^k(x)$ and either $\mathfrak{F}^f_{y_m}(x_m) = 0$, $\mathfrak{F}^b_{y_m}(x_m) = 0$ or $\mathfrak{F}^c_{y_m}(x_m) = 0$. Without loss of generality, we can assume that there exist subsequences $\{x_{m_q}\}_{q=1}^\infty$ and $\{y_{m_q}\}_{q=1}^\infty$ of $\{x_m\}_{m=1}^\infty$ and $\{y_m\}_{m=1}^\infty$ such that $\mathfrak{F}^f_{y_{m_q}}(x_{m_q}) = 0$. Since the sequence $\{y_{m_q}\}_{q=1}^\infty$ is a subset of $\mathcal{S}^k(x)$, which is closed and hence compact, there exists a subset of $\{y_{m_q}\}_{q=1}^\infty$, still denoted by $\{y_{m_q}\}_{q=1}^\infty$, such that this sequence has a limit $y_* \in \mathcal{S}^k(x)$ as q tends to infinity. By the continuity of $\mathfrak{F}^f_y(x)$ in both x and y, $\lim_{q\to\infty} \mathfrak{F}^f y_{m_q}(x_{m_q}) = \mathfrak{F}^f_{y_*}(x_*)$. That implies $\mathfrak{F}^f_{y_*}(x_*) = 0$ and hence $x_* \in \mathcal{S}^{k+1}(x)$. We finally conclude that the set $\mathcal{S}^{k+1}(x)$ is closed. By induction $\mathcal{S}^n(x)$ is closed for all $n \in \mathbb{N}\backslash\{0\}$. □

Corollary 11 *The set $\mathcal{S}(x)$ is Lebesgue measurable.*

Proof The proof of this corollary follows directly from Proposition 10 and the definition of $\mathcal{S}(x)$. □

Remark 12 The two sets $\mathcal{S}_f^1(x)$ and $\mathcal{S}_b^1(x)$ defined in (38) and (39) are indeed disjoint. This can be seen by a proof of contradiction. Suppose that y is a common wave vector of both $\mathcal{S}_f^1(x)$ and $\mathcal{S}_b^1(x)$. This means

$$\sum_{i=1}^{3} 2[\cos(2\pi(y_i - x_i)) + \cos(2\pi x_i) - \cos(2\pi y_i)] = 6 + \omega_0,$$

and

$$\sum_{i=1}^{3} 2[\cos(2\pi(x_i - y_i)) + \cos(2\pi y_i) - \cos(2\pi x_i)] = 6 + \omega_0.$$

Taking the sum of the above two identities yields

$$\sum_{i=1}^{3} 2\cos(2\pi(y_i - x_i)) = 6 + \omega_0.$$

The left hand side is smaller than or equal to 6, while the right hand side is strictly greater than 6 due to the fact that $\omega_0 > 0$. This leads to a contradiction; and thus, $\mathcal{S}_f^1(x)$ and $\mathcal{S}_b^1(x)$ are disjoint. However, $\mathcal{S}_c^1(x)$ can have common wave vectors with both $\mathcal{S}_f^1(x)$ and $\mathcal{S}_b^1(x)$.

4.1.2 Continuity of set index functionals

In the study of the wave kinetic equation, we frequently encounter integrals of the types

$$\int_{\mathbb{T}^3} \delta(\omega(x) - \omega(x-y) - \omega(y)) f(y) \mathrm{d}y, \tag{41}$$

$$\int_{\mathbb{T}^3} \delta(\omega(y) - \omega(y-x) - \omega(x)) f(y) \mathrm{d}y, \tag{42}$$

and

$$\int_{\mathbb{T}^3} \delta(\omega(x+y) - \omega(x) - \omega(y)) f(y) \mathrm{d}y. \tag{43}$$

Special cases of (41)–(42)–(43) involve $f(y) = \chi_A(y)$, the characteristic function of a Lebesgue measurable set A.

Definition 1 (Index functionals of sets) Let A be a Lebesgue measurable set, we define the following three functionals.

(I) The "forward collision" index of the set A:

$$\mu_1[A](x) := \int_{\mathbb{R}} \int_{\mathbb{T}^3} e^{it(\omega(x)-\omega(x-y)-\omega(y))} \chi_A(y) \mathrm{d}y \mathrm{d}t, \tag{44}$$

where χ_A is the characteristic function of the set A.

(II) The "backward collision" index of the set A:

$$\mu_2[A](x) := \int_{\mathbb{R}} \int_{\mathbb{T}^3} e^{it(\omega(y)-\omega(y-x)-\omega(x))} \chi_A(y) \mathrm{d}y \mathrm{d}t, \tag{45}$$

where χ_A is the characteristic function of the set A.

(III) The "central collision" index of the set A:

$$\mu_3[A](x) := \int_{\mathbb{R}}\int_{\mathbb{T}^3} e^{it(\omega(x+y)-\omega(x)-\omega(y))}\chi_A(y)\mathrm{d}y\mathrm{d}t, \tag{46}$$

where χ_A is the characteristic function of the set A.

For the sake of simplicity, in this section, we denote $\mu_1(\mathbb{T}^3)$, $\mu_2(\mathbb{T}^3)$ and $\mu_3(\mathbb{T}^3)$ by $F(x)$, $G(x)$ and $H(x)$.

Proposition 13 *The functions $F(x)$, $G(x)$ and $H(x)$ are continuous on the set*

$$\mathfrak{S} = \left\{x = (x^1, x^2, x^3) \in \mathbb{T}^3 \text{ in which } x^i \neq \pm\frac{1}{2}, 0, \text{ for all } i = 1, 2, 3\right\}.$$

Proof Notice that

$$\begin{aligned}\omega(x) - \omega(x-y) - \omega(y) \ &= -\omega_0 - 6 + \sum_{i=1}^{3} 2\Big[\cos(2\pi x^i - 2\pi y^i) + \cos(2\pi y^i) \\ &\quad - \cos(2\pi x^i)\Big],\end{aligned} \tag{47}$$

where $x = (x^1, x^2, x^3)$, $y = (y^1, y^2, y^3)$.
We will need to bound

$$\begin{aligned}\mathcal{J} &= \int_{\mathbb{T}^3} e^{it(\sum_{i=1}^3 2[\cos(2\pi x^i - 2\pi y^i)+\cos(2\pi y^i)])}\mathrm{d}y \\ &= \int_{\mathbb{T}} e^{it2[\cos(2\pi x^1-2\pi y^1)+\cos(2\pi y^1)]}\mathrm{d}y^1 \int_{\mathbb{T}} e^{it2[\cos(2\pi x^2-2\pi y^2)+\cos(2\pi y^2)]}\mathrm{d}y^2 \times \\ &\quad \times \int_{\mathbb{T}} e^{it2[\cos(2\pi x^3-2\pi y^3)+\cos(2\pi y^3)]}\mathrm{d}y^3 \\ &= \mathcal{J}_1 \times \mathcal{J}_2 \times \mathcal{J}_3\end{aligned} \tag{48}$$

which is a product of three oscillation integrals with phases $t\Phi_i(y)$, where $\Phi_i(y) = 2[\cos(2\pi x^i - 2\pi y^i) + \cos(2\pi y^i)]$, $i = 1, 2, 3$.

To estimate (48), we will use the method of stationary phase. Let us point out that in [21], the authors use different kinds of techniques, to estimate integrals of similar types but for different classes of dispersion relations. Notice that $\partial_{y^i}\Phi_i(y^i) = -4\pi\sin(2\pi y^i - 2\pi x^i) - 4\pi\sin(2\pi y^i) = 0$ when $y^i = \frac{x^i}{2}$, $y^i = \frac{1}{2} + \frac{x^i}{2}$, or $x^i = \pm\frac{1}{2}$. Observe that when $y^i = \frac{x^i}{2}$, $y^i = \frac{1}{2} + \frac{x^i}{2}$, we have $|\partial_{y^i y^i}\Phi_i(y^i)| = 8\pi^2|\cos(2\pi y^i - 2\pi x^i) + \cos(2\pi y^i)| = 16\pi^2|\cos(\pi x^i)| = 8\pi^2|1 + e^{i2\pi x^i}|$.

We observe that all x^i, $i = 1, 2, 3$, need to be different from $\pm\frac{1}{2}$. This fact could be seen by a proof of contradiction, in which we suppose that x^1 is equal to $\frac{1}{2}$ or $-\frac{1}{2}$ as follows. By Proposition 10, $\mathcal{S}(x)$ is non-empty, then either

$$\begin{aligned}0 \ &= \omega(x) - \omega(x-y) - \omega(y) \\ &= -\omega_0 - 6 + \sum_{i=1}^{3} 2[\cos(2\pi x^i - 2\pi y^i) + \cos(2\pi y^i) - \cos(2\pi x^i)], \\ 0 \ &= \omega(x+y) - \omega(x) - \omega(y) \\ &= -\omega_0 - 6 + \sum_{i=1}^{3} 2[\cos(2\pi x^i) + \cos(2\pi y^i) - \cos(2\pi x^i + 2\pi y^i)],\end{aligned}$$

or

$$\begin{aligned}0 &= \omega(y) - \omega(x) - \omega(y-x)\\ &= -\omega_0 - 6 + \sum_{i=1}^{3} 2[\cos(2\pi x^i) + \cos(2\pi y^i - 2\pi x^i) - \cos(2\pi y^i)],\end{aligned}$$

has to have a solution. Let us consider the first equation. Plugging the values $\pm\frac{1}{2}$ of x^1 into the equation yields

$$\omega_0 + 4 = \sum_{i=2}^{3} 2[\cos(2\pi x^i - 2\pi y^i) + \cos(2\pi y^i) - \cos(2\pi x^i)],$$

which has no solutions since $\omega_0 + 4 > 6$ and $[\cos(2\pi\alpha - 2\pi\beta) + \cos(2\pi\beta) - \cos(2\pi\alpha)] \leq \frac{3}{2}$ for all $\alpha, \beta \in \mathbb{T}$. Now, we consider the second equation, and plug the values $\pm\frac{1}{2}$ of x^1 into the equation to get

$$\omega_0 + 8 - 4\cos(2\pi y^1) = \sum_{i=2}^{3} 2[\cos(2\pi x^i) + \cos(2\pi y^i) - \cos(2\pi x^i + 2\pi y^i)],$$

which also has no solution since $\omega_0 + 8 - 4\cos(2\pi y^1) > 6$ and $[\cos(2\pi\alpha) + \cos(2\pi\beta) - \cos(2\pi\alpha + 2\pi\beta)] \leq \frac{3}{2}$ for all $\alpha, \beta \in \mathbb{T}$. Finally, in the last case, the same argument gives

$$\omega_0 + 8 + 4\cos(2\pi y^1) = \sum_{i=2}^{3} 2[\cos(2\pi x^i) + \cos(2\pi y^i - 2\pi x^i) - \cos(2\pi y^i)],$$

which again has no solution.

Since x^i is different from $\pm\frac{1}{2}$, it is clear that $\partial_{y^i}\Phi_i(y^i) = -4\pi\sin(2\pi y^i - 2\pi x^i) - 4\pi\sin(2\pi y^i) = 0$ when $y^i = \frac{x^i}{2}$ and $y^i = \frac{1}{2} + \frac{x^i}{2}$. By the method of stationary phase

$$\mathcal{J}_i \lesssim \frac{1}{\langle t\rangle^{\frac{1}{2}}\sqrt{|1 + e^{i2\pi x^i}|}}, \tag{49}$$

when x^i is different from $\pm\frac{1}{2}$.

Multiplying all inequalities (49) for $i = 1, 2, 3$ yields

$$\mathcal{J} \lesssim \frac{1}{\langle t\rangle^{\frac{3}{2}}\sqrt{|1 + e^{i2\pi x^1}||1 + e^{i2\pi x^2}||1 + e^{i2\pi x^3}|}}. \tag{50}$$

Let x be a point in $\mathfrak{S}$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathfrak{S}$ such that $\lim_{n\to\infty} x_n = x$. Since the set $\mathbb{T}^3 \backslash \mathfrak{S}$ is closed, without loss of generality, we suppose that there exists a ball $B(x, r)$ with radius r and centered at x such that $B(x, r) \cap (\mathbb{T}^3 \backslash \mathfrak{S}) = \emptyset$ and then $\{x_n\}_{n=1}^{\infty} \subset B(x, r)$. From the assumption $B(x, r) \cap (\mathbb{T}^3 \backslash \mathfrak{S}) = \emptyset$, it follows

$$\left|\int_{\mathbb{T}^3} e^{it(\omega(x)-\omega(x-y)-\omega(y))}\mathrm{d}y\right| \lesssim \frac{1}{\langle t\rangle^{\frac{3}{2}}\sqrt{|1 + e^{2\pi x^1}||1 + e^{2\pi x^2}||1 + e^{2\pi x^3}|}} \lesssim 1. \tag{51}$$

By the Lebesgue dominated convergence theorem, $\lim_{n\to\infty} F(x_n) = F(x)$ and the function F is then continuous on $\mathfrak{S}$. By the same argument, G, H are also continuous. □

Corollary 14 *The edges, i.e. the set $\mathbb{T}^3\backslash\mathfrak{S}$ of all wave vectors $y = (y^1, y^2, y^3)$ in which there is an index $i \in \{1, 2, 3\}$ such that $y^i = \pm\frac{1}{2}$ or 0, is a subset of the no-collision region $\mathfrak{I}$.*

Proof The corollary follows directly from the proof of Proposition 13. □

4.1.3 Restrictions on $\mathcal{S}(x)$

Proposition 15 *Given any function $f \in \mathbb{L}^1(\mathbb{T}^3)$ and a collisional invariant region $\mathcal{S}(x)$. Define restriction of f on $\mathcal{S}(x)$ as follows*

$$f_{|\mathcal{S}(x)}(y) = f(y) \text{ if } y \in \mathcal{S}(x) \text{ and } f_{|\mathcal{S}(x)}(y) = 0 \text{ if } y \in \mathbb{T}^3\backslash\mathcal{S}(x). \tag{52}$$

Then, in the distributional sense, we have

$$\int_{\mathbb{T}^3} \delta(\omega(x) - \omega(x-y) - \omega(y)) f(y)\mathrm{d}y = \int_{\mathbb{T}^3} \delta(\omega(x) - \omega(x-y) - \omega(y)) f_{|\mathcal{S}(x)}(y)\mathrm{d}y, \tag{53}$$

$$\int_{\mathbb{T}^3} \delta(\omega(y) - \omega(y-x) - \omega(x)) f(y)\mathrm{d}y = \int_{\mathbb{T}^3} \delta(\omega(y) - \omega(y-x) - \omega(x)) f_{|\mathcal{S}(x)}(y)\mathrm{d}y, \tag{54}$$

and

$$\int_{\mathbb{T}^3} \delta(\omega(x+y) - \omega(x) - \omega(y)) f(y)\mathrm{d}y = \int_{\mathbb{T}^3} \delta(\omega(x+y) - \omega(x) - \omega(y)) f_{|\mathcal{S}(x)}(y)\mathrm{d}y. \tag{55}$$

Proof We only prove (53), as the proofs of (54)–(55) follow by the same argument. For a fixed value of x, we denote by A_θ with $\theta > 0$ the set of all z in A such that

$$|\omega(x) - \omega(z) - \omega(x-z)| > \theta > 0 \tag{56}$$

for all z in A.

Let us introduce the following approximation

$$\int_{\mathbb{R}} \int_{\mathbb{T}^3} e^{it(\omega(x)-\omega(x-y)-\omega(y))-\epsilon^2t^2} \chi_{A_\theta}(y) f(t)\mathrm{d}y\mathrm{d}t. \tag{57}$$

Integrating in t, we obtain from (57)

$$\frac{C}{\epsilon} \int_{\mathbb{T}^3} e^{-\frac{\pi(\omega(x)-\omega(x-y)-\omega(y)^2}{\epsilon^2}} \chi_{A_\theta}(y) f(y)\mathrm{d}y, \tag{58}$$

for some universal positive constant C.

Combining (56) with the approximation (57), we find

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{T}^3} e^{it(\omega(x)-\omega(x-z)-\omega(z))-\epsilon^2t^2} \chi_{A_\theta}(z) f(z)\mathrm{d}y\mathrm{d}t \quad &= \frac{C}{\epsilon} \int_{\mathbb{T}^3} e^{-\frac{\pi(\omega(x)-\omega(x-z)-\omega(z))^2}{\epsilon^2}} \chi_{A_\theta}(z) f(z)\mathrm{d}z \\ &\lesssim \frac{1}{\epsilon} \int_{\mathbb{T}^3} e^{-\frac{\pi\theta^2}{\epsilon^2}} \chi_{A_\theta}(z) f(z)\mathrm{d}z.\end{aligned}$$

Using the fact that χ_{A_θ} is a subset of $\mathbb{T}^3$, we deduce

$$\int_{\mathbb{R}} \int_{\mathbb{T}^3} e^{it(\omega(x)-\omega(x-z)-\omega(z))-\epsilon^2t^2} \chi_{A_\theta}(z) f(z)\mathrm{d}z\mathrm{d}t \lesssim \frac{e^{-\frac{\pi\theta^2}{\epsilon^2}}}{\epsilon} \to 0 \text{ as } \epsilon \to 0. \tag{59}$$

Let $\varphi(x)$ be a test function in $C^\infty(\mathbb{T}^d)$. Again, the same stationary phase argument used in Proposition 13 can be applied to show that

$$\Big|\int_{\mathbb{R}}\int_{\mathbb{T}^3} e^{it(\omega(x)-\omega(x-z)-\omega(z))-\epsilon^2t^2}\varphi(x)\mathrm{d}z\mathrm{d}t\Big| \lesssim 1, \tag{60}$$

uniformly in ϵ. By the Lebesgue dominated convergence theorem, we find

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{T}^6} e^{it(\omega(x)-\omega(x-z)-\omega(z))}\chi_A(z)\varphi(x)\mathrm{d}z\mathrm{d}x\mathrm{d}t\\ &\quad=\lim_{\theta\to 0}\lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{T}^6} e^{it(\omega(x)-\omega(x-z)-\omega(z))-\epsilon^2t^2}\chi_{A_\theta}(z)f(z)\varphi(x)\mathrm{d}z\mathrm{d}x\mathrm{d}t=0.\end{aligned} \tag{61}$$

□

4.1.4 Weak formulation, local conservation of energy on collisional invariant regions

Lemma 16 *For any smooth function $f(k)$, there holds*

$$\begin{aligned}\int_{\mathbb{T}^3} Q_c[f](k)\varphi(k)\mathrm{d}k &= \iiint_{\mathbb{T}^9}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\times\\ &\quad\times[f_1f_2-ff_1-ff_2]\Big(\varphi(k)-\varphi(k_1)-\varphi(k_2)\Big)\,\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\end{aligned}$$

for any smooth test function φ.

If φ is supported in a collisional invariant region $\mathcal{S}(x)$, then, we also have

$$\begin{aligned}\int_{\mathbb{T}^3} Q_c[f](k)\varphi(k)\mathrm{d}k &= \iiint_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\\ &\quad\times[f_1f_2-ff_1-ff_2]\Big(\varphi(k)-\varphi(k_1)-\varphi(k_2)\Big)\,\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.\end{aligned}$$

Proof We have

$$\begin{aligned}&\int_{\mathbb{T}^3} Q[f](k)\varphi(k)\mathrm{d}k\\ &\quad=\int_{\mathbb{T}^9}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[f_1f_2-ff_1-ff_2]\varphi(k)\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\ &\quad-\int_{\mathbb{T}^9}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_2f-ff_1-f_1f_2]\varphi(k)\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\ &\quad-\int_{\mathbb{T}^9}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_2f-ff_1-f_1f_2]\varphi(k)\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2,\end{aligned}$$

by switching the variables $k\leftrightarrow k_1$ and $k\leftrightarrow k_2$ in the second and third integrals, respectively, the first identity follows. The second identity follows straightforwardly from Corollary 15 and the first identity. □

As a consequence, we obtain the following corollary.

Corollary 17 *(Conservation of energy on collisional invariant regions) Smooth solutions $f(t,k)$ of (2), with initial data $f(0,k)=f_0(k)$, satisfy*

$$\int_{\mathcal{S}(x)} f(t,k)\omega(k)\mathrm{d}k=\int_{\mathcal{S}(x)} f_0(k)\omega(k)\mathrm{d}k. \tag{62}$$

for all $t\geq 0$ and for all $x\in\mathfrak{V}$, defined in Proposition 8.

Proof This follows from Lemma 16 by taking $\varphi(k)=\omega(k)$ with $k=(k^1,k^2,k^3)$. □

4.1.5 Local equilibria on collisional invariant regions

In this section, we establish the form of local equilibria on collisional invariant regions. The key different between these local equilibria and the equilibria of classical kinetic equations is that these equilibria are only defined locally on collisional invariant regions. This is a very special feature of the 3-wave kinetic equation.

Lemma 18 (C^2-collisional invariants) *Let $\psi \in C^2(\mathcal{S}(x))$ be a collisional invariant on the collisional invariant region $\mathcal{S}(x)$, in the following sense. For any wave vectors $k, k_1, k_2 \in \mathcal{S}(x)$,*

$$k = k_1 + k_2 + z, \text{ for some } z \in \mathbb{Z}^d, \qquad \omega(k) = \omega(k_1) + \omega(k_2),$$

we have

$$\psi(k) = \psi(k_1) + \psi(k_2).$$

Then there exist a constant $a_x \in \mathbb{R}$, such that

$$\psi(k) = a_x \omega(k).$$

Proof Let us first prove that for $k \in \mathcal{S}(x)$, the partial derivatives $\partial_{k^j}\psi(k)$, with $k = (k^1, k^2, k^3)$, are well-defined. Without loss of generality, we only prove that the partial derivative with respect to the first component $\partial_{k^1}\psi(k)$ is well-defined. Since $k \in \mathcal{S}(x)$, there are two wave vectors k_1, k_2 such that either $k = k_1 + k_2$ and $\omega(k) = \omega(k_1) + \omega(k_2)$; or $k + k_1 = k_2$ and $\omega(k) + \omega(k_1) = \omega(k_2)$.

Case 1: $k = k_1 + k_2$ **and** $\omega(k) = \omega(k_1) + \omega(k_2)$**.** Since $\psi \in C^2(\mathbb{T}^3)$, in order to show that $\partial_{k^1}\psi(k)$ is well-defined at $k^1 \in \mathbb{T}$, we only have to prove that there exists $\epsilon > 0$ such that for each $\bar{k}^1 \in (k^1 - \epsilon, k^1 + \epsilon)$ there are $\bar{k}^2, \bar{k}^3 \in \mathbb{T}^3$, $\bar{k} = (\bar{k}^1, \bar{k}^2, \bar{k}^3) \in \mathcal{S}(x)$. For any $x, y \in \mathbb{T}$, define

$$F(x, y) = \cos(2\pi(x + y)) - \cos(2\pi x) - \cos(2\pi y).$$

Since $k = (k^1, k^2, k^3) = k_1 + k_2 = (k_1^1, k_1^2, k_1^3) + (k_2^1, k_2^2, k_2^3)$, we then have

$$F(k_1^1, k_2^1) + F(k_1^2, k_2^2) + F(k_1^3, k_2^3) = -\omega_0/2 - 3.$$

Now, we develop

$$\begin{aligned} F(x, y) + 1 =& -\cos(2\pi x) - \cos(2\pi y) + 1 + \cos(2\pi(x + y)) \\ =& 2\cos(\pi(x + y))[-\cos(\pi(x - y)) + \cos(\pi(x + y))]. \\ =& -4\cos(\pi(x + y))\sin(\pi x)\sin(\pi y) \le 4. \end{aligned}$$

Hence $\max_{x,y\in\mathbb{T}} F(x, y) = 3$ when $(x, y) = \left(\frac{1}{2}, -\frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$. We observe that the sum $F(k_1^2, k_2^2) + F(k_1^3, k_2^3)$ must be strictly smaller than 6; otherwise, $F(k_1^1, k_2^1) = -\omega_0/2 - 9 < -9$, which is a contradiction.

Since $F(k_1^2, k_2^2) + F(k_1^3, k_2^3) < 6$, then for any δ small, either positive or negative, there exist δ_1, δ_2, either positive or negative, such that

$$F(k_1^1 + \delta, k_2^1) + F(k_1^2 + \delta_1, k_2^2) + F(k_1^3 + \delta_2, k_2^3) = -\omega_0/2 - 3,$$

due to the continuity of F. If $\bar{k}^1 = k^1 + \delta$, then we choose $\bar{k}^2 = k^1 + \delta_1$ and $\bar{k}^3 = k^3 + \delta_2$.

Case 2: $k + k_1 = k_2$ **and** $\omega(k) + \omega(k_1) = \omega(k_2)$**.** Similar as Case 1, we only need to show that, for each $k^1 \in \mathbb{T}$, there exists $\epsilon > 0$ such that for each $\bar{k}^1 \in (k^1 - \epsilon, k^1 + \epsilon)$

there are $\bar{k}^2, \bar{k}^3 \in \mathbb{T}^3$, $\bar{k} = (\bar{k}^1, \bar{k}^2, \bar{k}^3) \in \mathcal{S}(x)$. Since $k_2 = (k_2^1, k_2^2, k_2^3) = k_1 + k = (k_1^1, k_1^2, k_1^3) + (k^1, k^2, k^3)$, we then have

$$F(k_1^1, k^1) + F(k_1^2, k^2) + F(k_1^3, k^3) = -\omega_0/2 - 3.$$

Since $F(k_1^2, k^2) + F(k_1^3, k^3) < 6$, then for any δ small, either positive or negative, there exist δ_1, δ_2, either positive or negative, such that

$$F(k_1^1, k^1 + \delta) + F(k_1^2, k^2 + \delta_1) + F(k_1^3, k^3 + \delta_2) = -\omega_0/2 - 3,$$

due to the continuity of F. If $\bar{k}^1 = k^1 + \delta$, then we choose $\bar{k}^2 = k^1 + \delta_1$ and $\bar{k}^3 = k^3 + \delta_2$.

Since on $\mathcal{S}(x)$, $\psi(k)$ is a function of $\omega(k)$ and k, there exists a twice differentiable continuous function $\phi \in C^2(\mathbb{R}_+ \times \mathbb{T}^3)$ such that $\psi(k) = \varphi(\omega(k), k)$.

For $k \in \mathcal{S}(x)$, there exist two wave vectors $k_1, k_2 \in \mathbb{T}^3$, such that either $k = k_1 + k_2$ and $\omega(k) = \omega(k_1) + \omega(k_2)$, or $k + k_1 = k_2$ and $\omega(k) + \omega(k_1) = \omega(k_2)$. We assume that $k = k_1 + k_2$ and $\omega(k) = \omega(k_1) + \omega(k_2)$, $k_1, k_2 \in \mathbb{T}^3$, the other case can be consider with exactly the same argument. As we observe before, k_1, k_2 also belong to $\mathcal{S}(x)$ due to the fact that k is connected to both k_1, k_2 by one-collisions. We have

$$\psi(k_1) + \psi(k_2) = \psi(k) = \varphi(\omega(k), k) = \varphi(\omega(k_1) + \omega(k_2), k_1 + k_2).$$

We now follow the strategy of [10] and [51]. Differentiating the above identity with respect to k_1^j and k_2^j yields

$$\begin{aligned}\partial_{k_1^j}\psi(k_1) =& \partial_r\varphi(\omega(k), k)\partial_{k_1^j}\omega(k_1) + \partial_{k_1^j}\varphi(\omega(k), k),\\ \partial_{k_2^j}\psi(k_2) =& \partial_r\varphi(\omega(k), k)\partial_{k_2^j}\omega(k_2) + \partial_{k_2^j}\varphi(\omega(k), k).\end{aligned}$$

Letting $i \in \{1, 2, 3\}$ be a different index, we manipulate the above identity as

$$\begin{aligned}&(\partial_{k_1^j}\psi(k_1) - \partial_{k_2^j}\psi(k_2))(\partial_{k_1^i}\omega(k_1) - \partial_{k_2^i}\omega(k_2))\\ &\quad = (\partial_{k_1^i}\psi(k_1) - \partial_{k_2^i}\psi(k_2))(\partial_{k_1^j}\omega(k_1) - \partial_{k_2^j}\omega(k_2)).\end{aligned}$$

We differentiate the above identity in k_1, with l being an index in $\{1, 2, 3\}$

$$\begin{aligned}&\partial_{k_1^j}\partial_{k_1^l}\psi(k_1)(\partial_{k_1^i}\omega(k_1) - \partial_{k_2^i}\omega(k_2)) + (\partial_{k_1^j}\psi(k_1) - \partial_{k_2^j}\psi(k_2))\partial_{k_1^i}\partial_{k_1^l}\omega(k_1)\\ &\quad = \partial_{k_1^i}\partial_{k_1^l}\psi(k_1)(\partial_{k_1^j}\omega(k_1) - \partial_{k_2^j}\omega(k_2)) + (\partial_{k_1^i}\psi(k_1) - \partial_{k_2^i}\psi(k_2))\partial_{k_1^j}\partial_{k_1^l}\omega(k_1),\end{aligned}$$

and now in k_2, with h being an index in $\{1, 2, 3\}$

$$\begin{aligned}&\partial_{k_1^j}\partial_{k_1^l}\psi(k_1)\partial_{k_2^i}\partial_{k_2^h}\omega(k_2) + \partial_{k_2^j}\partial_{k_2^h}\psi(k_2)\partial_{k_1^i}\partial_{k_1^l}\omega(k_1)\\ &\quad = \partial_{k_1^i}\partial_{k_1^l}\psi(k_1)\partial_{k_2^j}\partial_{k_2^h}\omega(k_2) + \partial_{k_2^i}\partial_{k_2^h}\psi(k_2)\partial_{k_1^j}\partial_{k_1^l}\omega(k_1).\end{aligned}$$

A particular case of the above identity is the following

$$\partial^2_{k_1^i}\psi(k_1)\partial^2_{k_2^j}\omega(k_2) = \partial^2_{k_1^j}\psi(k_1)\partial^2_{k_2^i}\omega(k_2),$$

which implies

$$\partial^2_{k_1^i}\psi(k_1)\cos(k_2^j) = \partial^2_{k_2^i}\psi(k_1)\cos(k_1^j),$$

for any $k_1, k_3 \in \mathcal{S}(x)$, and k_1, k_2 are connected to $k_1 + k_2$ by one collision.

Hence $\psi(k) = a_x\omega(k) + b_x \cdot k + c_x$, with $a_x, c_x \in \mathbb{R}$, $b_x \in \mathbb{R}^3$ for any $k \in \mathcal{S}(x)$. By the fact $\psi(k) = \psi(k_1) + \psi(k_2)$ whenever k is connected to k_1, k_2 by one-collisions, it is straightforward that $c_x = b_x = 0$. □

Proposition 19 *(L^1-collisional invariants) Let $\psi \in L^1(\mathcal{S}(x))$ be a collisional invariant on the collisional invariant region $\mathcal{S}(x)$, in the following sense. For any $k \in \mathcal{S}(x)$, such that*

$$k = k_1 + k_2, \text{ for some } z \in \mathbb{Z}^d, \qquad \omega(k) = \omega(k_1) + \omega(k_2),$$

we have

$$\psi(k) = \psi(k_1) + \psi(k_2).$$

Then there exist a constant $a_x \in \mathbb{R}$, such that

$$\psi(k) = a_x\omega(k).$$

Proof For any function $\phi \in C^\infty(\mathbb{T}^3)$, we define the standard mollifier $\phi_\delta(k) = \delta^{-3}\phi\left(\frac{k}{\delta}\right)$ and the standard approximation $\psi_\delta = \psi * \phi_\delta$ with $\delta > 0$. It is then classical that $\lim_{\delta\to 0} \|\psi_\delta - \psi\|_{L^1(\mathcal{S}(x))} = 0$.

Since $\psi(k) = \psi(k_1) + \psi(k_2)$, we also have $\psi_\delta(k) = \psi_\delta(k_1) + \psi_\delta(k_2)$. Lemma 18 can be applied to ψ_δ, yielding $\psi_\delta(k) = a_x^\delta\omega(k)$ for some constant $a_x^\delta \in \mathbb{R}$. The conclusion of the Proposition then follows after passing δ to 0, while taking into account the limit $\lim_{\delta\to 0} \|\psi_\delta - \psi\|_{L^1(\mathcal{S}(x))} = 0$. □

Proposition 20 *(Equilibria in Collisional Invariant Regions) Given a collisional invariant region $\mathcal{S}(x)$, a function $\mathcal{F}^c(k) \in C(\mathcal{S}(x))$ is said to be a local equilibrium of Q_c on $\mathcal{S}(x)$ if and only if $Q_c[\mathcal{F}^c](k) = 0$ and $\mathcal{F}^c(k) > 0$ for all $k \in \mathcal{S}(x)$.*

Let $E_x \in \mathbb{R}_+$ and assume

$$\int_{\mathcal{S}(x)} \frac{1}{a_x}\mathrm{d}k = E_x, \tag{63}$$

with $a_x \in \mathbb{R}_+$; the local equilibrium on $\mathcal{S}(x)$ of Q_c can be uniquely determined as

$$\mathcal{F}^c(k) = \frac{1}{a_x\omega(k)}, \tag{64}$$

subjected to the local energy constraint

$$\int_{\mathcal{S}(x)} \mathcal{F}^c(k)\omega(k)\mathrm{d}k = E_x. \tag{65}$$

Proof Since $Q_c[\mathcal{F}^c](k) = 0$ for all $k \in \mathcal{S}(x)$, using $\frac{1}{\mathcal{F}^c}$ as a test function, we obtain

$$\begin{aligned}
0 &= \int_{\mathcal{S}(x)} Q_c[\mathcal{F}^c](k)\frac{1}{\mathcal{F}^c(k)}\mathrm{d}k \\
&= \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[\mathcal{F}_1^c\mathcal{F}_2^c - \mathcal{F}_1^c\mathcal{F}^c - \mathcal{F}_2^c\mathcal{F}^c] \\
&\quad \times \left[\frac{1}{\mathcal{F}^c} - \frac{1}{\mathcal{F}_1^c} - \frac{1}{\mathcal{F}_2^c}\right]\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2 \\
&= \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\mathcal{F}^c\mathcal{F}_1^c\mathcal{F}_2^c\left[\frac{1}{\mathcal{F}^c} - \frac{1}{\mathcal{F}_1^c} - \frac{1}{\mathcal{F}_2^c}\right]^2\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned} \tag{66}$$

which implies $\frac{1}{\mathcal{F}^c} - \frac{1}{\mathcal{F}_1^c} - \frac{1}{\mathcal{F}_2^c} = 0$ for all $k, k_1, k_2 \in \mathcal{S}(x)$ satisfying $k = k_1 + k_2$ in the periodic sense (i.e. there exists some $z \in \mathbb{Z}^d$ such that $k = k_1 + k_2 + z$) and $\omega = \omega_1 + \omega_2$. Therefore $\frac{1}{\mathcal{F}^c}$ is a collisional invariant; and by Proposition 19, $\mathcal{F}^c$ takes the form (64), given that the system (63) has a unique solution a_x. □

4.1.6 Entropy formulation on the collisional invariant region $\mathcal{S}(x)$

Let f be a positive solution of (2), we define the local entropy on the collisional invariant region $\mathcal{S}(x)$ as follows

$$S_{c,\mathcal{S}(x)}[f] = \int_{\mathcal{S}(x)} s_c[f]\mathrm{d}k = \int_{\mathcal{S}(x)} \ln(f)\mathrm{d}k. \tag{67}$$

In the sequel, we only consider the local entropy on one collisional invariant region, then, for the sake of simplicity, we denote $S_{c,\mathcal{S}(x)}[f]$ by $S_c[f]$.

Now, we take the derivative in time of $S_c[f]$

$$\partial_t S_c[f] = \int_{\mathcal{S}(x)} \frac{\partial_t f}{f}\mathrm{d}k. \tag{68}$$

Replacing the quantity $\partial_t f$ in the above formulation by the right hand side of (2), we find

$$\begin{aligned}\partial_t S_c[f] = &\iiint_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\times\\ &\times [f_1f_2 - ff_1 - ff_2]\frac{1}{f}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\ &-2\iiint_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\times\\ &\times [f_2f - ff_1 - f_1f_2]\frac{1}{f}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.\end{aligned} \tag{69}$$

We now apply Lemma 16 to the above identity to get

$$\begin{aligned}\partial_t S_c[f] = &\iiint_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[f_1f_2 - ff_1 - ff_2]\\ &\times\left[\frac{1}{f_2}+\frac{1}{f_1}-\frac{1}{f}\right]\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.\end{aligned} \tag{70}$$

By noting that

$$f_1f_2 - ff_1 - ff_2 = ff_1f_2\left[\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f}\right],$$

we obtain from (70) the following entropy identity

$$\begin{aligned}\partial_t S_c[f] = &\int_{\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)ff_1f_2\times\\ &\times\left[\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f}\right]^2\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\ &=: D_c[f].\end{aligned} \tag{71}$$

It is clear that the quantity $D_c[f]$ is positive. Borrowing the idea of [12, 54], we now define the reciprocal, of f

$$g = \frac{1}{f}. \tag{72}$$

As a consequence, the formula (71) can be expressed in the following form

$$\begin{aligned}\partial_t S_c[f] = D_c[f] = \mathbb{D}_c[g] := \iiint_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\\ \times \frac{[g_1+g_2-g]^2}{gg_1g_2}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.\end{aligned} \tag{73}$$

4.1.7 Cutting off and splitting the collision operator on the collisional invariant region $\mathcal{S}(x)$

In this subsection, we follow the idea of [12] to introduce a cut-off version for the collision operator $Q_c[f]$. The intuition behind this cut-off operator is explained below. We expect that as t tends to infinity, the solution f of (2) converges to an equilibrium, which is a function bounded from above and below by positive constants. Since the equilibrium is bounded from above and below, it is not affected by the cut-off operator. As a result, the solution f is expected to be unchanged, under the effect of the cut-off operator, as t goes to infinity.

Let ϱ_N (for $0 < N \leq \infty$) be a function in $C^1(\mathbb{R}_+)$ satisfying $\varrho_N[z] = 1$ when $\frac{1}{N} \leq z \leq N$, $\varrho_N[z] = 0$ when $0 \leq z \leq \frac{1}{2N}$ and $z \geq 2N$, and $0 \leq \varrho_N[z] \leq 1$ when $\frac{1}{2N} \leq z \leq \frac{1}{N}$ and $N \leq z \leq 2N$. For $f \in C^1(\mathcal{S}(x))$ and $0 < N \leq \infty$, define the cut-off function

$$\chi_N[f] = \varrho_N[f]\varrho_N[|\nabla f|]. \tag{74}$$

Note that $\chi_\infty[f] = 1$ for all $f \in C^1(\mathcal{S}(x))$.

We set the cut-off collision operator on the collisional invariant region $\mathcal{S}(x)$ for f and for g defined in (72)

$$\begin{aligned}Q_c^N[f](k) =& \int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\\ & [f_1f_2 - ff_1 - ff_2]\mathrm{d}k_1\mathrm{d}k_2\\ & -2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\\ & [f_2f - ff_1 - f_1f_2]\mathrm{d}k_1\mathrm{d}k_2\\ =& \int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\chi_N^*[gg_1g_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\\ & [g-g_1-g_2]\mathrm{d}k_1\mathrm{d}k_2\\ & -2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\chi_N^*[gg_1g_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\\ & [g_1-g_2-g]\mathrm{d}k_1\mathrm{d}k_2,\end{aligned} \tag{75}$$

in which

$$\chi_N^* = \chi_N[f]\chi_N[f_1]\chi_N[f_2] = \chi_N[1/g]\chi_N[1/g_1]\chi_N[1/g_2]. \tag{76}$$

When $N = \infty$, we have that

$$\begin{aligned}
Q_c^N[f](k) &= Q_c^\infty[f](k)\\
&= \int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[f_1f_2-ff_1-ff_2]\mathrm{d}k_1\mathrm{d}k_2\\
&\quad -2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_2f-ff_1-f_1f_2]\mathrm{d}k_1\mathrm{d}k_2\\
&= \int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}[gg_1g_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[g-g_1-g_2]\mathrm{d}k_1\mathrm{d}k_2\\
&\quad -2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}[gg_1g_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[g_1-g_2-g]\mathrm{d}k_1\mathrm{d}k_2.
\end{aligned} \tag{77}$$

We also define the splitting collision operators on $\mathcal{S}(x)$, in which the kernel $[gg_1g_2]^{-1}$ is removed

$$\begin{aligned}
\mathbb{Q}_c^{N,-}[g](k) &= \int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[g_1+g_2]\mathrm{d}k_1\mathrm{d}k_2\\
&\quad +2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_1\mathrm{d}k_1\mathrm{d}k_2\\
&\quad -2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_2\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned} \tag{78}$$

$$\begin{aligned}
\mathbb{Q}_c^{N,+}[g](k) &= g\mathbb{L}_c^N(k)\\
&= g\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}k_1\mathrm{d}k_2\\
&\quad +2g\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned} \tag{79}$$

and

$$\mathbb{Q}_c^N[g] = \mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_0^{N,-}[g]. \tag{80}$$

Due to the symmetry of k_1 and k_2, $\mathbb{Q}_c^{N,-}[g](k)$ can be rewritten as

$$\begin{aligned}
\mathbb{Q}_c^{N,-}[g](k) &= \mathbb{Q}_c^{N,-,1}[g](k) + \mathbb{Q}_c^{N,-,2}[g](k) + \mathbb{Q}_c^{N,-,3}[g](k) :=\\
&= 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)g_1\mathrm{d}k_1\mathrm{d}k_2\\
&\quad +2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_1\mathrm{d}k_1\mathrm{d}k_2\\
&\quad -2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N^*[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_2\mathrm{d}k_1\mathrm{d}k_2.
\end{aligned} \tag{81}$$

Note that in all of the above definitions, the cut-off parameter N takes values in the interval $(0, \infty]$. We then have the following lemma.

Lemma 21 *Given a collisional invariant region $\mathcal{S}(x)$, a function $\mathcal{F}^c(k) \in C(\mathcal{S}(x))$ is said to be a local equilibrium of Q_c^N on $\mathcal{S}(x)$ if and only if $Q_c^N[\mathcal{F}^c](k) = 0$ and $\mathcal{F}^c(k) > 0$ for all $k \in \mathcal{S}(x)$.*

Under the local energy constraint

$$\int_{\mathcal{S}(x)} \mathcal{F}^c(k)\omega(k)\mathrm{d}k = E_x \tag{82}$$

where E_x is a given positive constant. Suppose that $E_x \in \mathbb{R}_+$ and

$$\int_{\mathcal{S}(x)} \frac{1}{a_x} \mathrm{d}k = E_x, \tag{83}$$

with $a_x \in \mathbb{R}_+$; the local equilibrium on $\mathcal{S}(x)$ can be uniquely determined, when N is sufficiently large, as

$$\mathcal{F}^c(k) = \frac{1}{a_x \omega(k)}. \tag{84}$$

Similarly, a function $\mathcal{E}^c(k)$ is said to be a local equilibrium of $\mathbb{Q}_c^N$ on $\mathcal{S}(x)$ if and only if $\mathbb{Q}_c^N[\mathcal{F}^c](k) = 0$ and

$$\mathcal{E}^c(k) = a_x \omega(k).$$

Proof The proof follows from the same lines of arguments used in the proof of Proposition 20. □

4.2 The long time dynamics of solutions to the 3-wave kinetic equation on non-collision and collisional invariant regions

4.2.1 An estimate on the distance between f and $\mathcal{F}^c$

This section is devoted to the estimate of the difference between a function f and a local equilibrium $\mathcal{F}^c$, defined on the same collisional invariant region. The two functions f and $\mathcal{F}^c$ are supposed to have the same energy.

Proposition 22 *Let $\mathcal{S}(x)$ be a collisonal invariant region and f be a positive function such that $f \in L^1(\mathcal{S}(x))$. Let*

$$\mathcal{F}^c(k) = \frac{1}{a_x \omega(k)} =: \frac{1}{\mathcal{E}^c(k)}, \tag{85}$$

where $a_x \in \mathbb{R}$ satisfying $\mathcal{F}^c(k) > 0$ for all $k \in \mathcal{S}(x)$.

In addition, we assume

$$\int_{\mathcal{S}(x)} f(k)\omega(k)\mathrm{d}k = \int_{\mathcal{S}(x)} \mathcal{F}(k)\omega(k)\mathrm{d}k. \tag{86}$$

We also define g using (72).

Then, the following inequalities always hold true for $0 \le N \le \infty$

$$\int_{\mathcal{S}(x)} \sqrt{f \left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right|} \mathrm{d}k \lesssim \left[\int_{\mathcal{S}(x)} f \mathrm{d}k\right]^{\frac{1}{2}} \times$$
$$\times \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g-g_1-g_2|^2 \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{4}}, \tag{87}$$

and

$$\left\|\sqrt{\mathbb{L}_c^N \mathcal{E}^c |f - \mathcal{F}^c|}\right\|_{L^1(\mathcal{S}(x))} \lesssim \left[\int_{\mathcal{S}(x)} f \mathrm{d}k\right]^{\frac{1}{2}} \left\{ \|g - \mathcal{E}^c\|_{L^1(\mathcal{S}(x))}^{\frac{1}{2}} + \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \; \chi_N^* \delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g-g_1-g_2|^2 \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{4}} \right\} \tag{88}$$

in which the constants on the right hand sides do not depend on f.

Proof Considering the difference between f and $\mathcal{F}^c$ on $\mathcal{S}(x)$, we find

$$|f - \mathcal{F}^c| = \left|\frac{1}{g} - \frac{1}{\mathcal{E}^c}\right| = \frac{|g - \mathcal{E}^c|}{g\mathcal{E}^c},$$

which then implies

$$\mathcal{E}^c|f - \mathcal{F}^c| = f|g - \mathcal{E}^c|.$$

Multiplying both sides with $\mathbb{L}_c^N$ and taking the square yields

$$\sqrt{\mathbb{L}_c^N \mathcal{E}^c|f - \mathcal{F}^c|} = \sqrt{\mathbb{L}_c^N f|g - \mathcal{E}^c|},$$

which, by the fact that $\mathbb{L}_c^N g = \mathbb{Q}_c^{N,+}[g]$ and $\mathbb{L}_c^N \mathcal{E}^c = \mathbb{Q}_c^{N,+}[\mathcal{E}^c]$, implies

$$\sqrt{\mathbb{L}_c^N \mathcal{E}^c|f - \mathcal{F}^c|} = \sqrt{f\left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,+}[\mathcal{E}^c]\right|}.$$

Applying the triangle inequality to the right hand side gives

$$\sqrt{\mathbb{L}_c^N \mathcal{E}^c|f - \mathcal{F}^c|} \lesssim \sqrt{f\left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right|} + \sqrt{f\left|\mathbb{Q}_c^{N,-}[g] - \mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|} + \sqrt{f\left|\mathbb{Q}_c^{N,+}[\mathcal{E}^c] - \mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|}.$$

By Lemma 21, the last term on the right hand side of the above inequality vanishes, yielding

$$\sqrt{\mathbb{L}_c^N \mathcal{E}^c|f - \mathcal{F}^c|} \lesssim \sqrt{f\left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right|} + \sqrt{f\left|\mathbb{Q}_c^{N,-}[g] - \mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|}. \tag{89}$$

Integrating the first term on the right hand side and using Hölder's inequality leads to

$$\left(\int_{\mathcal{S}(x)} \sqrt{f\left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right|}\mathrm{d}k\right)^2 \leq \left(\int_{\mathcal{S}(x)} f\mathrm{d}k\right)\left(\int_{\mathcal{S}(x)} \left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right| \mathrm{d}k\right). \tag{90}$$

Observe that

$$\begin{aligned}\left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right| &\leq \int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g-g_1-g_2|\mathrm{d}k_1\mathrm{d}k_2 \\ &\quad + 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-g_2-g|\mathrm{d}k_1\mathrm{d}k_2,\end{aligned}$$

which, after integrating in k and taking into account the symmetry of k, k_1, k_2, yields

$$\int_{\mathcal{S}(x)} \left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right| \mathrm{d}k$$
$$\leq 3 \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)|g - g_1 - g_2| \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.$$

Applying Hölder's inequality again to the right hand side implies

$$\begin{aligned}
&\int_{\mathcal{S}(x)} \left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right| \mathrm{d}k \leq \\
&\leq 3 \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2) \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{2}} \\
&\quad \times \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)|g - g_1 - g_2|^2 \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{2}}.
\end{aligned} \tag{91}$$

Using the fact that $\chi_N^* \leq 1$, Corollary 14 and Proposition 15 to bound the integral containing only $[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)$, we derive from the above inequality the following estimate

$$\begin{aligned}
&\int_{\mathcal{S}(x)} \left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right| \mathrm{d}k \\
&\leq 3 \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2) \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{2}} \times \\
&\times \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)|g - g_1 - g_2|^2 \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{2}} \\
&\lesssim \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)|g - g_1 - g_2|^2 \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{2}}.
\end{aligned} \tag{92}$$

Putting (90) and (92) together, we obtain

$$\begin{aligned}
&\int_{\mathcal{S}(x)} \sqrt{f \left|\mathbb{Q}_c^{N,+}[g] - \mathbb{Q}_c^{N,-}[g]\right|} \mathrm{d}k \lesssim \left[\int_{\mathcal{S}(x)} f \mathrm{d}k\right]^{\frac{1}{2}} \\
&\times \left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \chi_N^* \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2)|g - g_1 - g_2|^2 \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac{1}{4}}.
\end{aligned} \tag{93}$$

Integrating the second term on the right hand side of (89) and using Hölder's inequality

$$\left(\int_{\mathcal{S}(x)} \sqrt{f \left|\mathbb{Q}_c^{N,-}[g] - \mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|} \mathrm{d}k\right)^2 \leq \left(\int_{\mathcal{S}(x)} f \mathrm{d}k\right) \left(\int_{\mathcal{S}(x)} \left|\mathbb{Q}_c^{N,-}[g] - \mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right| \mathrm{d}k\right). \tag{94}$$

It is straightforward that

$$\begin{aligned}
&\left|\mathbb{Q}_c^{N,-}[g]-\mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|\\
&\quad\le\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[|g_1-\mathcal{E}_1^c|+|g_2-\mathcal{E}_2^c|]\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_2-\mathcal{E}_2^c|\mathrm{d}k_1\mathrm{d}k_2.
\end{aligned}$$

Integrating in k, we immediately find

$$\begin{aligned}
&\int_{\mathcal{S}(x)}\left|\mathbb{Q}_c^{N,-}[g]-\mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|\mathrm{d}k\\
&\quad\le\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)[|g_1-\mathcal{E}_1^c|+|g_2-\mathcal{E}_2^c|]\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_2-\mathcal{E}_2^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned}$$

which, by the symmetry between k_1 and k_2 and the fact that $\chi_N^*\le 1$, implies

$$\begin{aligned}
&\int_{\mathcal{S}(x)}\left|\mathbb{Q}_c^{N,-}[g]-\mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|\mathrm{d}k\\
&\quad\le 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_2-\mathcal{E}_2^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.
\end{aligned}$$

Now, we can also combine the last and the first terms on the right hand side using the change of variables between k, k_1, k_2 to get

$$\begin{aligned}
&\int_{\mathcal{S}(x)}\left|\mathbb{Q}_c^{N,-}[g]-\mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|\mathrm{d}k\\
&\quad\le 4\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\
&\qquad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.
\end{aligned}\tag{95}$$

Let us estimate each term on the right hand side of (95).

Taking the integration in k_2 of the first term yields

$$\begin{aligned}
&4\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\
&\quad=4\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega(k)\omega(k_1)\omega(k-k_1)]^{-1}\delta(\omega(k)-\omega(k_1)-\omega(k-k_1))|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1.
\end{aligned}$$

Observing that $\omega(k) \geq \omega_0 > 0$ for all $k \in \mathbb{T}^3$, we find

$$\begin{aligned}&4\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\&\lesssim\int_{\mathcal{S}(x)\times\mathcal{S}(x)}\delta(\omega(k)-\omega(k_1)-\omega(k-k_1))|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1,\end{aligned}$$

which, after integrating with respect to k_1, leads to

$$\begin{aligned}&4\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\&\lesssim\int_{\mathcal{S}(x)}\left[\int_{\mathcal{S}(x)}\delta(\omega(k)-\omega(k_1)-\omega(k-k_1))\mathrm{d}k\right]|g_1-\mathcal{E}_1^c|\mathrm{d}k_1.\end{aligned}$$

Note that the integration with respect to k is uniformly bounded in $k_1 \in \mathbb{T}^3$ by Corollary 14 and Proposition 15, we then get

$$\begin{aligned}&4\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\&\lesssim\int_{\mathcal{S}(x)}|g_1-\mathcal{E}_1^c|\mathrm{d}k_1=\|g-\mathcal{E}^c\|_{L^1(\mathcal{S}(x))}.\end{aligned}\tag{96}$$

The second term on the right hand side of (95) can also be estimated in the same way. Taking the integration in k_2 of the second term yields

$$\begin{aligned}&2\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\&=2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega(k)\omega(k_1)\omega(k-k_1)]^{-1}\delta(\omega(k_1)-\omega(k)-\omega(k_1-k))|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1,\end{aligned}$$

which, similarly as above, can be bounded as

$$\begin{aligned}&2\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\&\lesssim\int_{\mathcal{S}(x)}\left[\int_{\mathcal{S}(x)}\delta(\omega(k_1)-\omega(k)-\omega(k_1-k))\mathrm{d}k\right]|g_1-\mathcal{E}_1^c|\mathrm{d}k_1.\end{aligned}$$

Again, the integration with respect to k is bounded, we therefore have

$$\begin{aligned}&4\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)|g_1-\mathcal{E}_1^c|\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\\&\lesssim\int_{\mathcal{S}(x)}|g_1-\mathcal{E}_1^c|\mathrm{d}k_1=\|g-\mathcal{E}^c\|_{L^1(\mathcal{S}(x))}.\end{aligned}\tag{97}$$

Now, combining (94), (95), (96), (97) leads to

$$\begin{aligned}&\int_{\mathcal{S}(x)}\sqrt{f\left|\mathbb{Q}_c^{N,-}[g]-\mathbb{Q}_c^{N,-}[\mathcal{E}^c]\right|}\mathrm{d}k\lesssim\\&\lesssim\left[\int_{\mathcal{S}(x)}f\mathrm{d}k\right]^{\frac{1}{2}}\left[\int_{\mathcal{S}(x)}|g_1-\mathcal{E}_1^c|\mathrm{d}k_1\right]^{\frac{1}{2}}=\left[\int_{\mathcal{S}(x)}f\mathrm{d}k\right]^{\frac{1}{2}}\|g-\mathcal{E}^c\|_{L^1(\mathcal{S}(x))}^{\frac{1}{2}}.\end{aligned}\tag{98}$$

Putting together the three estimates (89), (93) and (98) yields

$$\begin{aligned}\left\|\sqrt{\mathbb{L}_c^N\mathcal{E}^c|f-\mathcal{F}^c|}\right\|_{L^1(\mathcal{S}(x))} \lesssim \left[\int_{\mathcal{S}(x)} f\mathrm{d}k\right]^{\frac12}\|g-\mathcal{E}^c\|^{\frac12}_{L^1(\mathcal{S}(x))} + \left[\int_{\mathcal{S}(x)} f\mathrm{d}k\right]^{\frac12}\times \\ \times\left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\chi_N^*\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)|g-g_1-g_2|^2\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right]^{\frac14}\end{aligned} \tag{99}$$

□

4.2.2 A lower bound on the solution of the equation with the cut-off collision operator on the collisional invariant region $\mathcal{S}(x)$

The following Proposition provides a uniform lower bound to classical solutions of the wave kinetic equation on $\mathcal{S}(x)$, under the effect of the cut-off operator χ_N.

Proposition 23 *Suppose that the initial condition f_0 of (2) is bounded from below by a strictly positive constant f_0^*, and $f_0 \in C(\mathcal{S}(x))$. Let f be a classical solution in $C^0([0,\infty), C(\mathcal{S}(x)))$ $\cap\, C^1((0,\infty), C(\mathcal{S}(x)))$ to (2). There exists a strictly positive function $f^*(t) > 0$, which is non-increasing in t, such that $f(t,k) > f^*(t) > 0$ for all $k \in \mathcal{S}(x)$ and for all $t \geq 0$. To be more precise, there exists a universal constant $f_* > 0$ such that*

$$f(t,k) > f^*(t) = \frac{f_*}{\sup_{s\in[0,t]}\|f(s,\cdot)\|_{C(\mathcal{S}(x))}}.$$

Proof Rearranging the equation, one finds

$$\begin{aligned}\partial_t f =& \int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)f_1f_2\mathrm{d}k_1\mathrm{d}k_2 \\ &+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_1f_2+ff_1]\mathrm{d}k_1\mathrm{d}k_2 \\ &-f\left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)(f_1+f_2)\mathrm{d}k_1\mathrm{d}k_2\right. \\ &\left.+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)f_2\mathrm{d}k_1\mathrm{d}k_2\right].\end{aligned}$$

Using the symmetry of f_1 and f_2 in the term containing f_1+f_2, we can turn this term into a new term, in which f_1+f_2 is replaced by $2f_1$

$$\begin{aligned}\partial_t f =& \int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)f_1f_2\mathrm{d}k_1\mathrm{d}k_2 \\ &+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_1f_2+ff_1]\mathrm{d}k_1\mathrm{d}k_2 \\ &-2f\left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)f_1\mathrm{d}k_1\mathrm{d}k_2\right. \\ &\left.+\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)f_2\mathrm{d}k_1\mathrm{d}k_2\right].\end{aligned} \tag{100}$$

Now, let us consider the term with the minus sign

$$2f\Bigg[\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)f_1\mathrm{d}k_1\mathrm{d}k_2 \\ +\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)f_2\mathrm{d}k_1\mathrm{d}k_2\Bigg]. \tag{101}$$

We define the function $\mathbb{B}:\mathbb{R}_+\to\mathbb{R}_+$

$$\mathbb{B}(t)=\sup_{s\in[0,t]}\|f(s,\cdot)\|_{C(\mathcal{S}(x))}, \tag{102}$$

which is an increasing function in t. Using the fact that $\omega\geq\omega_0>0$ and the function $\mathbb{B}(t)$, we can bound (101) from above by

$$\frac{2\mathbb{B}(t)}{\omega_0^3}f\Bigg[\int_{\mathcal{S}(x)\times\mathcal{S}(x)}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}k_1\mathrm{d}k_2 \\ +\int_{\mathcal{S}(x)\times\mathcal{S}(x)}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\mathrm{d}k_1\mathrm{d}k_2\Bigg].$$

Integrating in k_2 and using the definite of the two delta functions $\delta(k-k_1-k_2)$ and $\delta(k_1-k-k_2)$

$$\frac{2\mathbb{B}(t)}{\omega_0^3}f(k)\Bigg[\int_{\mathcal{S}(x)}\delta(\omega(k)-\omega(k_1)-\omega(k-k_1))\mathrm{d}k_1 \\ +\int_{\mathcal{S}(x)}\delta(\omega(k)-\omega(k_1)-\omega(k-k_1))\mathrm{d}k_1\Bigg]\leq\frac{2\mathbb{B}(t)}{\omega_0^3}\mathfrak{C}_1f(k)=:\mathcal{C}(t)f(k).$$

We therefore obtain the following bound for $\partial_t f$

$$\partial_t f\geq\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)f_1f_2\mathrm{d}k_1\mathrm{d}k_2 \\ +2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)[f_1f_2+ff_1]\mathrm{d}k_1\mathrm{d}k_2 \\ -\mathcal{C}(t)f. \tag{103}$$

Define the positive terms on the right hand side by $K[f]$, we then have the simplified equation

$$\partial_t f\geq K[f]-\mathcal{C}(t)f, \tag{104}$$

which, by Duhamel's formula and the mononicity in t of $\mathcal{C}(t)$, gives

$$f(t,k)\geq f_0(k)e^{-\mathcal{C}(T)t}+\int_0^t K[f](t-s,k)e^{-\mathcal{C}(T)(t-s)}\mathrm{d}s, \tag{105}$$

Using the fact that $f_0(k)\geq f_0^*>0$, we deduce from (105) the following estimate

$$f(t,k)\geq f_0^*e^{-\mathcal{C}(T)t}+\int_0^t K[f](t-s,k)e^{-\mathcal{C}(T)(t-s)}\mathrm{d}s. \tag{106}$$

We observe that the second term on the right hand side is always positive, since it contains only positive components. This implies

$$f(t,k)\geq f_0^*e^{-\mathcal{C}(T)t}, \tag{107}$$

for all $t \in [0, T]$.

Now, let us examine the operator $K[f]$ in details. Using the fact $\omega \le \omega_0 + 12$, we can bound $K[f]$ as

$$K[f] \ge [\omega_0 + 12]^{-3} \Bigg[\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2) f_1 f_2 \mathrm{d}k_1 \mathrm{d}k_2 + 2 \int_{\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1} \delta(k_1 - k - k_2)\delta(\omega_1 - \omega - \omega_2)[f_1 f_2 + f f_1] \mathrm{d}k_1 \mathrm{d}k_2 \Bigg].$$

From which, we can use (107), to bound f, f_1, f_2 from below

$$K[f] \ge [\omega_0 + 12]^{-3} \Bigg[\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2) f_0^{*2} e^{-2\mathcal{C}(T)t} \mathrm{d}k_1 \mathrm{d}k_2 + 4 \int_{\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k_1 - k - k_2)\delta(\omega_1 - \omega - \omega_2) f_0^{*2} e^{-2\mathcal{C}(T)t} \mathrm{d}k_1 \mathrm{d}k_2 \Bigg],$$

for all $t \in [0, T]$.

The above inequality leads to

$$\begin{aligned} K[f] \ge& \frac{f_0^{*2} e^{-2\mathcal{C}(T)t}}{[\omega_0 + 12]^3} \Bigg[\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k - k_1 - k_2)\delta(\omega - \omega_1 - \omega_2) \mathrm{d}k_1 \mathrm{d}k_2 \\ &+ 4 \int_{\mathcal{S}(x)\times\mathcal{S}(x)} \delta(k_1 - k - k_2)\delta(\omega_1 - \omega - \omega_2) \mathrm{d}k_1 \mathrm{d}k_2 \Bigg] \\ \ge& \frac{f_0^{*2} e^{-2\mathcal{C}(T)t}}{[\omega_0 + 12]^3} \mathfrak{C}_2 \ge \mathcal{C}_1 e^{-2\mathcal{C}(T)t}, \end{aligned} \tag{108}$$

for all $t \in [0, T]$. Note that $\mathcal{C}_1$ is a universal strictly positive constant.

We follow the strategy of [42] by plugging (108) into (106)

$$\begin{aligned} f(t,k) \ge& f_0^* e^{-\mathcal{C}(T)t} + \mathcal{C}_1 \int_0^t e^{-3\mathcal{C}(T)(t-s)} \mathrm{d}s \\ \ge& f_0^* e^{-\mathcal{C}(T)t} + \frac{\mathcal{C}_1}{3\mathcal{C}(T)} [1 - e^{-3\mathcal{C}(T)t}], \end{aligned} \tag{109}$$

for all $t \in [0, T]$.

We define the time-dependent function

$$F(t) = f_0^* e^{-\mathcal{C}(T)t} + \frac{\mathcal{C}_1}{3\mathcal{C}(T)} [1 - e^{-3\mathcal{C}(T)t}],$$

which is continuous and non-negative.

Pick a finite time $t_0 = \frac{c}{\mathcal{C}(T)} > 0$, in which c is a fixed constant to be determined later. For $t \in [0, t_0]$, it is clear that $F(t) \ge f_0^* e^{-\mathcal{C}(T)t} = f_0^* e^{-c} > 0$. When $t > t_0$, then $F(t) \ge \frac{\mathcal{C}_1}{3\mathcal{C}(T)} + f_0^* e^{-3\mathcal{C}(T)t}[e^{2\mathcal{C}(T)t} - \frac{\mathcal{C}_1}{3\mathcal{C}(T) f_0^*}] > \frac{\mathcal{C}_1}{3\mathcal{C}(T)} + f_0^* e^{-3\mathcal{C}(T)t}[e^{2c} - \frac{\mathcal{C}_1}{3\mathcal{C}(T) f_0^*}]$. For a suitable choice of c, $e^{2c} = \frac{\mathcal{C}_1}{3\mathcal{C}(T) f_0^*}$. It then follows that $F(t) > \frac{\mathcal{C}_1}{3\mathcal{C}(T)}$, for all $t \in [0, T]$.

As a consequence, $f(t,k)$ is bounded from below by a strictly positive function $\frac{\mathcal{C}_1}{3\mathcal{C}(t)}$ for $k \in \mathcal{S}(x)$. Since $\mathbb{B}(t)$ is an non-decreasing function of time, it follows that $\frac{\mathcal{C}_1}{3\mathcal{C}(t)}$ is a non-increasing function of time. □

4.2.3 Convergence to equilibrium of the solution of the equation with the cut-off collision operator on the collisional invariant region $\mathcal{S}(x)$

The below proposition shows the convergence to equilibrium of the equation with cut-off operators. This contains the main ingredients of the proof of the convergence in the non cut-off case.

Proposition 24 *Let f be a positive, classical solution in $C([0,\infty), C^1(\mathcal{S}(x))) \cap C^1((0,\infty), C^1(\mathcal{S}(x)))$ of (2) on $\mathcal{S}(x)$, with the initial condition $f_0 \in C(\mathcal{S}(x))$, $f_0 \geq 0$. Let $E_x \in \mathbb{R}_+$ be a constant and*

$$\int_{\mathcal{S}(x)} \frac{1}{a_x} \mathrm{d}k = E_x = \int_{\mathcal{S}(x)} \omega(k) f_0(k) \mathrm{d}k, \tag{110}$$

has a unique solution $a_x \in \mathbb{R}_+$; the local equilibrium on $\mathcal{S}(x)$ can be uniquely determined as

$$\mathcal{F}^c(k) = \frac{1}{a_x \omega(k)}. \tag{111}$$

Then, the following limits always hold true,

$$\lim_{t\to\infty} \left\| f(t,\cdot) - \mathcal{F}^c \right\|_{L^1(\mathcal{S}(x))} = 0. \tag{112}$$

and

$$\lim_{t\to\infty} \left| \int_{\mathcal{S}(x)} \ln[f] \mathrm{d}k - \int_{\mathcal{S}(x)} \ln\left[\mathcal{F}^c\right] \mathrm{d}k \right| = 0. \tag{113}$$

If, in addition, there is a positive constant $M^* > 0$ such that $f(t,k) < M^*$ for all $t \in [0,\infty)$ and for all $k \in \mathcal{S}(x)$, then

$$\lim_{t\to\infty} \left\| f(t,\cdot) - \mathcal{F}^c \right\|_{L^p(\mathcal{S}(x))} = 0, \qquad \forall p \in [1,\infty). \tag{114}$$

If we suppose further that $f_0(k) > 0$ for all $k \in \mathcal{S}(x)$, there exists a constant M_* such that $f(t,k) > M_*$ for all $t \in [0,\infty)$ and for all $k \in \mathcal{S}(x)$.

We need the following Lemma, whose proof could be found in the Appendix.

Lemma 25 *Let $\mathcal{S}(x)$ be a collisonal invariant region and f be a positive function such that $f\omega \in L^1(\mathcal{S}(x))$. Let*

$$\mathcal{F}^c(k) = \frac{1}{a_x \omega(k)} =: \frac{1}{\mathcal{E}^c(k)}, \tag{115}$$

where the constant $a_x \in \mathbb{R}_+$ such that $\mathcal{F}^c(k) > 0$ for all $k \in \mathcal{S}(x)$.

Suppose, in addition, that

$$\int_{\mathcal{S}(x)} f(k)\omega(k) \mathrm{d}k = \int_{\mathcal{S}(x)} \mathcal{F}^c(k)\omega(k) \mathrm{d}k. \tag{116}$$

Then, the following inequalities always hold true

$$0 \leq S_c[\mathcal{F}^c] - S_c[f], \tag{117}$$

and

$$\|f - \mathcal{F}^c\|_{L^1(\mathcal{S}(x))} \lesssim \left[S_c[\mathcal{F}^c] - S_c[f]\right]^{\frac{1}{2}}, \tag{118}$$

in which the constant on the right hand side does not depend on f; $S_c[f]$ is defined in (67).

Proof We divide the proof in to several steps.

Step 1: Entropy estimates. Let us first recall (73), which is written as follows

$$\partial_t \int_{\mathcal{S}(x)} \ln(f)\mathrm{d}k = \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\times \\ \times \frac{[g_1+g_2-g]^2}{gg_1g_2}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2.$$

The above identity shows that $\int_{\mathcal{S}(x)} \ln(f)\mathrm{d}k$ is an increasing function of time. In particular $\int_{\mathcal{S}(x)} \ln(f)\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0)\mathrm{d}k \geq 0$. Picking $n \in \mathbb{N}$ and considering the difference of the entropy at two times n and $n+1$ yields

$$\left(\int_{\mathcal{S}(x)} \ln(f(2^{n+1},k))\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0(k))\mathrm{d}k\right) - \left(\int_{\mathcal{S}(x)} \ln(f(2^n,k))\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0(k))\mathrm{d}k\right) \\ = \int_{2^n}^{2^{n+1}} \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\times \\ \times \frac{[g_1+g_2-g]^2}{gg_1g_2}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2 dt.$$

Since the quantity $\int_{\mathcal{S}(x)} \ln(f(2^n,k))\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0(k))\mathrm{d}k$ is always positive, we deduce from the above that

$$\int_{\mathcal{S}(x)} \ln(f(2^{n+1},k))\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0(k))\mathrm{d}k \\ \geq \int_{2^n}^{2^{n+1}} \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\frac{[g_1+g_2-g]^2}{gg_1g_2}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2 dt.$$

By Lemma 25, applied to the left hand side of the above inequality, we find

$$\int_{\mathcal{S}(x)} \ln(\mathcal{F}^c(k))\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0(k))\mathrm{d}k \geq \\ \geq \int_{2^n}^{2^{n+1}} \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2) \\ \frac{[g_1+g_2-g]^2}{gg_1g_2}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2 dt, \tag{119}$$

which, after dividing both sides by 2^n, implies

$$\frac{1}{2^n}\left[\int_{\mathcal{S}(x)} \ln(\mathcal{F}^c(k))\mathrm{d}k - \int_{\mathcal{S}(x)} \ln(f_0(k))\mathrm{d}k\right] \geq \\ \geq \frac{1}{2^n}\int_{2^n}^{2^{n+1}} \int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2) \\ \frac{[g_1+g_2-g]^2}{gg_1g_2}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2 dt. \tag{120}$$

As a consequence, there exists a sequence of times $t_n \in [2^n, 2^{n+1}]$ such that

$$\lim_{n\to\infty}\left[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)} [\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2) \\ \times \frac{[g_1(t_n)+g_2(t_n)-g(t_n)]^2}{g(t_n)g_1(t_n)g_2(t_n)}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\right] = 0. \tag{121}$$

For the sake of simplicity, we denote $g(t_n)$ and $f(t_n)$ by g^n and f^n.

Step 2: The convergence.

Taking advantage of the fact $g^n \le 2N$ in the cut-off region of the operator χ_N^*, the following limit can be deduced from (121)

$$\lim_{n\to\infty}\Big[\int_{\mathcal{S}(x)\times\mathcal{S}(x)\times\mathcal{S}(x)}[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\chi_N^*\times \times\left[g_1^n+g_2^n-g^n\right]^2\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\Big]=0, \tag{122}$$

in which the product $g^n g_1^n g_2^n$ has been eliminated. Since $g^n g_1^n g_2^n$ is removed, the inequality (87) can be applied, leading to another limit

$$\lim_{n\to\infty}\int_{\mathcal{S}(x)}\sqrt{f^n\left|\mathbb{Q}_c^{N,+}[g^n]-\mathbb{Q}_c^{N,-}[g^n]\right|}\mathrm{d}k=0. \tag{123}$$

The above expression contains f^n, which can be, again, eliminated using the lower bound $f^n \ge \frac{1}{2N}$ in the cut-off region, yielding

$$\lim_{n\to\infty}\int_{\mathcal{S}(x)}\sqrt{\left|\mathbb{Q}_c^{N,+}[g^n]-\mathbb{Q}_c^{N,-}[g^n]\right|}\mathrm{d}k=0. \tag{124}$$

Replacing $\mathbb{Q}_c^{N,+}[g^n]=g^n\mathbb{L}_c^N[g^n]$ in the above formula leads to

$$\lim_{n\to\infty}\int_{\mathcal{S}(x)}\sqrt{\left|g^n\mathbb{L}_c^N-\mathbb{Q}_c^{N,-}[g^n]\right|}\mathrm{d}k=0. \tag{125}$$

Notice that $g^n\mathbb{L}_c^N=g^n\chi_N[g^n]\tilde{\mathbb{L}}_c^N$, in which $\tilde{\mathbb{L}}_c^N$ takes the following form

$$\begin{aligned}\tilde{\mathbb{L}}_c^N &:= \mathcal{G}_1^N[g^n]+\mathcal{G}_2^N[g^n]\\ &:= \int_{\mathcal{S}(x)\times\mathcal{S}(x)}\chi_N[g^n(k_1)]\chi_N[g^n(k_2)]\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}k_1\mathrm{d}k_2\\ &\quad +2\int_{\mathcal{S}(x)\times\mathcal{S}(x)}\chi_N[g^n(k_1)]\chi_N[g^n(k_2)]\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\mathrm{d}k_1\mathrm{d}k_2.\end{aligned} \tag{126}$$

Let us consider the first sequence $\{\mathcal{G}_1^N[g^n]\}$. We will show that this sequence is equicontinuous in all $L^p(\mathcal{S}(x))$ with $1\le p<\infty$. This, by the Kolmogorov-Riesz theorem [23] implies the strong convergence of $\{\mathcal{G}_1^N[g^n]\}$ towards a function $\mathcal{G}_1$ in $L^p(\mathcal{S}(x))$ with $1\le p<\infty$. To see this, let us consider any vector k' belonging to a ball $B(O,\delta)$ centered at the origin and with radius δ, and estimate the difference $\mathcal{G}_1^N[g^n](\cdot+k')-\mathcal{G}_1^N[g^n](\cdot)$ in the L^p-norm

$$\begin{aligned}&\int_{\mathcal{S}(x)}|\mathcal{G}_1^N[g^n](k+k')-\mathcal{G}_1^N[g^n](k)|^p\mathrm{d}k\\ &\quad=\int_{\mathcal{S}(x)}\Big|\int_{\mathcal{S}(x)}\Big[\chi_N[g^n(k'+k-k_1)]\delta(\omega(k')-\omega(k_1)-\omega(k'+k-k_1))-\\ &\quad-\chi_N[g^n(k-k_1)]\delta(\omega(k)-\omega(k_1)-\omega(k-k_1))\Big]\chi_N[g^n(k_1)]\mathrm{d}k_1\Big|^p\mathrm{d}k.\end{aligned} \tag{127}$$

To estimate the above quantity, we will use the triangle inequality, as follows

$$\begin{aligned}
&\int_{\mathcal{S}(x)} |\mathcal{G}_1^N[g^n](k+k') - \mathcal{G}_1^N[g^n](k)|^p \mathrm{d}k \\
&\lesssim \int_{\mathcal{S}(x)} \Big| \int_{\mathcal{S}(x)} |\chi_N[g^n(k'+k-k_1)] - \chi_N[g^n(k-k_1)]| \times \\
&\quad \times \delta(\omega(k'+k) - \omega(k_1) - \omega(k'+k-k_1)) \chi_N[g^n(k_1)] \mathrm{d}k_1 \\
&\quad + \int_{\mathcal{S}(x)} \chi_N[g^n(k-k_1)] |\delta(\omega(k'+k) - \omega(k_1) - \omega(k'+k-k_1)) \\
&\quad - \delta(\omega(k) - \omega(k_1) - \omega(k-k_1))| \chi_N[g^n(k_1)] \mathrm{d}k_1 \Big|^p \mathrm{d}k.
\end{aligned} \tag{128}$$

In the right hand side of this equality, we have the sum of two integrals inside the power of order p. To facilitate the computations, we use Young's inequality to split this into two separate integrals as

$$\begin{aligned}
&\int_{\mathcal{S}(x)} |\mathcal{G}_1^N[g^n](k+k') - \mathcal{G}_1^N[g^n](k)|^p \mathrm{d}k \\
&\lesssim \int_{\mathcal{S}(x)} \Big| \int_{\mathcal{S}(x)} |\chi_N[g^n(k'+k-k_1)] - \chi_N[g^n(k-k_1)]| \times \\
&\quad \times \delta(\omega(k'+k) - \omega(k_1) - \omega(k'+k-k_1)) \chi_N[g^n(k_1)] \mathrm{d}k_1 \Big|^p \mathrm{d}k \\
&\quad + \int_{\mathcal{S}(x)} \Big| \int_{\mathcal{S}(x)} \chi_N[g^n(k-k_1)] |\delta(\omega(k'+k) - \omega(k_1) - \omega(k'+k-k_1)) \\
&\quad - \delta(\omega(k) - \omega(k_1) - \omega(k-k_1))| \chi_N[g^n(k_1)] \mathrm{d}k_1 \Big|^p \mathrm{d}k.
\end{aligned} \tag{129}$$

We can choose δ small such that $\chi_N[g^n(k'+k-k_1)] - \chi_N[g^n(k-k_1)]$ is small, uniformly in k and k_1, thanks to the cut-off property $\frac{1}{N} \leq |f^n(k)|, |\nabla f^n(k)| \leq N$ in the cut-off region. Combining this observation, with Proposition 15, Corollary 14 and the boundedness of $\chi_N[g^n(k_1)]$, we can choose δ small enough, depending on a small $\epsilon > 0$, such that the first term on the right hand side is smaller than $\epsilon^p/2$. The second term on the right hand side can also be bounded by $\epsilon^p/2$ using Proposition 13 and the fact that $\chi_N[g^n(k-k_1)]$ and $\chi_N[g^n(k_1)]$ are both bounded by 1. As a result, for any small constant $\epsilon > 0$, we can choose δ such that for any $k' \in B(O, \delta)$,

$$\int_{\mathcal{S}(x)} |\mathcal{G}_1^N[g^n](k+k') - \mathcal{G}_1^N[g^n](k)|^p \mathrm{d}k \lesssim \epsilon^p, \tag{130}$$

which shows that the sequence $\mathcal{G}_1^N[g^n]$ is indeed equicontinuous in $L^p(\mathcal{S}(x))$ and the existence of $\sigma_1 \in L^p(\mathcal{S}(x))$ satisfying $\lim_{n\to\infty} \mathcal{G}_1^N[g^n] = \sigma_1$ in $L^p(\mathcal{S}(x))$ for all $p \in [1, \infty)$ is guaranteed by the Kolmogorov-Riesz theorem [23].

The same argument can be applied to $\mathcal{G}_2^N[g^n]$, leading to the existence of $\sigma_2 \in L^p(\mathcal{S}(x))$ satisfying $\lim_{n\to\infty} \mathcal{G}_2^N[g^n] = \sigma_2$ in $L^p(\mathcal{S}(x))$ for all $p \in [1, \infty)$ by the Kolmogorov-Riesz theorem [23]. As a result $\lim_{n\to\infty} \tilde{\mathbb{L}}_c^N = \sigma = \sigma_1 + \sigma_2$ in $L^p(\mathcal{S}(x))$ for all $p \in [1, \infty)$.

Similarly, if we define

$$\begin{aligned}
&\tilde{\mathbb{Q}}_c^{N,-}[g](k) = \tilde{\mathbb{Q}}_c^{N,-,1}[g](k) + \tilde{\mathbb{Q}}_c^{N,-,2}[g](k) + \tilde{\mathbb{Q}}_c^{N,-,3}[g](k) := \\
&= 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N[1/g](k_1)\chi_N[1/g](k_2)[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)g_1\mathrm{d}k_1\mathrm{d}k_2 \\
&+ 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N[1/g](k_1)\chi_N[1/g](k_2)[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_1\mathrm{d}k_1\mathrm{d}k_2 \\
&- 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \chi_N[1/g](k_1)\chi_N[1/g](k_2)[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_2\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned} \tag{131}$$

the Kolmogorov-Riesz theorem [23] can be used in the same manner to deduce the existence of a function ς such that we also have $\lim_{n\to\infty}\tilde{\mathbb{Q}}_c^{N,-}[g^n] = \varsigma$ in $L^p(\mathcal{S}(x))$ for all $p\in[1,\infty)$.

Now, the fact that $\lim_{n\to\infty}\tilde{\mathbb{Q}}_c^{N,-}[g^n] = \varsigma$ and $\lim_{n\to\infty}\tilde{\mathbb{L}}_c^N = \sigma$ can be used to replace the quantity $\mathbb{Q}_c^{N,-}[g^n]$ by ς and the quantity $\tilde{\mathbb{L}}_c^N$ by σ in (123) and (125) to have

$$\lim_{n\to\infty}\int_{\mathcal{S}(x)}\sqrt{|\sigma\chi_N[f^n] - f^n\chi_N[f^n]\varsigma|}\mathrm{d}k = 0, \tag{132}$$

and

$$\lim_{n\to\infty}\int_{\mathcal{S}(x)}\sqrt{|g^n\chi_N[g^n]\sigma - \varsigma\chi_N[f^n]|}\mathrm{d}k = 0. \tag{133}$$

Due to its boundedness, the sequences $\{g^n\chi_N[f^n]\}$, $\{f^n\chi_N[f^n]\}$ and $\{\chi_N[f^n]\}$ converge weakly to g_N^∞, f_N^∞ and ξ_N^∞ in $L^1(\mathcal{S}(x))$, it follows immediately that $g_N^\infty\sigma = \xi_N^\infty\varsigma$ and $\xi_N^\infty\sigma = f_N^\infty\varsigma$.

By a similar argument as above, $\{\chi_N[f^n]\}$ is also equicontinuous in $L^p(\mathcal{S}(x))$ and then $\lim_{n\to\infty}\chi_N[f^n] = \xi_N^\infty$ in $L^p(\mathcal{S}(x))$ for all $p\in[1,\infty)$ by the Kolmogorov-Riesz theorem [23]. As a consequence,

$$\begin{aligned}
\varsigma(k) =& 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k_1)\xi_N^\infty(k_2)[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2) \\
& g_N^\infty(k_1)\mathrm{d}k_1\mathrm{d}k_2 \\
& + 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k_1)\xi_N^\infty(k_2)[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2) \\
& g_N^\infty(k_1)\mathrm{d}k_1\mathrm{d}k_2 \\
& - 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k_1)\xi_N^\infty(k_2)[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2) \\
& g_N^\infty(k_2)\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned}$$

and

$$\begin{aligned}
\sigma(k) =& \int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k_1)\xi_N^\infty(k_2)\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}k_1\mathrm{d}k_2 \\
& + 2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k_1)\xi_N^\infty(k_2)\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned}$$

which can be combined with (133) and the fact that $\{g^n\chi_N[f^n]\}$, $\{f^n\chi_N[f^n]\}$ converge weakly to g_N^∞, f_N^∞ to give

$$\begin{aligned}
&\int_{\mathcal{S}(x)\times\mathcal{S}(x)} g_N^\infty(k)\xi_N^\infty(k)\xi_N^\infty(k_1)\xi_N^\infty(k_2)\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\mathrm{d}k_1\mathrm{d}k_2\\
&\quad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} g_N^\infty(k)\xi_N^\infty(k)\xi_N^\infty(k_1)\xi_N^\infty(k_2)\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)\mathrm{d}k_1\mathrm{d}k_2\\
&\quad=2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k)\xi_N^\infty(k_1)\xi_N^\infty(k_2)[\omega\omega_1\omega_2]^{-1}\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)g_N^\infty(k_1)\mathrm{d}k_1\mathrm{d}k_2\\
&\quad+2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k)\xi_N^\infty(k_1)\xi_N^\infty(k_2)[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_N^\infty(k_1)\mathrm{d}k_1\mathrm{d}k_2\\
&\quad-2\int_{\mathcal{S}(x)\times\mathcal{S}(x)} \xi_N^\infty(k)\xi_N^\infty(k_1)\xi_N^\infty(k_2)[\omega\omega_1\omega_2]^{-1}\delta(k_1-k-k_2)\delta(\omega_1-\omega-\omega_2)g_N^\infty(k_2)\mathrm{d}k_1\mathrm{d}k_2,
\end{aligned} \tag{134}$$

for a.e. k in $\mathcal{S}(x)$.

From (134), we deduce that

$$g_N^\infty(k)\xi_N^\infty(k)=g_N^\infty(k_1)\xi_N^\infty(k_1)+g_N^\infty(k_2)\xi_N^\infty(k_2),$$

when $k=k_1+k_2$ and $\omega(k)=\omega(k_1)+\omega(k_2)$, for a.e. k in $\mathcal{S}(x)$. The proofs of Proposition 19 and Lemma 21 can then be redone, yielding $g_N^\infty(k)\xi_N^\infty(k)=A_N\omega(k)=:\mathcal{E}^c(k)>0$ for some constant $A_N\in\mathbb{R}$. These constants are subjected to the conservation of energy

$$\int_{\mathcal{S}(x)}\frac{\omega(k)}{A_N\omega(k)}\mathrm{d}k=\lim_{n\to\infty}\int_{\mathcal{S}(x)}\omega(k)f^n\chi_N[f^n]\mathrm{d}k=:E_x^N. \tag{135}$$

In addition, we have $f_N^\infty=\frac{1}{A_N\omega(k)}$. Since $\lim_{N\to\infty}E_x^N=E_x$, when N is large enough $\frac{1}{N}<g_N^\infty(k),\ f_N^\infty(k)<N$ for all $k\in\mathcal{S}(x)$. As a consequence, g^n and f^n converge almost everywhere to $g_N^\infty(k)$, and $f_N^\infty(k)$.

The fact that f^n converges to $f_N^\infty(k)$ almost everywhere, when N is sufficiently large, ensures the existence of $N_0>0$ such that $f_N^\infty(k)=f_M^\infty(k)$ for all $N,M>N_0$. Passing to the limits $N\to\infty$ in (136), we find $A_N=A$ for all $N>N_0$, with

$$\int_{\mathcal{S}(x)}\frac{\omega(k)}{A\omega(k)}\mathrm{d}k\ =E_x. \tag{136}$$

As a result,

$$\lim_{n\to\infty}f^n(k)=\frac{1}{A\omega(k)}=:\mathcal{F}^c$$

almost everywhere on $\mathcal{S}(x)$, which then implies

$$\liminf_{n\to\infty}\int_{\mathcal{S}(x)}\ln[f]\mathrm{d}k\geq\int_{\mathcal{S}(x)}\ln[\mathcal{F}^c]\mathrm{d}k,$$

by Fatou's Lemma. Therefore, due to Lemma 25

$$\lim_{n\to\infty}[S_c[\mathcal{F}^c]-S_c[f^n]]=0,$$

leading to

$$\lim_{t\to\infty}[S_c[\mathcal{F}^c]-S_c[f(t)]]=0.$$

By (118), we finally obtain

$$\lim_{t\to\infty} \|f - \mathcal{F}^c\|_{L^1(\mathcal{S}(x))} = 0.$$

Step 3: Additional assumption $f(t,k) < M^*$ **for all** $t \in [0,\infty)$ **and for all** $k \in \mathcal{S}(x)$**.** Suppose, in addition, that $f(t,k) < M^*$ for all $t \in [0,\infty)$. By Egorov's theorem, for all $\delta > 0$, there exists a set $\mathcal{V}_\delta$, whose measure $m(\mathcal{V}_\delta)$ is smaller than δ and f^n converges uniformly to $f^\infty(k)$ on $\mathcal{S}(x)\backslash\mathcal{V}_\delta$. Since $\frac{1}{N} < f_N^\infty(k) < N$, there exists an integer n_δ such that for all $n > n_\delta$, the inequality $\frac{1}{N} < f^n(k) < N$ holds true for all $k \in \mathcal{S}(x)\backslash\mathcal{V}_\delta$. As a consequence, for each $\epsilon > 0$

$$\|f - \mathcal{F}^c\|_{L^p(\mathcal{S}(x))} \le C\|f - \mathcal{F}^c\|_{L^\infty(\mathcal{S}(x)\backslash\mathcal{V}_\delta)} + Cm(\mathcal{V}_\delta)^{\frac{1}{p}} \le C\|f - \mathcal{F}^c\|_{L^\infty(\mathcal{S}(x)\backslash\mathcal{V}_\delta)} + C\delta^{\frac{1}{p}},$$

where C is a universal constant, for all $1 < p < \infty$.

For any $\epsilon > 0$, we can choose $\delta > 0$ and a time t_δ such that for $t > t_\delta$, $C\delta^{\frac{1}{p}} < \epsilon/2$ and $C\|f - \mathcal{F}^c\|_{L^\infty(\mathcal{S}(x)\backslash\mathcal{V}_\delta)} < \epsilon/2$. That implies the strong convergence of f towards $\mathcal{F}^c$ in $L^p(\mathcal{S}(x)$ for all $1 < p < \infty$.

Now, if $f_0(k) > 0$ for all $k \in \mathcal{S}(x)$ and $f(t,k) < M^*$ for all $t \in [0,\infty)$ and for all $k \in \mathcal{S}(x)$, by Proposition 23, there exists a constant M_* such that $f(t,k) > M_*$ for all $t \in [0,\infty)$ and for all $k \in \mathcal{S}(x)$. □

4.3 Proof of Theorem 3

The proof of Theorem 3 follows from Proposition 24 and Proposition 7.

5 Appendix

5.1 Appendix A: Proof of Lemma 25

Define the functional

$$\Psi_t(f,\mathcal{F}^c) = [\mathcal{F}^c + t(f - \mathcal{F}^c)]^2.$$

It follows from the mean value theorem that

$$0 \le \int_0^1 \frac{(1-t)(f - \mathcal{F}^c)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t = s_c[\mathcal{F}^c] - s_c[f] + s_c'[\mathcal{F}^c](f - \mathcal{F}^c).$$

Since $s'(y) = 1/y$, we find $s'[\mathcal{F}^c(k)] = a_x\omega(k)$. That leads to

$$0 \le \int_0^1 \frac{(1-t)(f - \mathcal{F}^c)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t = s_c[\mathcal{F}^c] - s_c[f] + (a_x\omega(k))(f - \mathcal{F}^c).$$

Integrating both sides of the above inequality on $\mathcal{S}(x)$ yields

$$\begin{aligned}0 &\le \int_{\mathcal{S}(x)}\int_0^1 \frac{(1-t)(f - \mathcal{F}^c)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t\mathrm{d}k\\ &= \int_{\mathcal{S}(x)} s_c[\mathcal{F}^c]\mathrm{d}k - \int_{\mathcal{S}(x)} s_c[f]\mathrm{d}k + \int_{\mathcal{S}(x)} (a_x\omega(k))(f - \mathcal{F}^c)\mathrm{d}k,\end{aligned}$$

which, by the fact that

$$\int_{\mathcal{S}(x)} (a_x\omega(k))(f-\mathcal{F}^c)\mathrm{d}k = 0,$$

implies

$$0 \leq \int_{\mathcal{S}(x)} \int_0^1 \frac{(1-t)(f-\mathcal{F}^c)^2}{\Psi_t(f,\mathcal{F}^c)} \mathrm{d}t\mathrm{d}k \leq S_c[\mathcal{F}^c] - S_c[f]. \tag{137}$$

Observing that

$$(\mathcal{F}^c - f)_+ = 2\int_0^1 \frac{\sqrt{1-t}(\mathcal{F}^c-f)_+}{\sqrt{\Psi_t(f,\mathcal{F}^c)}} \sqrt{(1-t)\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t,$$

and applying Hölder's inequality to the right hand side, we obtain the following inequality

$$(\mathcal{F}^c - f)_+ \leq 2\left[\int_0^1 \frac{(1-t)(\mathcal{F}^c-f)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t\right]^{\frac{1}{2}} \left[\int_0^1 (1-t)\Psi_t(f,\mathcal{F}^c)\mathrm{d}t\right]^{\frac{1}{2}}.$$

Now, observe that for $k \in \mathcal{S}(x)$ satisfying $\mathcal{F}^c(k) > f(k)$, then

$$0 < \Psi_t(f,\mathcal{F}^c)(k) \leq [\mathcal{F}^c(k)]^2$$

for all $t \in [0, 1]$. This fact can reduce the above inequality to

$$(\mathcal{F}^c - f)_+ \leq 2\left[\int_0^1 \frac{(1-t)(\mathcal{F}^c-f)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t\right]^{\frac{1}{2}} \left[\int_0^1 (1-t)[\mathcal{F}^c(k)]^2\mathrm{d}t\right]^{\frac{1}{2}},$$

which, by integrating in k

$$\int_{\mathcal{S}(x)} (\mathcal{F}^c - f)_+\mathrm{d}k \leq 2\int_{\mathcal{S}(x)} \left[\int_0^1 \frac{(1-t)(\mathcal{F}^c-f)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t\right]^{\frac{1}{2}} \left[\int_0^1 (1-t)[\mathcal{F}^c(k)]^2\mathrm{d}t\right]^{\frac{1}{2}} \mathrm{d}k,$$

and applying Hölder's inequality to the right hand side, gives

$$\int_{\mathcal{S}(x)} (\mathcal{F}^c - f)_+\mathrm{d}k \leq 2\left[\int_{\mathcal{S}(x)}\int_0^1 \frac{(1-t)(\mathcal{F}^c-f)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t\mathrm{d}k\right]^{\frac{1}{2}} \left[\int_{\mathcal{S}(x)}\int_0^1 (1-t)[\mathcal{F}^c(k)]^2\mathrm{d}t\mathrm{d}k\right]^{\frac{1}{2}}.$$

Indeed, the second term with the bracket on the right hand side can be computed explicitly, that implies

$$\int_{\mathcal{S}(x)} (\mathcal{F}^c - f)_+\mathrm{d}k \lesssim \left[\int_{\mathcal{S}(x)}\int_0^1 \frac{(1-t)(\mathcal{F}^c-f)^2}{\Psi_t(f,\mathcal{F}^c)}\mathrm{d}t\mathrm{d}k\right]^{\frac{1}{2}}.$$

The above inequality can be combined with (137) to become

$$\int_{\mathcal{S}(x)} (\mathcal{F}^c - f)_+\mathrm{d}k \lesssim \left[S_c[\mathcal{F}^c] - S_c[f]\right]^{\frac{1}{2}}.$$

Using the boundedness of the dispersion relation $\omega(k)$, we find

$$\int_{\mathcal{S}(x)} (\mathcal{F}^c - f)_+\omega(k)\mathrm{d}k \lesssim \int_{\mathcal{S}(x)} (\mathcal{F}^c - f)_+\mathrm{d}k \lesssim \left[S_c[\mathcal{F}^c] - S_c[f]\right]^{\frac{1}{2}}.$$

Now, from the identity

$$|f - \mathcal{F}^c| = f - \mathcal{F}^c + 2(\mathcal{F} - f)_+,$$

the above gives

$$\int_{\mathcal{S}(x)} |f-\mathcal{F}^c|\omega(k)\mathrm{d}k = \int_{\mathbb{T}^3}(f-\mathcal{F}^c)\omega(k)\mathrm{d}k + \int_{\mathcal{S}(x)} 2(\mathcal{F}^c-f)_+\omega(k)\mathrm{d}k$$
$$\lesssim \int_{\mathcal{S}(x)}(f-\mathcal{F}^c)\omega(k)\mathrm{d}k + 2\left[S_c[\mathcal{F}^c]-S_c[f]\right]^{\frac{1}{2}}.$$

From the hypothesis

$$\int_{\mathcal{S}(x)}(f-\mathcal{F}^c)\omega(k)\mathrm{d}k = 0,$$

we then infer from the above inequality that

$$\int_{\mathcal{S}(x)} |f-\mathcal{F}^c|\omega(k)\mathrm{d}k \lesssim \left[S_c[\mathcal{F}^c]-S_c[f]\right]^{\frac{1}{2}}.$$

Using the fact that $\omega(k) \geq \omega_0$, we obtain

$$\int_{\mathcal{S}(x)} |f-\mathcal{F}^c|\mathrm{d}k \lesssim \left[S_c[\mathcal{F}^c]-S_c[f]\right]^{\frac{1}{2}}.$$

Acknowledgements B. R. is funded in part by a grant from the Simons Foundation (No. 430192). A. S. is in part by the Simons Foundation grant number 851844 and NSF-DMS grant number 220931. M.-B. T is funded in part by a Humboldt Fellowship, NSF CAREER DMS-2303146, and NSF Grants DMS-2204795, DMS-2305523, DMS-2306379. We would like to thank Prof. Herbert Spohn, Prof. Gigliola Staffilani and Prof. Enrique Zuazua for fruitful discussions on the topic. We would like to thank the referees for the important remarks.

Data Availability There is no Data used in this work.

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