FORMATION OF CONDENSATIONS FOR NON-RADIAL SOLUTIONS TO 3-WAVE KINETIC EQUATIONS

GIGLIOLA STAFFILANI AND MINH-BINH TRAN

ABSTRACT. We consider in this work a 2-dimensional 3-wave kinetic equation describing the dynamics of the thermal cloud outside a Bose-Einstein Condensate. We construct global non-radial mild solutions for the equation. Those mild solutions are the summation of Dirac masses on circles. We prove that in each spatial direction, either Dirac masses at the origin, which are the so-called Bose-Einstein condensates, can be formed in finite time or the solutions converge to Bose-Einstein condensates as time evolves to infinity. We also describe a dynamics of the formation of the Bose-Einstein condensates latter case. In this case, on each direction, the solutions accumulate around circles close to the origin at growth rates at least linearly in time.

CONTENTS

1. INTRODUCTION

1.1. **The 3-wave kinetic equation.** Quantum gases in lower dimensions [45, 55] have garnered significant attention as model systems for exploring diverse phenomena. Compared to their three-dimensional counterparts, dimensionally reduced systems can display remarkably novel properties. Recent experiments have demonstrated the occurrence of Bose-Einstein condensations (BECs) in several one- and two-dimensional systems [5, 32, 39, 52, 61]. In this

Date: March 21, 2025.

G.S. is funded in part by the NSF grants DMS-2052651, DMS-2306378 and the Simons Foundation through the Simons Collaboration on Wave Turbulence.

M.-B. T is funded in part by a Humboldt Fellowship, NSF CAREER DMS-2303146, and NSF Grants DMS-2204795, DMS-2305523, DMS-2306379.

work, we are interested in the analysis of the following 3-wave kinetic equation, coming from the theory of 2-dimensional low temperature trapped bose gases [56]

$$\frac{\partial f}{\partial \tau}(\tau,k) \;=\; Q[f](\tau,k)\,, \qquad f(0,k)=f_{in}(k) \quad k\in\mathbb{R}^2, \quad \tau\in[0,\infty). \tag{1}$$

where

$$\begin{aligned} Q[f] :=& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathrm{d}k_1\mathrm{d}k_2 \mathcal{W}(|k|,|k_1|,|k_2|)\delta\left(\omega(k)-\omega(k_1)-\omega(k_2)\right)\delta(k-k_1-k_2) \\ & \times \left[f(k_1)f(k_2)-(f(k_1)+f(k_2))f(k)\right], \\ & -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathrm{d}k_1\mathrm{d}k_2 2\mathcal{W}(|k|,|k_1|,|k_2|)\delta\left(\omega(k_1)-\omega(k)-\omega(k_2)\right)\delta(k_1-k-k_2) \\ & \times \left[f(k)f(k_2)-(f(k)+f(k_2))f(k_1)\right] \end{aligned} \tag{2}$$

The collision kernel is given by

$$\mathcal{W}(|k|,|k_1|,|k_2|) \;=\; [|k|+|k_1|+|k_2|]^{-1}\frac{1}{\pi}. \tag{3}$$

The dispersion relation $\omega(k)$ is given by the phonon dispersion relation

$$\omega(k)=|k|. \tag{4}$$

Existence and uniqueness of strong radial solutions to the 3-dimensional version of (1) have been studied in [2]. Maxwellian lower bounds and the convergence to equilibrium and the energy cascade phenomenon for the 3-dimensional case has also been done in [53, 28] and [63]. Numerical schemes for the radial and 3-dimensional version have also been provided in [1, 70, 71, 72]. Important and deep studies on the radial case of the 3-dimensional version, but for the linearized case and related models have been done in [13, 25, 26, 31]. However, all of the above mentioned works are for radial solutions.

In our previous works [65, 66], we consider the 4-wave kinetic equation

$$\partial_\tau f \;=\; \mathfrak{Q}\left[f\right], \quad f(0,k)=f_{in}(k), \quad k\in\mathbb{R}^3, \quad \tau\in[0,\infty), \tag{5}$$

$$\begin{aligned} \mathfrak{Q}\left[f\right] \;=\; & \iiint_{\mathbb{R}^9} \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3 \mathfrak{W}(|k|,|k_1|,|k_2|,|k_3|)\delta(k+k_1-k_2-k_3) \\ & \times\delta(\omega+\omega_1-\omega_2-\omega_3)[f_2f_3(f_1+f)-ff_1(f_2+f_3)], \end{aligned} \tag{6}$$

where $\omega,\omega_1,\omega_2,\omega_3$ is the shorthand notation for $\omega(k),\omega(k_1),\omega(k_2),\omega(k_3)$, and f,f_1,f_2,f_3 is the shorthand notation for $f(k),f(k_1),f(k_2),f(k_3)$. Extending of the pioneering work of Escobedo and Velazquez [29, 30], we prove that the solution of (5) forms a delta function at the origin (Bose-Einstein condensates) as time evolves, and the energy of the solution moves towards high frequencies, under the assumptions that (i) the kernel $\mathfrak{W}(|k|,|k_1|,|k_2|,|k_3|)$ is not singular, (ii) $\omega(|k|)$ is of the general form but different from $|k|$ (one example considered in these works is $\omega(k)=|k|^\alpha$ with $1<\alpha\le 2$), and (iii) the solution is isotropic $f(t,k)=f(t,|k|)$. Motivated by the previous works [65, 66] (and inspired by [29, 30]), we aim to remove those assumptions, but in the context of the 3-wave kinetic equation (1). It is therefore the goal of this work to construct non-radial solutions of (1), in which the kernel (3) is singular and the dispersion relation is of the phonon type (4). We put the initial condition f_{in} on circles (see Definition 1 for the definition of the circular lattice) and the origin is not in the support of f_{in}. We show that as time evolves, the solutions remain to be summations of Dirac masses on the circular lattice. Moreover, either the solutions form Bose-Einstein condensates in finite time, or when time goes to infinity, they converge to Dirac functions at the origin in all spatial directions, as they are non-radial. We are also able to describe a dynamics of the formation of those Bose-Einstein condensates in the latter case: The concentrations of the solutions around circles close to the origin for each direction are growing at least linearly in time (see Remark 3). To the best of our knowledge, this work appears to be the only available example for the formation of Bose-Einstein condensates and its dynamics for non-radial solutions of wave kinetic equations.

The 3-wave kinetic equations play a crucial role in the theory of weak turbulence and have been extensively studied in various contexts. They have been analyzed in [33] for stratified ocean flows, in [62] for Bose-Einstein condensates, in [14, 33, 67] for phonon interactions in anharmonic crystal lattices, and in [60] for beam waves. Additionally, the analysis of four-wave kinetic equations was pioneered by Escobedo and Velazquez in [29, 30] and has been further explored in several other works [3, 10, 20, 27, 34, 35, 36, 37, 51, 65, 66]. The 6-wave kinetic equations have also been analyzed recently in [54].

The rigorous derivation problem of wave kinetic equations has been recently studied in [8, 9, 12, 11, 19, 23, 24, 22, 21, 38, 40, 43, 44, 49, 50, 64] and the problem has been completely solved in the work of Deng and Hani [17, 16, 18].

1.2. **Physical context of the model** (1)-(2)-(3)-(4)**.** The realization of Bose-Einstein condensation (BEC) in trapped atomic vapors of ^{23}Na [15], ^{87}Rb [4] and ^{7}Li [6] has started a period of immense theoretical and experimental research. The experimental results need a theoretical support which is a coupling system between the coupled non-equilibrium dynamics of both the BEC and the thermal cloud of the trapped Bose gas. The spatially homogeneous quantum Boltzmann equation for the density $f(\tau,k)$ of the non-condensate atoms reads

$$\frac{\partial f}{\partial \tau} \;=\; C_{12}[f] \;+\; C_{22}[f] \;+\; C_{31}[f], \quad f(0,k) \;=\; f_{in}(k), \tag{7}$$

where the forms of C_{12}, C_{22}, C_{31} are given explicitly below

$$\begin{aligned} C_{12}[f](k) = {} & \frac{4\pi g^2 n_c}{\hbar(2\pi)^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\delta(k_1-k_2-k_3)\\ & \times(\delta(k-k_1)-\delta(k-k_2)-\delta(k-k_3))\delta(\omega_1-\omega_2-\omega_3)\\ & \times K^{12}(k_1,k_2,k_3)\Big[f_2f_3(f_1+1)-f_1(f_2+1)(f_3+1)\Big], \end{aligned} \tag{8}$$

$$\begin{aligned} C_{22}[f](k) = {} & \frac{\pi g^2}{\hbar(2\pi)^6}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\\ & \times(\delta(k-k_1)+\delta(k-k_2)-\delta(k-k_3)-\delta(k-k_4))\\ & \times\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(k_1+k_2-k_3-k_4)K^{22}(k_1,k_2,k_3,k_4)\\ & \times\Big[f_3f_4(f_2+1)(f_1+1)-f_1f_2(f_3+1)(f_4+1)\Big], \end{aligned} \tag{9}$$

and

$$\begin{aligned} C_{31}[f](t,p) = {} & \frac{3\pi g^2}{\hbar(2\pi)^6}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4 \qquad (10)\\ & \times(\delta(k-k_1)-\delta(k-k_2)-\delta(k-k_3)-\delta(k-k_4)) \end{aligned}$$

$$\times\delta(k_1-k_2-k_3-k_4)\delta(\omega_1-\omega_2-\omega_3-\omega_4)K^{31}(k_1,k_2,k_3,k_4)$$
$$\times\Big[f_3f_4f_2(f_1+1)-f_1(f_2+1)(f_3+1)(f_4+1)\Big],$$

in which ω_i, f_i stand for $\omega(k_i)$, $f(k_i)$, $k\in\mathbb{R}^d$ is the d-dimensional non-zero momentum variable. The dispersion relation $\omega(k)$ is the Bogoliubov dispersion relation $\left[\frac{gn_c\hbar^2}{m}p^2+(\frac{\hbar^2p^2}{2m})^2\right]^{\frac{1}{2}}$, but at very low temperature, can be approximated by the phonon dispersion relation (4) (see [57, 58]). The quantity $\hbar$ is the reduced Planck constant, g is the interaction coupling constant, n_c is the density of the condensate and m is the mass of the particles. The quantities

$$K^{12}(k_1,k_2,k_3)=K^{12}(k_2,k_1,k_3)=K^{12}(k_2,k_3,k_1),$$

$$K^{22}(k_1,k_2,k_3,k_4)=K^{22}(k_2,k_1,k_3,k_4)=K^{22}(k_2,k_3,k_1,k_4)=K^{22}(k_2,k_3,k_4,k_1),$$

$$K^{31}(k_1,k_2,k_3,k_4)=K^{31}(k_2,k_1,k_3,k_4)=K^{31}(k_2,k_3,k_1,k_4)=K^{31}(k_2,k_3,k_4,k_1)$$

are symmetric, explicit, positive functions that are the kernels of the collision operators.

Equation (7) has a long history. In the pioneering work [46, 48, 47], Kirkpatrick and Dorfman began developing a theoretical framework, which introduces a mean-field kinetic equation for the thermal cloud, describing relaxation through "collisions" between excitations. Kirkpatrick-Dorfman's framework was later extended by Zaremba, Nikuni, and Griffin [74], who formulated a fully coupled system consisting of a quantum Boltzmann equation for the density function of the thermal cloud and an equation for the BEC wavefunction. The Zaremba-Nikuni-Griffin model has been highly successful in decribing a wide range of BEC phenomena [41]. The model was also independently derived by Pomeau, Brachet, Metens, and Rica [56, 73]. In the model, two primary types of collisional processes are considered:

- The $1\leftrightarrow 2$ interactions between the condensate and excited atoms, which is described by the collision operator (8).
- The $2\leftrightarrow 2$ interactions among the excited atoms themselves, which is described by the collision operator (9).

However, in a later work [42], Reichl and Gust argued that a third, missing collisional process, involving $1\leftrightarrow 3$ interactions between excitations needs to be taken into account. However, a concise mathematical justification for the existence of the missing collision operator C_{31} was a challenging open problem for several years until it was resolved in a recent work by Pomeau and Tran [68] (see also the discussion in [69]). Experimental evidences for this new collision operator have also been done [58].

However, considering all the 3 collision operators C_{12},C_{22},C_{31} is a too complicated problem since both operators C_{22},C_{31} are 4-wave kinetic ones while C_{12} is of 3-wave kinetic type. It is therefore a common practice that one omits the 2 collision operators C_{22},C_{31} in (7) for mathematical purposes (see for instance [13, 25, 26, 31])

$$\frac{\partial f}{\partial\tau}(\tau,k) \;=\; n_c\overline{C}_{12}[f](\tau,k)\,,\qquad f(0,k)=f_{in}(k),\tag{11}$$

and the collision operator is defined as

$$\begin{aligned}\overline{C}_{12}[f] := & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}k_1 \mathrm{d}k_2 K^{12}(k_1,k_2,k_3)\frac{1}{\pi}\delta\left(\omega(k)-\omega(k_1)-\omega(k_2)\right)\delta(k-k_1-k_2) \\ & \times \left[f(k_1)f(k_2)-(f(k_1)+f(k_2)+1)f(k)\right] \\ & -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}k_1 \mathrm{d}k_2 K^{12}(k_1,k_2,k_3)\frac{1}{\pi}\delta\left(\omega(k_1)-\omega(k)-\omega(k_2)\right)\delta(k_1-k-k_2) \\ & \times \left[f(k)f(k_2)-(f(k)+f(k_2)+1)f(k_1)\right],\end{aligned} \tag{12}$$

where we have normalized all the constants to be zero.

Assuming that a lot of the particles are condensed into the BEC and the BEC is quite stable, we can suppose n_c is a (big) constant, that can be normalized to be 1 by a time scaling

$$\tau \to \frac{\tau}{n_c}.$$

Next, we perform a common simplification strategy to Equation (11), which is to keep only the second order terms while omitting first order terms in $\left[f(k_1)f(k_2)-(f(k_1)+f(k_2)+1)f(k)\right]$, $\left[f(k)f(k_2)-(f(k)+f(k_2)+1)f(k_1)\right]$ and reduce them to $\left[f(k_1)f(k_2)-(f(k_1)+f(k_2))f(k)\right]$ and $\left[f(k)f(k_2)-(f(k)+f(k_2))f(k_1)\right]$, we get (1)-(2), where d is replaced by 2 and $K^{12}(k_1,k_2,k_3)$ is replaced by

$$K^{12}(k_1,k_2,k_3) \longrightarrow \frac{1}{\pi}[|k|+|k_1|+|k_2|]^{-1}. \tag{13}$$

Note that the constant $\frac{1}{\pi}$ is put so that we simplify the unnecessary constants in later computations and does not play any significant role.

Acknowledgment We would like to express our gratitude to J. J.-L. Velazquez, H. Spohn, J. Lukkarinen for fruitful discussions on the topics. We would like to express our gratitude to R. Grande and M. Dolce for several useful suggestions to improve the quality of the manuscript.

2. The Settings and main results

2.1. Measure space. Let $\Xi \geq 3$ be a fixed prime number and set

$$\begin{aligned}\Upsilon_\eta(\mu,\nu) \ = \ \Xi^{-\eta}\Upsilon(\mu,\nu) \ = \ \Xi^{-\eta}(\Xi\mu-\nu) \quad ; \\ \mu \ = 1,2,3,\cdots \quad ; \quad \nu = 1,\cdots,\Xi-1; \quad \eta = 0,1,2,3,\cdots\end{aligned} \tag{14}$$

We also introduce the following sets

$$\begin{aligned}\Theta_\eta \ &= \ \{\Upsilon_\eta(\mu,\nu); \quad \mu = 1,2,3,\cdots \quad ; \quad \nu = 1,\cdots,\Xi-1\ \} \quad , \quad \eta = 0,1,2,3,\cdots; \\ \Lambda_\alpha \ &= \ \bigcup_{\alpha\geq\eta}\Theta_\eta, \quad \Theta_{-1} \ = \ \{0\}, \quad \Lambda \ = \ \Lambda_0 \bigcup \Theta_{-1}.\end{aligned} \tag{15}$$

We denote

$$\mathbb{S} \ = \left\{\widehat{k}\in\mathbb{R}^2 \mid |\widehat{k}| = 1\right\}. \tag{16}$$

For a vector $k \in \mathbb{R}^2$, we write $k = |k|\widehat{k}$, with $\widehat{k}\in\mathbb{S}$. We then can identify $\mathbb{R}^2$ with $[0,\infty)\times\mathbb{S}$.

Definition 1. *We call $\Lambda\times\mathbb{S}$ a circular lattice. Let $\mathfrak{C} > 1$, $\gamma \in (0,1]$ be fixed constants. We denote by $\mathcal{M}_+([0,\infty))$ the space of positive Radon measure defined on $[0,\infty)$ and we endow $\mathcal{M}_+([0,\infty))$ with the standard weak topology* [7, 59]. *We denote as $\mathfrak{S}$ the space of measures $f(|k|\widehat{k}) \in \mathcal{M}_+([0,\infty))\times L^\infty(\mathbb{S})$ defined on the circular lattice $(|k|,\widehat{k})\in\Lambda_0\times\mathbb{S}$ such that*

(i) *$\forall f\in\mathfrak{S}$, $k=|k|\widehat{k}\in\mathbb{R}^2=[0,\infty)\times\mathbb{S}$, we have*

$$f(k) \ = \ \sum_{\eta=0}^{\infty}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\delta_{\{|k|=\Upsilon_\eta(\mu,\nu)\}}f_{\Upsilon_\eta(\mu,\nu)}(\widehat{k}) \ + \ \delta_{\{|k|=0\}}f_{-1}(\widehat{k}), \tag{17}$$

for $f_{\Upsilon_\eta(\mu,\nu)}(\widehat{k}), f_{-1}(\widehat{k})\in L^\infty(\mathbb{S})$.

(ii) *We set*

$$f_\eta(|k|\widehat{k}) = \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\delta_{\{|k|=\Upsilon_\eta(\mu,\nu)\}}f_{\Upsilon_\eta(\mu,\nu)}(\widehat{k}). \tag{18}$$

Thus,

$$f \ = \ \sum_{\eta=0}^{\infty}f_\eta \ + \ f_{-1} \tag{19}$$

in which for $\eta\geq 0$

$$\int_{[0,\infty)\setminus\Theta_\eta}\mathrm{d}|k|f_\eta(|k|\widehat{k}) \ = 0, \tag{20}$$

for a.e. $\widehat{k}\in\mathbb{S}$.

We impose the condition,

$$\|f\|_{\mathfrak{S}}\equiv\max\left\{\sup_{\eta\geq 0}\mathfrak{C}^\eta\sup_{\widehat{k}\in\mathbb{S}}\left(\int_{\Theta_\eta}\mathrm{d}|k|\left|f_\eta(|k|\widehat{k})\right|\right),\sup_{\widehat{k}\in\mathbb{S}}\left|f_{-1}(\widehat{k})\right|\right\}<\infty. \tag{21}$$

We write, for $f_{in}(k)=f(0,k)\in\mathfrak{S}$

$$f_{in}(k) \ = \ \sum_{\eta=0}^{\infty}f_\eta(0,k) \tag{22}$$

in which

$$\int_{[0,\infty)\setminus\Theta_\eta}\mathrm{d}|k|f_\eta(0,|k|\widehat{k}) \ = 0, \tag{23}$$

for a.e. $\widehat{k}\in\mathbb{S}$. The following assumptions are imposed on the initial condition.

- There exists constants $\mathscr{C}_1>\mathscr{C}_2>0$ and $\mathscr{C}_3>0$ such that for $\rho\in\mathbb{Z},\rho\geq 0$

$$f_\rho(0,|k|\widehat{k}) = \delta_{\{|k|=\Xi^{-\rho}\}}f_{\Xi^{-\rho}}(0,\widehat{k})$$

  with

$$\mathscr{C}_2\frac{\mathscr{C}_3^\rho}{(\rho!)^\gamma} \ \leq \ f_{\Xi^{-\rho}}(0,\widehat{k})\leq \ \mathscr{C}_1\frac{\mathscr{C}_3^\rho}{(\rho!)^\gamma},$$

  for a.e. $\widehat{k}\in\mathbb{S}$.

  In other words,

$$\mathscr{C}_2\frac{\mathscr{C}_3^\rho}{(\rho!)^\gamma} \ \leq \ \int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(0,|k|\widehat{k}) \ = \ \int_{\{|k|=\Xi^{-\rho}\}}\mathrm{d}|k|f_\rho(0,|k|\widehat{k}) \ \leq \ \mathscr{C}_1\frac{\mathscr{C}_3^\rho}{(\rho!)^\gamma}, \tag{24}$$

  for a.e. $\widehat{k}\in\mathbb{S}$.

- Moreover, for a.e. $\widehat{k} \in \mathbb{S}$

$$\int_{\{|k|=0\}} \mathrm{d}|k| f_{in}(|k|\widehat{k}) \ = \ 0, \quad \frac{\mathscr{C}_1}{\mathscr{C}_2\mathscr{C}_3} > 10. \tag{25}$$

We have the following basic lemma, whose proof can be found in the Appendix.

Lemma 1. *For any positive constant $R > 0$, we define set*

$$A = \left\{ f \ \middle| \ \|f\|_{\mathfrak{S}} \le R \right\}.$$

Suppose that $\{f_n\}_{n=0}^{\infty}$ is a sequence in A that converges to f in the weak topology of $\mathcal{M}_+([0,\infty))$ as n goes to infinity, for a.e. $\widehat{k} \in \mathbb{S}$, then $f \in A$.

2.2. **Main theorem.**

Definition 2. *Performing the time change of variables $t = \tau|k|$ and $f(t,k) = f(\tau|k|, k)$, we obtain from* (1)

$$\frac{\partial f}{\partial t}(t,k)\frac{1}{|k|} \ = \ Q[f](t,k)\,, \qquad f(0,k) = f_{in}(k). \tag{26}$$

We say that $f(t,k)$ is a local mild solution of (1) *and* (26) *with a (non-radial) initial condition $f_{in}(k) \ge 0$, $f_{in} \in \mathfrak{S}$ if $f(t,k) \ge 0$ and there exists $T > 0$ such that $f(t,k) \in C([0,T),\mathfrak{S})$ and for all $\phi \in C(\mathbb{R}^2)$, for all $t \in [0,T)$, we have*

$$\int_{\mathbb{R}^2} \mathrm{d}k f(t,k)\phi(k)\frac{1}{|k|} \ = \ \int_{\mathbb{R}^2} \mathrm{d}k f_{in}(k)\phi(k)\frac{1}{|k|} \ + \ \int_0^t \int_{\mathbb{R}^3} \mathrm{d}k Q\,[f]\,\phi(k). \tag{27}$$

We say that $f(t,k) \in C([0,\infty),\mathfrak{S})$ is a global mild solution of (1) *and* (26) *with a (non-radial) initial condition $f_{in}(k) \ge 0$, $f_{in} \in \mathfrak{S}$ if $f(t,k) \ge 0$ and for all $\phi \in C(\mathbb{R}^2) \cap L^\infty_{|k|}L^1_{\widehat{k}}$ and for all $t \ge 0$,* (27) *holds true.*

Theorem 2. *Under Assumptions* (24)-(25), (26) *(and* (1)*) has a global mild solution in $f(t,k) \in C([0,\infty),\mathfrak{S})$ in the sense of Definition 2. Moreover, for a.e. $\widehat{k} \in \mathbb{S}$, one of following alternatives holds true.*

- *(I) Finite time condensation: There exists a finite time $0 < T(\widehat{k}) < \infty$ such that*

$$\int_{\{0\}} \mathrm{d}|k| f(T(\widehat{k}), |k|\widehat{k}) \quad > 0, \tag{28}$$

  with $k = |k|\widehat{k}$.
- *(II) Infinite time condensation:*

$$\lim_{t\to\infty} \int_{\{|k|=0\}} \mathrm{d}|k|\, f(t,k)\varphi(|k|) = \varphi(0) \int_{\mathbb{R}_+} \mathrm{d}|k| f_{in}(k), \tag{29}$$

  for any test function $\varphi(|k|) \in C([0,\infty))$.

  There exist a time sequence $\{\tau_n\}_{n=1}^{\infty}$, with $\tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ and $\lim_{n\to\infty} \tau_n = \infty$, and a constant $N_0 > 1$ such that for all $n > N_0$ and for a.e. $\widehat{k} \in \mathbb{S}$,)

$$\int_{\{|k|=\Xi^{-n}\}} \mathrm{d}|k| f(t,k) \quad > \ \mathfrak{C}_1(t+1) \int_{\{|k|\le \Xi^{-n}\}} \mathrm{d}|k| f(0,k), \tag{30}$$

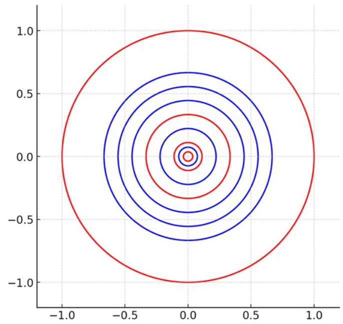

Figure 1. The circular lattice Λ. The initial conditions are supported on the red circles.

for all $t \in [\tau_{n-1}, \tau_n)$*, where* $\mathfrak{C}_1 > 0$ *is a constant independent of* $t, \widehat{k}$ *and* n, N_0*.*

Remark 3. In the above theorem, the initial condition concentrates on the circles $|k| = \Xi^{-\rho}$, $\rho \geq 0$, with the magnitude $\mathcal{O}\Big((\rho!)^{-\gamma}\Big)$ (see Figure 1). That means we construct initial data such that the smaller the radius of the circle is, the smaller the initial is assumed to be. The initial condition decays on circles near the origin.

The theorem shows that the support of the solution remains on the circular lattice $\Lambda \times \mathfrak{S}$ as time evolves. Moreover, from (29), even though for a.e. $\widehat{k} \in \mathbb{S}$

$$\int_{\{|k|=0\}} \mathrm{d}|k| f_{in}(|k|\widehat{k}) \ = \ 0, \tag{31}$$

the solution $f(t,k)$ either forms a Dirac mass - Bose-Eistein Condensate $\int_{\mathbb{R}_+} \mathrm{d}|k| f_{in}(|k|\widehat{k})\delta_{\{|k|=0\}}$ in finite time or converges weakly to the Bose-Eistein Condensate as time goes to infinity for a.e. direction $\widehat{k} \in \mathbb{S}$. In the latter case, from (30), one can see that the solution accumulates at circles $|k| = \Xi^{-n}$ with a growth rate at least linearly in time $\mathfrak{C}_1(t+1)$.

3. Estimates for the collision operator and a local existence result

Proposition 4. *For any suitable test function* $\varphi \in L^\infty_{|k|} L^1_{\widehat{k}}$*, the following weak formulation holds for the collision operator* Q

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) = & \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\, [|k|+|k_1|+|k_2|]^{-1}\frac{1}{\pi}\delta(k-k_1-k_2)\\
& \times \delta(|k|-|k_1|-|k_2|)\Big[f(k_1)f(k_2) - f(k_1)f(k) - f(k_2)f(k)\Big]\\
& \times \Big[\varphi(k) - \varphi(k_1) - \varphi(k_2)\Big]\\
= & \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{\mathbb{R}_+} |k_2|\mathrm{d}|k_2|\, \frac{1}{|k_1|}\mathcal{W}\big(|k_1|+|k_2|, |k_1|, |k_2|\big)\Big[f(k_1)f(|k_2|\widehat{k_1})\\
& - f(k_1)f(k_1+|k_2|\widehat{k_1}) - f(|k_2|\widehat{k_1})f(k_1+|k_2|\widehat{k_1})\Big]\\
& \times \Big[\varphi(k_1+|k_2|\widehat{k_1}) - \varphi(k_1) - \varphi(|k_2|\widehat{k_1})\Big],
\end{aligned} \tag{32}$$

where, for $k \in \mathbb{R}^2$, we denote $k = |k|\widehat{k}$, $\widehat{k} \in \mathbb{S}$.

When $\varphi(k) \in L^\infty_{|k|} L^1_{\widehat{k}}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) \\
&= 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1}) \Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \\
&\qquad + \varphi(-|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) - 2\varphi(|k_2|\widehat{k_1})\Big] \\
&\qquad + \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|=|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1}) \Big[\varphi(2|k_1|\widehat{k_1}) - 2\varphi(|k_1|\widehat{k_1}) + 2\varphi(0)\Big]\,.
\end{aligned} \tag{33}$$

In addition, when $\varphi(k) = \tilde{\varphi}(\widehat{k})\chi_{(0,\infty)}(|k|)$, $\tilde{\varphi}(\widehat{k}) \in L^1(\mathbb{S})$, we have,

$$\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) \ \le\ 0. \tag{34}$$

Proof. The proof of (32) follows a standard argument (see [63]) by performing the change of variables $k \leftrightarrow k_1$ and $k \leftrightarrow k_2$

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathrm{d}k\, \mathrm{d}k_1\, \mathrm{d}k_2\, (|k| + |k_1| + |k_2|)^{-1} \frac{1}{\pi} \delta(k - k_1 - k_2) \\
&\times \delta(|k| - |k_1| - |k_2|) \Big[f(k_1) f(k_2) - f(k_1) f(k) - f(k_2) f(k)\Big] \\
&\times \Big[\varphi(k) - \varphi(k_1) - \varphi(k_2)\Big].
\end{aligned} \tag{35}$$

Next, we develop

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) = &\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (2|k_1| + 2|k_2|)^{-1} \frac{1}{\pi} \delta(|k_1 + k_2| - |k_1| - |k_2|) \\
&\times \Big[f(k_1) f(k_2) - f(k_1) f(k_1 + k_2) - f(k_2) f(k_1 + k_2)\Big] \\
&\times \Big[\varphi(k_1 + k_2) - \varphi(k_1) - \varphi(k_2)\Big] \mathrm{d}k_1\, \mathrm{d}k_2\,.
\end{aligned} \tag{36}$$

We compute

$$|k_1 + k_2| - |k_1| - |k_2| = \big(|k_1|^2 + |k_2|^2 + 2|k_1||k_2|\widehat{k_1}\cdot\widehat{k_2}\big)^{1/2} - |k_1| - |k_2|\,,$$

yielding $|k_1 + k_2| - |k_1| - |k_2| = 0$ if and only if $\widehat{k_1}\cdot\widehat{k_2} = 1$. By a polar change of variable, we obtain, for any continuous function $G(k_2)$

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{d}k_2\, G(k_2)\, \delta\big(|k_1 + k_2| - |k_1| - |k_2|\big) \\
&= \int_{\mathbb{R}_+} |k_2| \mathrm{d}|k_2| \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^1 \mathrm{d}s\, G\big(k_2(s, \sin(\phi))\big)\delta(y(s)) \\
&= \int_{\mathbb{R}_+} |k_2| \mathrm{d}|k_2|\, \frac{G(|k_2|\widehat{k_1})}{y'(1)} = \int_{\mathbb{R}_+} |k_2| \mathrm{d}|k_2|\, G(|k_2|\widehat{k_1}) \frac{|k_1| + |k_2|}{|k_1||k_2|}\,,
\end{aligned}$$

where $y(s) = \left(|k_1|^2 + |k_2|^2 + 2|k_1||k_2|s\right)^{1/2} - |k_1| - |k_2|$. Applying the above identity to (35) proves the second equality in (32).

We proceed further by supposing $\varphi(k) = \varphi(|k|\widehat{k})$

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) \ =\ & \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{\mathbb{R}_+} \mathrm{d}|k_2|\, \frac{1}{|k_1|}\Big[f(k_1)f(|k_2|\widehat{k_1}) \\
& - f(k_1)f(k_1 + |k_2|\widehat{k_1}) - f(|k_2|\widehat{k_1})f(k_1 + |k_2|\widehat{k_1})\Big] \\
& \times \Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) - \varphi(|k_1|\widehat{k_1}) - \varphi(|k_2|\widehat{k_1})\Big] \\
=\ & \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{\mathbb{R}_+} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \ - \varphi(|k_1|\widehat{k_1}) - \varphi(|k_2|\widehat{k_1})\Big] \\
& - 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{\mathbb{R}_+} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(k_1 + |k_2|\widehat{k_1})\Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \\
& - \varphi(|k_1|\widehat{k_1}) - \varphi(|k_2|\widehat{k_1})\Big]
\end{aligned} \tag{37}$$

which gives

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) \ =\ & \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{\mathbb{R}_+} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \\
& - \varphi(|k_1|\widehat{k_1}) - \varphi(|k_2|\widehat{k_1})\Big] \\
& - 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|\geq|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(|k_2|\widehat{k_1}) \\
& - \varphi(|k_1|\widehat{k_1}) - \varphi(-|k_1|\widehat{k_1} + |k_2|\widehat{k_1})\Big] \ ,
\end{aligned} \tag{38}$$

in which we perform the change of variable $k_1 + |k_2|\widehat{k_1} \to k_2$ in the second integral. Next, we split the two integrals and regroup the terms as follows

$$\begin{aligned}
& \int_{\mathbb{R}^2} \mathrm{d}k\, Q[f](k)\varphi(k) \\
& = 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \ - \varphi(|k_1|\widehat{k_1}) - \varphi(|k_2|\widehat{k_1})\Big] \\
& \quad + \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|=|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(2|k_1|\widehat{k_1}) - 2\varphi(|k_1|\widehat{k_1}) + 2\varphi(0)\Big] \\
& \quad - 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(|k_2|\widehat{k_1}) \\
& \quad - \varphi(|k_1|\widehat{k_1}) - \varphi(-|k_1|\widehat{k_1} + |k_2|\widehat{k_1})\Big]
\end{aligned} \tag{39}$$

$$
\begin{aligned}
&= 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1}) \Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \\
&\quad - 2\varphi(|k_2|\widehat{k_1}) + \varphi(-|k_1|\widehat{k_1} + |k_2|\widehat{k_1})\Big] \\
&\quad + \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|=|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1}) \Big[\varphi(2|k_1|\widehat{k_1}) - 2\varphi(|k_1|\widehat{k_1}) + 2\varphi(0)\Big],
\end{aligned}
$$

yielding (33).

Finally, (34) follows from straightforward computations. □

Proposition 5. *For all $\varphi(k) \in C(\mathbb{R}^2) \cap L^\infty_{|k|} L^1_{\widehat{k}}$, we have*

$$
\int_{\left([0,\infty)\setminus\left(\bigcup_{\eta=0}^\infty \Theta_\eta \bigcup\{0\}\right)\right)\times\mathbb{S}} \mathrm{d}k\, Q[f](k)\varphi(k) \;=\; 0, \tag{40}
$$

for $f \in \mathfrak{S}$.

We have the estimate

$$
\Big\| |Q[f](k)||k| \Big\|_{\mathfrak{S}} \;\le C_Q \|f\|^2_{\mathfrak{S}}, \tag{41}
$$

for some constant $C_Q > 0$ independent of f. Moreover, the equation (26) *(and* (1)*) has a local mild solution in $C([0,T),\mathfrak{S})$, for some $T > 0$, in the sense of Definition 2.*

Proof. First, we prove (40). For $M \in \mathbb{N}$, we set

$$
f^M \;=\; \sum_{\eta=-1}^{M} f_\eta, \tag{42}
$$

where we have used (18) and (33), and

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \mathrm{d}k\, Q_M[f](k)\varphi(k) \\
&= 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, f^M(k_1) f^M(|k_2|\widehat{k_1}) \frac{1}{|k_1|} \Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \\
&\quad - 2|k_2|\varphi(|k_2|\widehat{k_1}) + \varphi(-|k_1|\widehat{k_1} + |k_2|\widehat{k_1})\Big] \\
&\quad + \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|=|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f^M(k_1) f^M(|k_2|\widehat{k_1}) \Big[\varphi(2|k_1|\widehat{k_1}) - 2\varphi(|k_1|\widehat{k_1}) + 2\varphi(0)\Big] \\
&= \sum_{\eta,\xi=-1}^{M} 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f_\eta(k_1) f_\xi(|k_2|\widehat{k_1}) \frac{1}{|k_1||k_2|} \Big[\varphi(|k_1|\widehat{k_1} + |k_2|\widehat{k_1}) \\
&\quad - 2\varphi(|k_2|\widehat{k_1}) + \varphi(-|k_1|\widehat{k_1} + |k_2|\widehat{k_1})\Big] \\
&\quad + \sum_{\eta,\xi=-1}^{M} \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|=|k_1|} \mathrm{d}|k_2|\, f_\eta(k_1) f_\xi(|k_2|\widehat{k_1}) \frac{1}{|k_1|} \Big[\varphi(2|k_1|\widehat{k_1}) - 2\varphi(|k_1|\widehat{k_1}) + 2\varphi(0)\Big],
\end{aligned} \tag{43}
$$

for $\varphi(k) \in C(\mathbb{R}^2) \cap L^\infty_{|k|} L^1_{\widehat{k}}$.

We can bound

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}\mathrm{d}k\left[Q[f](k)-Q_M[f](k)\right]\varphi(k)\Big|\\
\lesssim\ &\|\varphi\|_{L^\infty_{|k|}L^1_{\widehat{k}}}\left(\sup_{\widehat{k}\in\mathbb{S}}\int_{\bigcup_{\eta\geq M+1}\Theta_\eta}\mathrm{d}|k|f(|k|\widehat{k})\right)\left(\sup_{\widehat{k}\in\mathbb{S}}\int_{\bigcup_{\eta\geq -1}\Theta_\eta}\mathrm{d}|k|f(|k|\widehat{k})|\right),
\end{aligned}\tag{44}$$

yielding

$$\lim_{M\to\infty}\Big|\int_{\mathbb{R}^2}\mathrm{d}k\left[Q[f](k)-Q_M[f](k)\right]\varphi(k)\Big| \ = 0.\tag{45}$$

Therefore, we can write

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbb{S}}\mathrm{d}k\,Q[f](k)\varphi(k)\\
=\ &\sum_{\eta,\xi=-1}^{\infty}2\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\Big[\varphi(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\\
&-2\varphi(|k_2|\widehat{k_1})+\varphi(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big]\\
&+\sum_{\eta,\xi=-1}^{\infty}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\Big[\varphi(2|k_1|\widehat{k_1})-2\varphi(|k_1|\widehat{k_1})+2\varphi(0)\Big].
\end{aligned}\tag{46}$$

By the same argument, similarly, we also obtain

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbb{S}}\mathrm{d}k\,\Big|Q[f](k)\varphi(k)\Big|\\
\leq\ &\sum_{\eta,\xi=-1}^{\infty}2\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\Big[\Big|\varphi(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big|\\
&+2\Big|\varphi(|k_2|\widehat{k_1})\Big|+2\Big|\varphi(|k_1|\widehat{k_1})\Big|+\Big|\varphi(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big|\Big]\\
&+\sum_{\eta,\xi=-1}^{\infty}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\Big[\Big|\varphi(2|k_1|\widehat{k_1})\Big|+4\Big|\varphi(|k_1|\widehat{k_1})\Big|+2\Big|\varphi(0)\Big|\Big].
\end{aligned}\tag{47}$$

Next, we will prove that

$$\int_{\left([0,\infty)\setminus\left(\bigcup_{\eta=0}^{\infty}\Theta_\eta\bigcup\{0\}\right)\right)\times\mathbb{S}}\mathrm{d}k\,Q[f](k)\varphi(k) \ = \ 0.\tag{48}$$

To this end, we estimate for a sufficiently large number M and a sufficiently small constant $\varepsilon>0$

$$
\begin{aligned}
&\left|\int_{\left([0,\infty)\setminus\left(\bigcup_{\eta=0}^{\infty}\Theta_\eta\bigcup\{0\}\right)\right)\times\mathbb{S}}\mathrm{d}k\,Q[f](k)\varphi(k)\right| \ \le\ \int_{([0,\infty)\setminus(\Lambda_M\bigcup\{0\}))\times\mathbb{S}}\mathrm{d}k\left|Q[f](k)\varphi(k)\right|\\
\le\ &\left|\int_{\left([0,\infty)\setminus\bigcup_{h\in(\Lambda_M\bigcup\{0\})}\left(h-\frac{\varepsilon}{\Xi 2M},h+\frac{\varepsilon}{\Xi 2M}\right)\right)\times\mathbb{S}}\mathrm{d}k\left|Q[f](k)\varphi(k)\right|\right| \qquad (49)\\
&+\left|\int_{\left(\bigcup_{h\in(\Lambda_M\bigcup\{0\})}\left(h-\frac{\varepsilon}{\Xi 2M},h\right)\bigcup_{h\in\Lambda_M}\left(h,h+\frac{\varepsilon}{\Xi 2M}\right)\right)\times\mathbb{S}}\mathrm{d}k\left|Q[f](k)\varphi(k)\right|\right| \ =:\ A+B.
\end{aligned}
$$

We first estimate the first term on the right hand side, A. Let $\phi_\varepsilon^M(|k|)$ be a bounded nonnegative and continuous test function being 1 in the set $\mathbb{R}_+\setminus\left[h-\frac{\varepsilon}{\Xi 2M},h+\frac{\varepsilon}{\Xi 2M}\right]$ and vanishes in small neighbourhoods of all $h\in\Lambda_M\bigcup\{0\}$. We set $\tilde\varphi_\varepsilon^M=\phi_\varepsilon^M\varphi$ and bound

$$
\begin{aligned}
A\ \lesssim\ &\Big|\sum_{\max\{\eta,\xi\}\le M}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\\
&\times\Big[\left|\tilde\varphi_\varepsilon^M(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(|k_2|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(|k_1|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\right|\Big]\\
&+\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\\
&\times\Big[\left|\tilde\varphi_\varepsilon^M(2|k_1|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(|k_1|\widehat{k_1})\right|\Big]+\left|\tilde\varphi_\varepsilon^M(0)\right|\Big]\ \Big|\\
&+\Big|\sum_{\max\{\eta,\xi\}\ge M+1}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\\
&\times\Big[\left|\tilde\varphi_\varepsilon^M(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(|k_2|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(|k_1|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\right|\Big]\\
&+\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\\
&\times\Big[\left|\tilde\varphi_\varepsilon^M(2|k_1|)\right|+\left|\tilde\varphi_\varepsilon^M(|k_1|\widehat{k_1})\right|+\left|\tilde\varphi_\varepsilon^M(0)\right|\Big]\ \Big|\ =:\ A_1\ +\ A_2.
\end{aligned}
\qquad (50)
$$

We now study the first term on the right hand side A_1. By the choice of the function ϕ_ε^M, in order for $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(|k_1|)$, or $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(|k_2|)$ in the above expression not to vanish, $|k_1|$ or $|k_2|$ has to be in $\bigcup_{\alpha=M+1}^\infty\Theta_\alpha$. However $|k_1|\in\Theta_\eta$, $|k_2|\in\Theta_\xi$, by the definition of f_η and f_ξ. Therefore, both $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(|k_1|)$ and $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(|k_2|)$ vanish. By the choice of the function ϕ_ε^M, in order for $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(|k_1|+|k_2|)$ not to vanish, $|k_1|+|k_2|\in\bigcup_{\rho=M+1}^\infty\Theta_\rho$, while $|k_1|\in\Theta_\eta$, $|k_2|\in\Theta_\xi$, by the definition of f_η and f_ξ. It is clear that $|k_1||k_2|\neq 0$. We suppose $|k_1|=\Upsilon_\eta(\mu,\nu)$, $|k_2|=\Upsilon_\xi(\mu',\nu')$, $|k_1|+|k_2|=\Upsilon_\rho(\mu'',\nu'')$, that leads to

$$\Upsilon_\eta(\mu,\nu)\ +\ \Upsilon_\xi(\mu',\nu')\ =\ \Upsilon_\rho(\mu'',\nu''),$$

yielding

$$\Xi^{-\eta}\Upsilon(\mu,\nu)\ +\ \Xi^{-\xi}\Upsilon(\mu',\nu')\ =\ \Xi^{-\rho}\Upsilon(\mu'',\nu''),$$

then

$$\Xi^{\rho-\eta}\Upsilon(\mu,\nu)\ +\ \Xi^{\rho-\xi}\Upsilon(\mu',\nu')\ =\ \Upsilon(\mu'',\nu'').$$

Since $\rho > M \geq \max\{\xi,\eta\}$, the left hand side of the above equation is divisible by Ξ, while the right hand side is not. As thus, $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(|k_1|+|k_2|)$ vanishes. By the choice of the function ϕ_ε^M, in order for $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(-|k_1|+|k_2|)$ not to vanish, $-|k_1|+|k_2| \in \bigcup_{\rho=M+1}^\infty \Theta_\alpha$, while $|k_1|\in\Theta_\eta$, $|k_2|\in\Theta_\xi$, by the definition of f_η and f_ξ. Arguing similarly as above, we also deduce that $f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\phi_\varepsilon^M(-|k_1|+|k_2|)$ vanishes. As a result, $A_1 = 0$.

Next we will establish a bound for A_2. Similar with (44), we bound

$$\Big|A_2\Big| \lesssim \|\varphi\|_{L^\infty_{|k|}L^1_{\widehat{k}}}\left(\sup_{\widehat{k}\in\mathbb{S}}\int_{\bigcup_{\eta\geq 0}\Theta_\eta\bigcup\{0\}}\mathrm{d}|k|f(|k|,\widehat{k})\right)\left(\sup_{\widehat{k}\in\mathbb{S}}\int_{\bigcup_{\eta\geq M}\Theta_\eta}\mathrm{d}|k|f(|k|,\widehat{k})\right) \longrightarrow 0 \text{ as } M\to\infty. \tag{51}$$

Therefore, we find

$$\Big|A\Big| \longrightarrow 0 \text{ as } M\to\infty. \tag{52}$$

Let $\varpi > 0$ be a small constant satisfying $\ln\left(\frac{1}{\varpi}\right) > M\ln(\Xi)$. We define a nonnegative continuous test function φ_ϖ satisfying $\varphi_\varpi = 1$ in $\left[h-\frac{\varpi}{2}, h+\frac{\varpi}{2}\right]$, $\varphi_\varpi = 0$ in $\mathbb{R}_+\setminus(h-\varpi,h+\varpi)$ with $h\in\Lambda_M\bigcup\{0\}$. We set $\tilde\varphi_\varpi = \varphi_\varpi\varphi$ and bound

$$\begin{aligned} B \lesssim \Big| &\sum_{\eta,\xi=-1}^\infty \int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\, f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|} \\ &\times\Big[\Big|\tilde\varphi_\varpi(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big| + \Big|\tilde\varphi_\varpi(|k_2|\widehat{k_1})\Big| + \Big|\tilde\varphi_\varpi(|k_1|\widehat{k_1})\Big| + \Big|\tilde\varphi_\varpi(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big|\Big] \\ &+\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\, f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|} \\ &\times\Big[\Big|\tilde\varphi_\varpi(2|k_1|\widehat{k_1})\Big| + \Big|\tilde\varphi_\varpi(|k_1|\widehat{k_1})\Big| + \Big|\tilde\varphi_\varpi(0)\Big|\Big]\Big|. \end{aligned} \tag{53}$$

We first prove that the first term in (53) tends to 0 as $\varpi\to 0$. Suppose the contrary, we bound this term as follows

$$\begin{aligned} &\sum_{\eta,\xi=-1}^\infty \int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\, f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\frac{1}{|k_1|}\Big|\tilde\varphi_\varpi(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big| \\ \lesssim &\sum_{h\in\Lambda_M}\sum_{\eta,\xi=-1}^\infty \|\varphi\|_{L^\infty_{|k|}L^1_{\widehat{k}}}\sup_{\widehat{k}\in\mathbb{S}^2}\iint_{0<||k_1|+|k_2|-h|<\varpi,|k_2|>|k_1|}\mathrm{d}|k_1|\mathrm{d}|k_2|\, f_\eta(k_1)f_\xi(|k_2|\widehat{k_1}). \end{aligned} \tag{54}$$

We first consider the case $h\in\Lambda_M$, and $|k_1||k_2|\neq 0$. From the above equation, we set $|k_1| = \Upsilon_\eta(\mu,\nu)$,$|k_2| = \Upsilon_\xi(\mu',\nu')$, $h = \Upsilon_\rho(\mu'',\nu'')$, and obtain

$$0 < |\Upsilon_\eta(\mu,\nu)\ +\ \Upsilon_\xi(\mu',\nu')\ -\ \Upsilon_\rho(\mu'',\nu'')| < \varpi,$$

yielding

$$0 < |\Xi^{-\eta}\Upsilon(\mu,\nu)\ +\ \Xi^{-\xi}\Upsilon(\mu',\nu')\ -\ \Xi^{-\rho}\Upsilon(\mu'',\nu'')| < \varpi.$$

We denote as $\delta = \max\{\eta, \xi, M\} \geq \rho$. Then

$$0 < |\Xi^{\delta-\eta}\Upsilon(\mu,\nu) \ + \ \Xi^{\delta-\xi}\Upsilon(\mu',\nu') \ - \ \Xi^{\delta-\rho}\Upsilon(\mu'',\nu'')| < \varpi\Xi^{\delta}.$$

Since $|\Xi^{\delta-\eta}\Upsilon(\mu,\nu) \ + \ \Xi^{\delta-\xi}\Upsilon(\mu',\nu') \ - \ \Xi^{\delta-\rho}\Upsilon(\mu'',\nu'')|$ is an integer, we bound $1 < \varpi\Xi^{\delta}$. We deduce

$$\ln(\Xi)\delta \ = \ \ln(\Xi)\max\{\eta,\xi,M\} \geq \ln\left(\frac{1}{\varpi}\right),$$

contradicting the fact that $\ln\left(\frac{1}{\varpi}\right) > M\ln(\Xi)$. We now consider the case $h \in \Lambda_M$, and $|k_1| = 0 \neq |k_2|$. Setting $|k_2| = \Upsilon_{\xi}(\mu',\nu')$, $h = \Upsilon_{\rho}(\mu'',\nu'')$, we find

$$0 < |\Xi^{-\xi}\Upsilon(\mu',\nu') \ - \ \Xi^{-\rho}\Upsilon(\mu'',\nu'')| < \varpi.$$

Denoting as $\delta = \max\{\xi, M\} \geq \rho$, we find

$$0 < |\Xi^{\delta-\xi}\Upsilon(\mu',\nu') \ - \ \Xi^{\delta-\rho}\Upsilon(\mu'',\nu'')| < \varpi\Xi^{\delta},$$

yielding $1 < \varpi\Xi^{\delta}$. This contradicts the fact that $\ln\left(\frac{1}{\varpi}\right) > M\ln(\Xi)$. Similarly, when $|k_1| = |k_2| = 0$, we also have a contradiction.

Next, we consider the case when $h = 0$. If $0 \neq |k_1||k_2|$ we set $|k_1| = \Upsilon_{\eta}(\mu,\nu)$,$|k_2| = \Upsilon_{\xi}(\mu',\nu')$, and obtain $0 < |\Upsilon_{\eta}(\mu,\nu) \ + \ \Upsilon_{\xi}(\mu',\nu')| < \varpi$. We set $\delta = \max\{\eta,\xi,M\}$ and also find $1 < \varpi\Xi^{\delta}$, leading to another contradiction. Suppose that $|k_1| = 0 \neq |k_2|$, we set $|k_2| = \Upsilon_{\xi}(\mu',\nu')$, and obtain $0 < |\Upsilon_{\xi}(\mu',\nu')| < \varpi$. We set $\delta = \max\{\xi, M\}$ and also obtain $1 < \varpi\Xi^{\delta}$, which is also a contradiction. Similarly, when $|k_1| = |k_2| = 0$, we also have a contradiction. Therefore, the first term in (53) tends to 0 as $\varpi \to 0$.

We can apply the same strategy to the other terms in B, and finally get

$$B \longrightarrow 0 \text{ as } \varpi \to 0. \tag{55}$$

Therefore (40) is proved and we only need to prove (41). Putting together (52) and (55), we obtain (48), which immediately leads to

$$\int_{[0,\infty)\times\mathbb{S}} \mathrm{d}k\, Q[f](k)\varphi(|k|\widehat{k}) \ = \ \int_{\left(\bigcup_{\rho=0}^{\infty}\Theta_{\rho}\bigcup\{0\}\right)\times\mathbb{S}} \mathrm{d}k\, Q[f](k)\varphi(|k|\widehat{k}). \tag{56}$$

We bound for $\rho \geq 0$, by similar arguments used to obtain (44)

$$\begin{aligned}
&\sup_{\widehat{k}\in\mathbb{S}} \mathfrak{C}^\rho \int_{\Theta_\rho} \mathrm{d}|k|\,|k|\Big|Q[f](k)\Big| \\
\lesssim\ & \mathfrak{C}^\rho \sum_{h\in\Theta_\rho}\sum_{\eta,\xi=-1}^{\infty} \sup_{\widehat{k}\in\mathbb{S}} \iint_{|k_1|+|k_2|=|h|,|k_2|\geq|k_1|} \mathrm{d}|k_1|\mathrm{d}|k_2|\, f_\eta(k_1) f_\xi(|k_2|\widehat{k_1}) \\
&+ \mathfrak{C}^\rho \sum_{h\in\Theta_\rho}\sum_{\eta,\xi=-1}^{\infty} \sup_{\widehat{k}\in\mathbb{S}} \iint_{-|k_1|+|k_2|=|h|,|k_2|\geq|k_1|} \mathrm{d}|k_1|\mathrm{d}|k_2|\, f_\eta(k_1) f_\xi(|k_2|\widehat{k_1}) \\
&+ \mathfrak{C}^\rho \sum_{h\in\Theta_\rho}\sum_{\eta,\xi=-1}^{\infty} \sup_{\widehat{k}\in\mathbb{S}} \iint_{|k_2|=|h|,|k_2|\geq|k_1|} \mathrm{d}|k_1|\mathrm{d}|k_2|\, f_\eta(k_1) f_\xi(|k_2|\widehat{k_1}) \\
&+ \mathfrak{C}^\rho \sum_{h\in\Theta_\rho}\sum_{\eta,\xi=-1}^{\infty} \sup_{\widehat{k}\in\mathbb{S}} \iint_{|k_1|=|h|,|k_2|\geq|k_1|} \mathrm{d}|k_1|\mathrm{d}|k_2|\, f_\eta(k_1) f_\xi(|k_2|\widehat{k_1}) \\
\lesssim\ & \mathfrak{C}^\rho \left(\sup_{\widehat{k}\in\mathbb{S}} \int_{\bigcup_{\eta\geq\rho}\Theta_\eta} \mathrm{d}|k| f(|k|,\widehat{k})\right)\left(\sup_{\widehat{k}\in\mathbb{S}} \int_{\bigcup_{\eta\geq-1}\Theta_\eta} \mathrm{d}|k| f(|k|,\widehat{k})\right) \\
\lesssim\ & \|f\|_{\mathfrak{S}}^2,
\end{aligned} \tag{57}$$

yielding

$$\sup_{\widehat{k}\in\mathbb{S}} \mathfrak{C}^\rho \int_{\Theta_\rho} \mathrm{d}|k|\,|k|\Big|Q[f](k)\Big| \lesssim \|f\|_{\mathfrak{S}}^2. \tag{58}$$

Similarly

$$\sup_{\widehat{k}\in\mathbb{S}} \int_{\{0\}} \mathrm{d}|k|\,|k|\Big|Q[f](k)\Big| \lesssim \|f\|_{\mathfrak{S}}^2. \tag{59}$$

We then deduce

$$\max\left\{\sup_{\widehat{k}\in\mathbb{S}} \int_{\{0\}} \mathrm{d}|k|\,|k|\Big|Q[f](k)\Big|, \sup_{\rho\geq0}\sup_{\widehat{k}\in\mathbb{S}} \left|\int_{\Theta_\rho} \mathrm{d}|k|\,|k|\mathfrak{C}^\rho |Q[f](k)|\right|\right\} \lesssim \|f\|_{\mathfrak{S}}^2, \tag{60}$$

which implies

$$\Big\||k||Q[f](k)|\Big\|_{\mathfrak{S}} \leq C_Q\|f\|_{\mathfrak{S}}^2, \tag{61}$$

for some constant $C_Q > 0$ independent of f.

For $g \in C([0,T),\mathfrak{S})$, we define

$$\|g\|_{\mathfrak{S},T} := \sup_{0\leq t<T} \|g(t)\|_{\mathfrak{S}}. \tag{62}$$

Let $T > 0$ be a sufficient small constant and let us consider the set

$$\mathscr{X}_T := \Big\{g \in C([0,T),\mathfrak{S}) \ \Big|\ \|g\|_{\mathfrak{S},T} \leq 2\|f_{in}\|_{\mathfrak{S}}\Big\}. \tag{63}$$

We also define the operator

$$\mathscr{O}[h] := f_{in} + \int_0^t \mathrm{d}s Q[h](s)|k|, \tag{64}$$

which can be bounded, when $h \in \mathscr{X}_T$, as

$$\|\mathscr{O}[h]\|_{\mathfrak{S},T} \leq \|f_{in}\|_{\mathfrak{S}} + TC_Q\|h\|^2_{\mathfrak{S},T} \leq \|f_{in}\|_{\mathfrak{S}} + T4C_Q\|f_{in}\|^2_{\mathfrak{S}} \leq 2\|f_{in}\|_{\mathfrak{S}},$$

when T is sufficiently small. Therefore the operator $\mathscr{O}$ maps $\mathscr{X}_T$ into $\mathscr{X}_T$.

It is also straightforward that, for any $h, g \in \mathfrak{S}$ and $\|h\|_{\mathfrak{S}}, \|g\|_{\mathfrak{S}} \leq 2\|f_{in}\|_{\mathfrak{S}}$, we have

$$\|\mathscr{O}[h] - \mathscr{O}[g]\|_{\mathfrak{S}} \leq C\|h-g\|_{\mathfrak{S}},$$

where C is a constant depending on $\|f_{in}\|_{\mathfrak{S}}$. Therefore, for h and $g \in \mathscr{X}_T$

$$\left\|\int_0^t \mathrm{d}sQ[h](s)|k| - \int_0^t \mathrm{d}sQ[g](s)|k|\right\|_{\mathfrak{S}} \leq CT\|h-g\|_{\mathfrak{S}}. \tag{65}$$

As a consequence, the operator $\mathscr{O}[h]$ is Lipschitz from $\mathscr{X}_T$ to $\mathscr{X}_T$.

Next, we will show that if $\{h_n\}_{n=0}^\infty$ is a Cauchy sequence in $\mathscr{X}_T$, it will have a limit h in $\mathscr{X}_T$ as n goes to infinity. This can be seen as follows, since $\{h_n\}_{n=0}^\infty$ is a Cauchy sequence in $\mathscr{X}_T$, then for all $t \in [0,T)$ and for a.e. $\widehat{k}$ in $\mathbb{S}$, $\{h_n\}_{n=0}^\infty$ is a Cauchy sequence in $\mathfrak{S}$. Therefore, for fixed $t \in [0,T)$ and $\widehat{k} \in \mathbb{S}$, there exists a subsequence $\{h_{n_i}\}_{i=0}^\infty$ such that $\{h_{n_i}\}_{i=0}^\infty$ converges weakly to a limit $h \in \mathcal{M}_+([0,\infty))$ as i goes to infinity in the weak topology of $\mathcal{M}_+([0,\infty))$. By Lemma 1, $h \in \mathfrak{S}$. Moreover, $\lim_{i\to\infty}\int_{\{\Upsilon_\eta(\mu,\nu)\}}\mathrm{d}|k|h_{n_i} = \int_{\{\Upsilon_\eta(\mu,\nu)\}}\mathrm{d}|k|h$, for each $\mu = 1,2,3,\cdots$, $\nu = 1,\cdots,\Xi-1$, $\eta = 0,1,2,3,\cdots$. Therefore, the whole sequence $\{h_n\}_{n=0}^\infty$ converges weakly to h in $\mathcal{M}_+([0,\infty))$ and $\lim_{n\to\infty}\int_{\{\Upsilon_\eta(\mu,\nu)\}}\mathrm{d}|k|h_n = \int_{\{\Upsilon_\eta(\mu,\nu)\}}\mathrm{d}|k|h$, for each $\mu = 1,2,3,\cdots$; $\nu = 1,\cdots,\Xi-1$; $\eta = 0,1,2,3,\cdots$. By Lemma 1 again, $h \in \mathscr{X}_T$ and hence h is also a strong limit of $\{h_n\}_{n=0}^\infty$ in $\mathscr{X}_T$. The claim is proved.

As a consequence, by the standard fixed point argument, the operator $\mathscr{O}[h]$ has a fixed point, which is a mild local solution of our equation.

□

4. Moment estimates

Lemma 6. *Suppose that $f \in C([0,\infty),\mathfrak{S})$ solves* (1) *and* (26) *in the sense of Definition 2.*

Then, the following inequality holds for $M = 0,1,2,\cdots$ and a.e. $\widehat{k} \in \mathbb{S}$

$$\sum_{\rho>M}\int_{\mathbb{R}_+}\mathrm{d}|k|\sup_{\widehat{k}\in\mathbb{S}} f_\rho(t,|k|\widehat{k}) \quad \leq e^{\mathcal{C}_1 t}\left(\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k| f_\rho(0,|k|\widehat{k})\right) \tag{66}$$

for some constant $\mathcal{C}_1 > 0$ independent of t, ρ, M.

Proof. We bound, for a positive test function $\varphi(k) \in L^\infty_{|k|}L^1_{\widehat{k}}$

$$\begin{aligned}
&\int_{[0,\infty)\times\mathbb{S}}\mathrm{d}k\, Q[f](k)\varphi(k) \\
\leq\ & \sum_{\eta,\xi=-1}^\infty 2\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\Big[\varphi(|k_1|\widehat{k_1}+|k_2|\widehat{k_1}) + \varphi(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big] \\
&+ \sum_{\eta,\xi=-1}^\infty \int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\Big[\varphi(2|k_1|\widehat{k_1}) + 2\varphi(|k_1|\widehat{k_1}) + 2\varphi(0)\Big].
\end{aligned} \tag{67}$$

For $\rho \geq 0$, we set $\varphi^\rho_{\mu,\nu}(k) = \delta_{|k|=\Upsilon_\rho(\mu,\nu)}\tilde{\varphi}(\widehat{k_1})$, for any $\tilde{\varphi}(\widehat{k_1}) \in L^1(\mathbb{S})$, $\tilde{\varphi}(\widehat{k_1}) \geq 0$ and use those functions as test functions in (67)

$$\begin{aligned}
&\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\int_{[0,\infty)\times\mathbb{S}} \mathrm{d}k\, Q[f](k)\varphi^\rho_{\mu,\nu}(k)\\
\leq\ &\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{\eta,\xi=-1}^{\infty}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\Big[\varphi^\rho_{\mu,\nu}(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\\
&+\varphi^\rho_{\mu,\nu}(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big]\\
&+\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{\eta,\xi=-1}^{\infty}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\\
&\times\Big[\varphi^\rho_{\mu,\nu}(2|k_1|\widehat{k_1})+2\varphi^\rho_{\mu,\nu}(0)\Big].
\end{aligned} \tag{68}$$

We bound

$$\begin{aligned}
&\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\int_{[0,\infty)\times\mathbb{S}} \mathrm{d}k\, Q[f](k)\varphi^\rho_{\mu,\nu}(k)\\
\lesssim\ &\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{\eta,\xi=-1}^{\infty}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|\geq|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\Big[\varphi^\rho_{\mu,\nu}(|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\\
&+\varphi^\rho_{\mu,\nu}(-|k_1|\widehat{k_1}+|k_2|\widehat{k_1})\Big] =: A+B.
\end{aligned} \tag{69}$$

We now estimate A

$$A \;=\; \sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{\eta,\xi=-1}^{\infty}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|\geq|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\xi(|k_2|\widehat{k_1})\delta_{|k_1|+|k_2|=\Upsilon_\rho(\mu,\nu)}\tilde{\varphi}(\widehat{k_1}). \tag{70}$$

Since $|k_1|+|k_2| = \Upsilon_\rho(\mu,\nu)$, we suppose $|k_1| = \Upsilon_\eta(\mu'',\nu)$, $|k_2| = \Upsilon_\xi(\mu',\nu)$, $|k_1|+|k_2| = \Upsilon_\rho(\mu,\nu)$, then

$$\Xi^{\rho-\eta}\Upsilon(\mu'',\nu) \;+\; \Xi^{\rho-\xi}\Upsilon(\mu',\nu) \;=\; \Upsilon(\mu,\nu). \tag{71}$$

Equation (71) only has a non-empty set of solutions only when $\eta=\xi\geq\rho$, $\eta=\rho\geq\xi$, or $\xi=\rho\geq\eta$. We then write

$$\begin{aligned}
A \;=\; &\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{\eta\geq\rho}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|\geq|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\eta(|k_2|\widehat{k_1})\delta_{|k_1|+|k_2|=\Upsilon_\rho(\mu,\nu)}\tilde{\varphi}(\widehat{k_1})\\
&+2\sum_{\rho>M}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{-1\leq\eta\leq\rho}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|\geq|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\rho(|k_2|\widehat{k_1})\\
&\times\delta_{|k_1|+|k_2|=\Upsilon_\rho(\mu,\nu)}\tilde{\varphi}(\widehat{k_1}) \;=:\; A_1 \;+\; A_2.
\end{aligned} \tag{72}$$

We now estimate A_1

$$
\begin{aligned}
A_1 \;\lesssim\; & \sum_{\rho>M}\sum_{\mu\,=\,1}^{\infty}\sum_{\nu=1}^{\Xi-1}\sum_{\eta\geq\rho}\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|\geq|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f_\eta(k_1)f_\eta(|k_2|\widehat{k_1})\\
& \times\delta_{|k_1|+|k_2|=\Upsilon_\rho(\mu,\nu)}\tilde{\varphi}(\widehat{k_1})\\
\lesssim\; & \sum_{\rho>M}\sum_{\eta\geq\rho}\left(\int_{\mathbb{S}}\mathrm{d}\widehat{k}\iint_{|k_2|\geq|k_1|}\mathrm{d}|k_2|\,\mathrm{d}|k_1|\,f_\eta(k_1)f_\eta(|k_2|\widehat{k_1})\tilde{\varphi}(\widehat{k_1})\right)\\
\lesssim\; & \left(\sum_{\eta>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\eta(|k_1|\widehat{k})\right)\left(\sum_{\eta>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{dk}f_\eta(|k_1|\widehat{k})\tilde{\varphi}(\widehat{k})\right)\\
\lesssim\; & \left(\sum_{\rho>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(|k|\widehat{k})\tilde{\varphi}(\widehat{k})\right),
\end{aligned}
\tag{73}
$$

and A_2

$$
\begin{aligned}
A_2 \;\lesssim\; & \sum_{\rho>M}\left(\sum_{-1\leq\eta\leq\rho}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\eta(t,|k|\widehat{k})\right)\left(\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(|k_1|\widehat{k})\tilde{\varphi}(\widehat{k})\right)\\
\lesssim\; & \left(\sum_{\rho>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(|k_1|\widehat{k})\tilde{\varphi}(\widehat{k})\right).
\end{aligned}
\tag{74}
$$

Similarly, we can also bound

$$
B \;\lesssim\; \left(\sum_{\rho>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(|k_1|\widehat{k})\tilde{\varphi}(\widehat{k})\right). \tag{75}
$$

Combining (68), (73), (74) we get

$$
\sum_{\rho>M}\int_{\mathbb{R}^2}\mathrm{d}k\partial_t f_\rho(t,|k|\widehat{k})\tilde{\varphi} \quad\leq \mathcal{C}_1\sum_{\rho>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(|k_1|\widehat{k})\tilde{\varphi}(\widehat{k}), \tag{76}
$$

for some constant $\mathcal{C}_1>0$ independent of t,ρ,M, yielding

$$
\sum_{\rho>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(t,|k_1|\widehat{k})\tilde{\varphi}(\widehat{k}) \quad\lesssim\quad \left(\sum_{\rho>M}\int_{\mathbb{S}}\mathrm{d}\widehat{k}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(0,|k_1|\widehat{k})\tilde{\varphi}(\widehat{k})\right)e^{\mathcal{C}_1 t}. \tag{77}
$$

Inequality (66) then follows from (77). □

Lemma 7. *Recalling* (24)-(25), *for* $M\in\mathbb{Z}, M>0$, $\frac{\epsilon\mathscr{C}_1(M+1)^\gamma}{\mathscr{C}_2\mathscr{C}_3}>1$ *and*

$$
t \;\leq\; \mathcal{T}_{\epsilon,M} \;:=\; \frac{1}{\mathcal{C}_1}\ln\left(\frac{\epsilon\mathscr{C}_1(M+1)^\gamma}{\mathscr{C}_2\mathscr{C}_3}\right) \tag{78}
$$

we have

$$\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(t,|k|\widehat{k}) \ \le \epsilon\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_M(0,|k|\widehat{k}), \tag{79}$$

for a.e. $\widehat{k}\in\mathbb{S}$.

Proof. From (66), we get

$$\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(t,|k_1|\widehat{k}) \ \le e^{\mathcal{C}_1 t}\left(\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(0,|k_1|\widehat{k})\right). \tag{80}$$

In order to have

$$\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(t,|k_1|\widehat{k}) \ \le \epsilon\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_M(0,|k_1|\widehat{k}), \tag{81}$$

we would need

$$e^{\mathcal{C}_1 t}\left(\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(0,|k_1|\widehat{k})\right) \ \le \epsilon\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_M(0,|k_1|\widehat{k}), \tag{82}$$

which is equivalent to

$$\begin{aligned} t \ &\le \frac{1}{\mathcal{C}_1}\ln\left(\frac{\epsilon\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_M(0,|k_1|\widehat{k})}{\sum_{\rho>M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(0,|k_1|\widehat{k})}\right)\\ &\le \frac{1}{\mathcal{C}_1}\ln\left(\frac{\epsilon\mathscr{C}_1\frac{\mathscr{C}_3^M}{(M!)^\gamma}}{\mathscr{C}_2\left(\frac{\mathscr{C}_3^{M+1}}{((M+1)!)^\gamma}\ +\ \frac{\mathscr{C}_3^{M+2}}{((M+2)!)^\gamma}\ +\ \cdots\right)}\right)\\ &\le \frac{1}{\mathcal{C}_1}\ln\left(\frac{\epsilon\mathscr{C}_1\frac{\mathscr{C}_3^M}{(M!)^\gamma}}{\mathscr{C}_2\frac{\mathscr{C}_3^{M+1}}{((M+1)!)^\gamma}}\right)\ \le\ \frac{1}{\mathcal{C}_1}\ln\left(\frac{\epsilon\mathscr{C}_1(M+1)^\gamma}{\mathscr{C}_2\mathscr{C}_3}\right) = \mathcal{T}_{\epsilon,M}, \end{aligned} \tag{83}$$

when $\frac{\epsilon\mathscr{C}_1(M+1)^\gamma}{\mathscr{C}_2\mathscr{C}_3}>1$. Note that we have used (24). □

Proposition 8. *We have the bound for f being a mild solution of* (1) *and* (26)

$$\|f(t)\|_{\mathfrak{S}} \ \le\ \mathcal{C}_3 e^{\mathcal{C}_1 t}, \quad \forall t\ge 0, \tag{84}$$

for some universal constant $\mathcal{C}_3>0$ independent of t. There exists a global mild solution of the equation (1) *and* (26) *in $C([0,\infty),\mathfrak{S})$ in the sense of Definition 2.*

Proof. Applying (66), we bound, for all $M\ge 0$

$$\begin{aligned} \int_{\mathbb{R}_+}\mathrm{d}|k|\sup_{\widehat{k}\in\mathbb{S}}f_M(t,|k|\widehat{k}) \ &\le\ e^{\mathcal{C}_1 t}\left(\sum_{\rho\ge M}\sup_{\widehat{k}\in\mathbb{S}}\int_{\mathbb{R}_+}\mathrm{d}|k|f_\rho(0,|k|\widehat{k})\right)\\ \le e^{\mathcal{C}_1 t}\left(\sum_{\rho\ge M}\mathscr{C}_1\frac{\mathscr{C}_3^\rho}{(\rho!)^\gamma}\right)\ &\le\ e^{\mathcal{C}_1 t}\left(\mathscr{C}_1\frac{\mathscr{C}_3^M}{(M!)^\gamma}S_\gamma(\mathscr{C}_3)\right), \end{aligned} \tag{85}$$

where

$$S_\gamma(x) \ := \ \sum_{n=0}^{\infty}\frac{x^n}{(n!)^\gamma}, \qquad \forall x>0. \tag{86}$$

As a result, we obtain

$$\mathfrak{C}^M\int_{\mathbb{R}_+}\mathrm{d}|k|\sup_{\widehat{k}\in\mathbb{S}}f_M(t,|k|\widehat{k}) \ \le \ \mathfrak{C}^M e^{\mathcal{C}_1 t}\left(\mathscr{C}_1\frac{\mathscr{C}_3^M}{(M!)^\gamma}S_\gamma(\mathscr{C}_3)\right) \ \le \ \mathcal{C}_3 e^{\mathcal{C}_1 t}, \tag{87}$$

for some universal constant $\mathcal{C}_3>0$ independent of M. From (87), we deduce

$$\|f(t)\|_{\mathfrak{S}} \ \le \ \mathcal{C}_3 e^{\mathcal{C}_1 t}, \tag{88}$$

for all $t\ge 0$. Therefore (84) is proved.

The global existence result follows from (84) and (25) as well as iterating the fixed point argument of Proposition 5.

□

5. Long time limit

Lemma 9. *Suppose that $f\in C([0,\infty),\mathfrak{S})$ solves (1) and (26) in the sense of Definition 2. For $\widehat{k}\in\mathbb{S}$ suppose Alternative (I) of Theorem 2 does not hold true. We bound $\forall\epsilon\in[0,1)$,*

$$\begin{aligned}&\left(\sum_{\eta\ge\rho}\int_{\Theta_\eta}\mathrm{d}|k|f_\eta(t,k)\Big((1+\epsilon)\Xi^{-\rho}-|k|\Big)\right) \ - \left(\sum_{\eta\ge\rho}\int_{\Theta_\eta}\mathrm{d}|k|f_\eta(0,k)\Big((1+\epsilon)\Xi^{-\rho}-|k|\Big)\right)\\ &\ge \mathcal{C}_2(1-\epsilon)\Xi^{-\rho}\int_0^t\mathrm{d}s\left(\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(s,\Xi^{-\rho}\widehat{k})\right)^2\end{aligned} \tag{89}$$

and for some constant $\mathcal{C}_2>0$ independent of t,ρ,g,ϵ.

Proof. Since Alternative (I) of Theorem 2 does not hold true, for all $t\ge 0$

$$\int_{\{0\}}\mathrm{d}|k|f(t,|k|\widehat{k}) \ = \ 0. \tag{90}$$

From (39), with $\varphi(k)=\varphi(|k|)$, we find

$$\begin{aligned}\partial_t\int_{\mathbb{R}^2}\mathrm{d}k\,\frac{f}{|k|}\varphi(k) \ =& \ 2\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|>|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(|k_1|+|k_2|)\\ &-2\varphi(|k_2|)+\varphi(-|k_1|+|k_2|)\Big]\\ &+\int_{\mathbb{R}^2}\mathrm{d}k_1\int_{|k_2|=|k_1|}\mathrm{d}|k_2|\,\frac{1}{|k_1|}f(k_1)f(|k_2|\widehat{k_1})\Big[\varphi(2|k_1|)-2\varphi(|k_1|)+2\varphi(0)\Big].\end{aligned} \tag{91}$$

By using the test function $\varphi_c(|k|)=(c-|k|)_+$, $c>0$, which satisfies $\varphi_c(|k|)=c-|k|$ when $|k|\le c$ and $\varphi_c(|k|)=0$ when $|k|>c$. Let us study the quantities

$$\mathcal{L}_c^1 \ := \varphi_c(|k_1|+|k_2|) \ - \ 2\varphi_c(|k_2|)+\varphi_c(-|k_1|+|k_2|), \tag{92}$$

and

$$\mathcal{L}_c^2 \ := \ \varphi_c(2|k_1|) - 2\varphi_c(|k_1|) + 2c. \tag{93}$$

We first prove that $\mathcal{L}_c^1 \geq 0$. We consider several cases.

- If $|k_2| \geq |k_2| - |k_1| \geq c$ or $|k_2| + |k_1| \leq c$, then

$$\mathcal{L}_c^1 \ = 0. \tag{94}$$

- If $|k_2| + |k_1| > c$ and $|k_2| \leq c$

$$\mathcal{L}_c^1 \ = \ - \ 2(c - |k_2|) + (c + |k_1| - |k_2|) > 0 \quad . \tag{95}$$

- If $|k_2| + |k_1| > c$ and $|k_2| > c$

$$\mathcal{L}_c^1 \ = \varphi_c(-|k_1| + |k_2|) \ \geq \ 0. \tag{96}$$

Next, it is straightforward that $\mathcal{L}_c^2 = (c - 2|k_1|)_+ - 2(c - |k_1|)_+ + 2c \geq 0$. Choosing $\varphi = \varphi_{(1+\epsilon)\Xi^{-\rho}}(|k|)\hat{\varphi}(\widehat{k})$, for $\rho \geq 0$, $\rho \in \mathbb{Z}$, $\hat{\varphi}(\widehat{k}) \in L^1(\mathbb{S})$, $\hat{\varphi}(\widehat{k}) \geq 0$, we bound

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^2} \mathrm{d}k \, \frac{f}{|k|}\Big((1+\epsilon)\Xi^{-\rho} - |k|\Big)_+ \hat{\varphi}(\widehat{k}) \\
&\geq 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|\geq|k_1|} \mathrm{d}|k_2| \, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1}) \Big[\Big((1+\epsilon)\Xi^{-\rho} - |k_1| - |k_2|\Big)_+ \\
&\quad - 2\Big((1+\epsilon)\Xi^{-\rho} - |k_2|\Big)_+ + \Big((1+\epsilon)\Xi^{-\rho} + |k_1| - |k_2|\Big)_+\Big]\hat{\varphi}(\widehat{k_1}) \\
&\gtrsim (1-\epsilon)\Xi^{-\rho} \int_{\mathbb{S}} \mathrm{d}\widehat{k} \left(\int_{|k|=\Xi^{-\rho}} \mathrm{d}|k| f_\rho(\Xi^{-\rho}\widehat{k})\right)^2 \hat{\varphi}(\widehat{k}),
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\sum_{\eta\geq\rho} \int_{\Theta_\eta} \mathrm{d}|k| f_\eta(t)\Big((1+\epsilon)\Xi^{-\rho} - |k|\Big)_+ - \sum_{\eta\geq\rho} \int_{\Theta_\eta} \mathrm{d}|k| f_\eta(0)\Big((1+\epsilon)\Xi^{-\rho} - |k|\Big)_+ \\
&\gtrsim (1-\epsilon)\Xi^{-\rho} \int_0^t \mathrm{d}s \left(\int_{|k|=\Xi^{-\rho}} \mathrm{d}|k| f_\rho(s)(\Xi^{-\rho}\widehat{k})\right)^2 .
\end{aligned} \tag{98}$$

□

Lemma 10. *Suppose that $f \in C([0,\infty), \mathfrak{S})$ solves (1) and (26) in the sense of Definition 2. For $\widehat{k} \in \mathbb{S}$ suppose Alternative (I) of Theorem 2 does not hold true. Then, the following inequalities hold for $t \in [0,\infty]$*

$$\int_{(0,\infty)} \mathrm{d}|k| \, f(t,k)\Big(c - |k|\Big)_+ \ \text{ is nondecreasing in } t, \quad \forall c > 0, \tag{99}$$

and

$$\int_{(0,c]} \mathrm{d}|k| \, f(t,k) \ \gtrsim \ \int_0^t \mathrm{d}s \left(\int_{c\geq|k|\geq\frac{3c}{4}} \mathrm{d}|k| f(s,k)\right)^2, \tag{100}$$

where the constant on the right hand side of (100) is independent of t, c.

Proof. By choosing $\varphi(k) = \varphi_c(k)\tilde{\varphi}(\widehat{k}) = \Big(c - |k|\Big)_+ \tilde{\varphi}(\widehat{k}) \geq 0$, $\tilde{\varphi}(\widehat{k}) \in L^1(\mathbb{S})$, we bound

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^2} \mathrm{d}k\, \frac{f}{|k|}\varphi(k) \\
&\geq 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|>|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1})\tilde{\varphi}(\widehat{k_1})\Big[\varphi_c(|k_1|+|k_2|) \\
&\quad - 2\varphi_c(|k_2|) + \varphi_c(-|k_1|+|k_2|)\Big] \\
&\quad + \int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|=|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1})\tilde{\varphi}(\widehat{k_1})\Big[\varphi_c(2|k_1|) - 2\varphi_c(|k_1|) + 2\varphi_c(0)\Big] \\
&\geq 2\int_{\mathbb{R}^2} \mathrm{d}k_1 \int_{|k_2|\geq|k_1|} \mathrm{d}|k_2|\, \frac{1}{|k_1|} f(k_1) f(|k_2|\widehat{k_1})\tilde{\varphi}(\widehat{k_1})\Big[\varphi_c(|k_1|+|k_2|) \\
&\quad - 2\varphi_c(|k_2|) + \varphi_c(-|k_1|+|k_2|)\Big] \\
&\gtrsim \int_{\mathbb{S}} \mathrm{d}\widehat{k_1} \iint_{c\geq|k_2|,|k_1|\geq\frac{3c}{4}} \mathrm{d}|k_1|\mathrm{d}|k_2|\, f(k_1) f(|k_2|\widehat{k_1})\tilde{\varphi}(\widehat{k_1})\Big[-\ 2(c-|k_2|) + (c+|k_1|-|k_2|)\Big],
\end{aligned} \tag{101}$$

yielding,

$$\partial_t \int_{[0,\infty)} \mathrm{d}|k|\, f\varphi_c(|k|) \ \gtrsim \frac{c}{2} \iint_{c\geq|k_2|,|k_1|\geq\frac{3c}{4}} \mathrm{d}|k_1|\mathrm{d}|k_2|\, f(k_1) f(|k_2|\widehat{k_1}), \tag{102}$$

which leads to (99) by (90). We now bound

$$\partial_t \int_{[0,\infty)} \mathrm{d}|k|\, f\varphi_c(|k|) \ \ \gtrsim \frac{c}{2}\left(\int_{c\geq|k|\geq\frac{3c}{4}} \mathrm{d}|k| f(k)\right)^2. \tag{103}$$

Since $\varphi_c(|k|) \leq c$, we bound using (90)

$$c\int_{(0,c]} \mathrm{d}|k|\, f(t,k) \ \ \gtrsim \frac{c}{2}\int_0^t \mathrm{d}s \left(\int_{c\geq|k|\geq\frac{3c}{4}} \mathrm{d}|k| f(s,k)\right)^2 + \int_{(0,\infty)} \mathrm{d}|k|\, f(0,k)\varphi_c(|k|), \tag{104}$$

yielding the desired estimate. □

Proposition 11. *Suppose that $f \in C\left([0,\infty), \mathfrak{S}\right)$ solves* (1) *and* (26) *in the sense of Definition 2. For $\widehat{k} \in \mathbb{S}$ suppose Alternative (I) of Theorem 2 does not hold true. Then for any $c > 0$*

$$\lim_{t\to\infty} \int_{(0,\infty)} \mathrm{d}|k|\, f(t)\Big(1 - \frac{|k|}{c}\Big)_+ = \int_{(0,\infty)} \mathrm{d}|k| f(0), \tag{105}$$

and

$$\lim_{t\to\infty} \int_{(0,c)} \mathrm{d}|k|\, f(t) = \int_{(0,\infty)} \mathrm{d}|k| f(0). \tag{106}$$

Moreover, there exist a time sequence $\{\tau_n\}_{n=1}^{\infty}$ and a constant $N_0 > 1$ such that $\tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ and $\lim_{n\to\infty} \tau_n = \infty$, and for all $n > N_0$ and

$$\int_{|k|=\Xi^{-n}} \mathrm{d}|k| f(t,k) \quad > \quad \mathfrak{C}_1(t+1)\int_{\{|k|\le \Xi^{-n}\}} \mathrm{d}|k| f(0,k), \tag{107}$$

for all $t \in [\tau_{n-1}, \tau_n)$, where $\mathfrak{C}_1 > 0$ is a constant independent of $t, \widehat{k}$ and n, N_0.)

Proof. By (34), we find

$$\int_{(0,\infty)} \mathrm{d}|k| f(t,|k|\widehat{k}) \ \le\ \int_{(0,\infty)} \mathrm{d}|k| f(0,|k|\widehat{k}).$$

Since the mapping $t \mapsto \int_{\mathbb{R}_+} \mathrm{d}|k|\, f(t)\Big(c-|k|\Big)_+$ is nondecreasing by (99) and

$$0 \ \le\ \int_{(0,\infty)} \mathrm{d}|k|\, f(t)\Big(c-|k|\Big)_+ \ \le\ c\int_{(0,\infty)} \mathrm{d}|k| f(t) \ \le\ c\int_{(0,\infty)} \mathrm{d}|k| f(0), \tag{108}$$

there exists a limit

$$\mathfrak{E}(\widehat{k}) = \frac{1}{c}\lim_{t\to\infty}\int_{(0,\infty)} \mathrm{d}|k|\, f(t)\Big(c-|k|\Big)_+ \le \int_{(0,\infty)} \mathrm{d}|k| f(0).$$

We will prove $\mathfrak{E}(\widehat{k}) = \int_{(0,\infty)} \mathrm{d}|k| f(0)$, using a proof by contradiction. Supposing $\int_{(0,\infty)} \mathrm{d}|k| f(0) - \mathfrak{E}(\widehat{k}) = \varepsilon(\widehat{k}) > 0$, for a.e. $\widehat{k} \in \mathbb{S}$, we will prove the existence of $0 < r_2 < r_1 < \infty$ such that

$$\int_{[r_1,r_2]} \mathrm{d}|k| f(t) \ge C(\varepsilon(\widehat{k})) \text{ for all } t \in [0,\infty), \tag{109}$$

for some constant $C(\varepsilon(\widehat{k})) > 0$.

For any $r_2 \in (0,c)$, we bound

$$\begin{aligned}\int_{(0,r_2]} \mathrm{d}|k| f(t) \ &\le\ \int_{(0,r_2]} \frac{\Big(c-|k|\Big)_+}{\Big(c-r_2\Big)_+} \mathrm{d}|k| f(t) \ \le\ \int_{(0,\infty)} \frac{\Big(c-|k|\Big)_+}{\Big(c-r_2\Big)_+} \mathrm{d}|k| f(t) \\ &\le\ \frac{c\mathfrak{E}(\widehat{k})}{c-r_2}.\end{aligned} \tag{110}$$

Next, we bound, using (99), for $r_1 > \max\{r_2, 1\}$

$$\begin{aligned} r_1\int_{(0,r_1]} \mathrm{d}|k| f(t) &\ge \int_{(0,\infty)} \mathrm{d}|k|\,(r_1-|k|)_+\, f(t) \ge \int_{(0,\infty)} (r_1-|k|)_+\, \mathrm{d}|k| f(0) \\ &\ge (r_1-1)\int_{(0,1]} \mathrm{d}|k| f(0) \ =\ (r_1-1)\int_{(0,\infty)} \mathrm{d}|k| f(0)\end{aligned} \tag{111}$$

with the notice that $f(0)$ is supported in $[0,1]$) for all $t \in [0,\infty)$, which implies

$$\int_{(0,r_1]} \mathrm{d}|k| f(t) \ge \frac{r_1-1}{r_1}\int_{(0,\infty)} \mathrm{d}|k| f(0) \text{ for all } t \in (0,\infty). \tag{112}$$

Choosing r_1, r_2 such that

$$\frac{c\mathfrak{E}(\widehat{k})}{c-r_2} < \frac{r_1-1}{r_1}\int_{(0,\infty)} \mathrm{d}|k| f(0), \tag{113}$$

and combining (110) and (112), we obtain (109). Let N be an integer and $r_0 < r_1$ such that $[r_1, r_2] \subset \left[r_0, \frac{4^N}{3^N} r_0\right]$, we bound, using (100)

$$\begin{aligned}
\int_{\left(0,\frac{4^N}{3^N}r_0\right]} \mathrm{d}|k| f(t) &\gtrsim \int_0^t \mathrm{d}s \left[\int_{\left[\frac{4^{N-1}}{3^{N-1}}r_0, \frac{4^N}{3^N}r_0\right]} \mathrm{d}|k| f(s)\right]^2, \\
\int_{\left(0,\frac{4^{N-1}}{3^{N-1}}r_0\right]} \mathrm{d}|k| f(t) &\gtrsim \int_0^t \mathrm{d}s \left[\int_{\left[\frac{4^{N-2}}{3^{N-2}}r_0, \frac{4^{N-1}}{3^{N-1}}r_0\right]} \mathrm{d}|k| f(s)\right]^2, \\
&\cdots \\
\int_{\left(0,\frac{4}{3}r_0\right]} \mathrm{d}|k| f(t) &\gtrsim \int_0^t \mathrm{d}s \left[\int_{\left[r_0, \frac{4}{3}r_0\right]} \mathrm{d}|k| f(s)\right]^2,
\end{aligned} \tag{114}$$

which, by summing all the inequalities and by the Cauchy-Schwarz inequality, yields

$$\begin{aligned}
\int_{(0,\infty)} \mathrm{d}|k| f(t) &\gtrsim r(N) \int_0^t \mathrm{d}s \left[\int_{\left[r_0, \frac{4^N}{3^N}r_0\right]} \mathrm{d}|k| f(s)\right]^2 \\
&\gtrsim r(N) \int_0^t \mathrm{d}s \left[\int_{[r_1,r_2]} \mathrm{d}|k| f(s)\right]^2 \gtrsim r(N) C(\varepsilon(\widehat{k}))^2 t,
\end{aligned} \tag{115}$$

where $r(N)$ is a constant depending on N and coming from the Cauchy-Schwarz inequality. Inequality (115) leads to

$$\int_{(0,\infty)} \mathrm{d}|k| f(0) \ge \int_{(0,\infty)} \mathrm{d}|k| f(t) \gtrsim C(\varepsilon(\widehat{k}))^2 r(N) t \to \infty$$

as t to ∞ which is a contradiction. As a conclusion, $\mathfrak{E}(\widehat{k}) = \int_{\mathbb{R}_+} \mathrm{d}|k| f(0)$ and that implies (105) of the Proposition.

The second limit (106) is a consequence of the first limit (105).

From (89), we obtain, for $0 < \epsilon < 1$

$$\begin{aligned}
&\sum_{\eta>\rho} \int_{\Theta_\eta} \mathrm{d}|k| f_\eta(t,k) \Big((1+\epsilon)\Xi^{-\rho} - |k|\Big)_+ + \epsilon\Xi^{-\rho} \int_{|k|=\Xi^{-\rho}} \mathrm{d}|k| f_\rho(t,k) \\
&\ge \mathcal{C}_2(1-\epsilon)\Xi^{-\rho} \int_0^t \mathrm{d}s \left(\int_{|k|=\Xi^{-\rho}} \mathrm{d}|k| f_\rho(s, \Xi^{-\rho}\widehat{k})\right)^2 \\
&\quad + \sum_{\eta>\rho} \int_{\Theta_\eta} \mathrm{d}|k| f_\eta(0,k) \Big((1+\epsilon)\Xi^{-\rho} - |k|\Big)_+ + \epsilon\Xi^{-\rho} \int_{|k|=\Xi^{-\rho}} \mathrm{d}|k| f_\rho(0,k).
\end{aligned} \tag{116}$$

Next, we bound, using (79)

$$\begin{aligned}\sum_{\eta>\rho}\int_{\Theta_\eta}\mathrm{d}|k|f_\eta(t,k)\Big((1+\epsilon)\Xi^{-\rho}-|k|\Big)_+ \ &\le\ \sup_{\widehat{k}\in\mathbb{S}}\Xi^{-\rho}\frac{\mathscr{C}_2\epsilon}{\mathscr{C}_1 2}\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(0,\Xi^{-\rho}\widehat{k})\\ &\le\ \Xi^{-\rho}\frac{\epsilon}{2}\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(0,\Xi^{-\rho}\widehat{k}),\end{aligned}\tag{117}$$

for $0\le\tau\le\mathcal{T}_{\frac{\mathscr{C}_2\epsilon}{\mathscr{C}_1 4},\rho}$. By (117), we estimate

$$\begin{aligned}\epsilon\Xi^{-\rho}\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(t,k)\ \ &\ge\ \mathcal{C}_2(1-\epsilon)\Xi^{-\rho}\int_0^t\mathrm{d}s\left(\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(s,\Xi^{-\rho}\widehat{k})\right)^2\\ &+\ \sum_{\eta>\rho}\int_{\Theta_\eta}\mathrm{d}|k|f_\eta(0,k)\Big((1+\epsilon)\Xi^{-\rho}-|k|\Big)_+\ +\ \frac{\epsilon}{2}\Xi^{-\rho}\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(0,k),\end{aligned}\tag{118}$$

yielding, for $\epsilon=\frac{1}{2}$,

$$\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(t,k)\ \ \ge\ \mathcal{C}_2\int_0^t\mathrm{d}s\left(\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(s,\Xi^{-\rho}\widehat{k})\right)^2\ +\ \frac{1}{2}\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(0,k).\tag{119}$$

Setting

$$X(t,\widehat{k})=\int_0^t\mathrm{d}s\left(\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(s,\Xi^{-\rho}\widehat{k})\right)^2$$

and $X_o(\widehat{k})=\ \int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(0,k)$, we bound

$$\sqrt{\dot{X}(t)}\ \ge\mathcal{C}_2X(t)\ +\ \frac{1}{2}X_o,\tag{120}$$

yielding

$$\dot{X}(t)\ \ge\Big(\mathcal{C}_2X(t)\ +\ \frac{1}{2}X_o\Big)^2.\tag{121}$$

Solving (121), we find

$$X(t)\ \ge\frac{t\frac{1}{2}X_0}{1-t\frac{1}{2}\mathcal{C}_2X_0},\tag{122}$$

for $0\le t\le\mathcal{T}_{\frac{\mathscr{C}_2\epsilon}{\mathscr{C}_1 4},\rho}$, which implies

$$\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(t,k)\ \ \ge\frac{t\frac{1}{2}X_0\mathcal{C}_2}{1-t\frac{1}{2}\mathcal{C}_2X_0}\ +\ \frac{1}{2}X_o.\tag{123}$$

When $t\in\big[\mathcal{T}_{\frac{\mathscr{C}_2\epsilon}{\mathscr{C}_1 4},\rho-1},\mathcal{T}_{\frac{\mathscr{C}_2\epsilon}{\mathscr{C}_1 4},\rho}\big)$, we can set $\tau_\rho=\mathcal{T}_{\frac{\mathscr{C}_2\epsilon}{\mathscr{C}_1 4},\rho}$ and bound for ρ sufficiently large

$$\int_{|k|=\Xi^{-\rho}}\mathrm{d}|k|f_\rho(t,k)\ \ >\ \mathfrak{C}_1(t+1)\int_{\{|k|\le\Xi^{-\rho}\}}\mathrm{d}|k|f_\rho(0,k),\tag{124}$$

for some constant $\mathfrak{C}_1>0$ independent of t,ρ. Therefore (107) is proved.

□

6. Proof of Theorem 2

The conclusion of the Theorem follows from Propositions 5, 8, 11.

7. Appendix

In this Appendix, we give the proof for Lemma 1. We bound, for a.e. $\widehat{k} \in \mathbb{S}$, $M \in \mathbb{S}$, $M \geq -1$

$$\int_{[0,\infty)\setminus\bigcup_{j=-1}^{M}\Theta_j} \mathrm{d}|k| f_n(|k|\widehat{k}) \ = \int_{\bigcup_{j=M+1}^{\infty}\Theta_j} \mathrm{d}|k| f_n(|k|\widehat{k}) \ \leq \quad R\mathfrak{C}^{-M-1}\frac{1}{1-\mathfrak{C}}. \tag{125}$$

Taking the limit $n \to \infty$, we bound

$$\int_{[0,\infty)\setminus\bigcup_{j=-1}^{M}\Theta_j} \mathrm{d}|k| f(|k|\widehat{k}) \ \leq R\mathfrak{C}^{-M-1}\frac{1}{1-\mathfrak{C}}, \tag{126}$$

which, after taking the limit $M \to \infty$, implies

$$\int_{[0,\infty)\setminus\bigcup_{j=-1}^{\infty}\Theta_j} \mathrm{d}|k| f(|k|\widehat{k}) \ = 0. \tag{127}$$

Moreover, since

$$\begin{aligned} \int_{\Theta_M} \mathrm{d}|k| f_n(|k|\widehat{k}) \ &\leq R\mathfrak{C}^{-M}, \quad \forall M \geq 0, \\ \text{and} \int_{\{|k|=0\}} \mathrm{d}|k| f_n(|k|\widehat{k}) \ &\leq R, \end{aligned} \tag{128}$$

it is clear that $\|f\|_{\mathfrak{S}} \leq R$. Therefore, $f \in A$.

References

[1] Das A. and M.-B. Tran. Numerical schemes for a fully nonlinear coagulation-fragmentation model coming from wave kinetic theory. *arXiv preprint arXiv:2412.05402*, 2024.

[2] R. Alonso, I. M. Gamba, and M.-B. Tran. The Cauchy problem and BEC stability for the quantum Boltzmann-Gross-Pitaevskii system for bosons at very low temperature. *arXiv preprint arXiv:1609.07467*, 2016.

[3] I. Ampatzoglou, J. K. Miller, N. Pavlović, and M. Tasković. Inhomogeneous wave kinetic equation and its hierarchy in polynomially weighted linfty spaces. *arXiv preprint arXiv:2405.03984*, 2024.

[4] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell. Observation of Bose-Einstein Condensation in a dilute atomic vapor. *Science*, 269(5221):198–201, 1995.

[5] D. J. Bishop and J. D. Reppy. Study of the superfluid transition in two-dimensional he 4 films. *Physical Review Letters*, 40(26):1727, 1978.

[6] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet. Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions. *Physical review letters*, 75(9):1687, 1995.

[7] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011.

[8] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. On the kinetic wave turbulence description for NLS. *Quart. Appl. Math.*, 78(2):261–275, 2020.

[9] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. *Inventiones mathematicae*, 225:787–855, 2021.

[10] C. Collot, H. Dietert, and P. Germain. Stability and cascades for the Kolmogorov–Zakharov spectrum of wave turbulence. *Archive for Rational Mechanics and Analysis*, 248(1):7, 2024.

[11] C. Collot and P. Germain. On the derivation of the homogeneous kinetic wave equation. *arXiv preprint arXiv:1912.10368*, 2019.

[12] C. Collot and P. Germain. Derivation of the homogeneous kinetic wave equation: longer time scales. *arXiv preprint arXiv:2007.03508*, 2020.

[13] E. Cortés and M. Escobedo. On a system of equations for the normal fluid-condensate interaction in a bose gas. *Journal of Functional Analysis*, 278(2):108315, 2020.

[14] G. Craciun and M.-B. Tran. A reaction network approach to the convergence to equilibrium of quantum Boltzmann equations for Bose gases. *ESAIM: Control, Optimisation and Calculus of Variations*, 2021.

[15] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle. Bose-Einstein condensation in a gas of sodium atoms. *Physical review letters*, 75(22):3969, 1995.

[16] Y. Deng and Z. Hani. Derivation of the wave kinetic equation: full range of scaling laws. *arXiv preprint arXiv:2110.04565*, 2021.

[17] Y. Deng and Z. Hani. Full derivation of the wave kinetic equation. *Inventiones mathematicae*, 233(2):543–724, 2023.

[18] Y. Deng and Z. Hani. Long time justification of wave turbulence theory. *arXiv preprint arXiv:2311.10082*, 2023.

[19] Yu Deng, Alexandru D Ionescu, and Fabio Pusateri. On the wave turbulence theory of 2d gravity waves, i: deterministic energy estimates. *Communications on Pure and Applied Mathematics*, 78(2):211–322, 2025.

[20] M. Dolce and R. Grande. On the convergence rates of discrete solutions to the wave kinetic equation. *Mathematics In Engineering*, 6(4):536–558, 2024.

[21] A. Dymov, S. Kuksin, A. Maiocchi, and S. Vladuts. The large-period limit for equations of discrete turbulence. *arXiv preprint arXiv:2104.11967*, 2021.

[22] Andrey Dymov and Sergei. Kuksin. On the Zakharov-L'vov stochastic model for wave turbulence. In *Doklady Mathematics*, volume 101, pages 102–109. Springer, 2020.

[23] Andrey Dymov and Sergei Kuksin. Formal expansions in stochastic model for wave turbulence 1: Kinetic limit. *Comm. Math. Phys.*, 382(2):951–1014, 2021.

[24] Andrey Dymov and Sergei Kuksin. Formal Expansions in Stochastic Model for Wave Turbulence 2: Method of Diagram Decomposition. *J. Stat. Phys.*, 190(1):Paper No. 3, 2023.

[25] M. Escobedo. On the linearized system of equations for the condensate–normal fluid interaction at very low temperature. *Studies in Applied Mathematics*, 150(2):448–456, 2023.

[26] M. Escobedo. On the linearized system of equations for the condensate-normal fluid interaction near the critical temperature. *Archive for Rational Mechanics and Analysis*, 247(5):92, 2023.

[27] M. Escobedo and A. Menegaki. Instability of singular equilibria of a wave kinetic equation. *arXiv preprint arXiv:2406.05280*, 2024.

[28] M. Escobedo and M.-B. Tran. Convergence to equilibrium of a linearized quantum Boltzmann equation for bosons at very low temperature. *Kinetic and Related Models*, 8(3):493–531, 2015.

[29] M. Escobedo and J. J. L. Velázquez. Finite time blow-up and condensation for the bosonic Nordheim equation. *Invent. Math.*, 200(3):761–847, 2015.

[30] M. Escobedo and J. J. L. Velázquez. On the theory of weak turbulence for the nonlinear Schrödinger equation. *Mem. Amer. Math. Soc.*, 238(1124):v+107, 2015.

[31] Miguel Escobedo, Federica Pezzotti, and Manuel Valle. Analytical approach to relaxation dynamics of condensed Bose gases. *Ann. Physics*, 326(4):808–827, 2011.

[32] D. S. Fisher and P. C. Hohenberg. Dilute bose gas in two dimensions. *Physical Review B*, 37(10):4936, 1988.

[33] I. M. Gamba, L. M. Smith, and M.-B. Tran. On the wave turbulence theory for stratified flows in the ocean. *M3AS: Mathematical Models and Methods in Applied Sciences. Vol. 30, No. 1 105-137*, 2020.

[34] P. Germain, A. D. Ionescu, and M.-B. Tran. Optimal local well-posedness theory for the kinetic wave equation. *Journal of Functional Analysis*, 279(4):108570, 2020.

[35] P. Germain, J. La, and A. Menegaki. Stability of rayleigh-jeans equilibria in the kinetic fpu equation. *arXiv preprint arXiv:2409.01507*, 2024.

[36] P. Germain, J. La, and K. Z. Zhang. Local well-posedness for the kinetic mmt model. *arXiv preprint arXiv:2310.11893*, 2023.

[37] P. Germain, J. La, and K. Z. Zhang. Local well-posedness for the kinetic mmt model. *Communications in Mathematical Physics*, 406(1):1–38, 2025.

[38] P. Germain and H. Zhu. On universality for the kinetic wave equation. *arXiv preprint arXiv:2402.14773*, 2024.

[39] A. Görlitz, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, et al. Realization of bose-einstein condensates in lower dimensions. *Physical review letters*, 87(13):130402, 2001.
[40] R. Grande and Z. Hani. Rigorous derivation of damped-driven wave turbulence theory. *arXiv preprint arXiv:2407.10711*, 2024.
[41] A. Griffin, T. Nikuni, and E. Zaremba. *Bose-condensed gases at finite temperatures.* Cambridge University Press, Cambridge, 2009.
[42] E. D Gust and L. E. Reichl. Collision integrals in the kinetic equations ofdilute bose-einstein condensates. *arXiv:1202.3418*, 2012.
[43] Z. Hani, J. Shatah, and H. Zhu. Inhomogeneous turbulence for wick NLS. *Communications on Pure and Applied Mathematics*, 77(11):4100–4162, 2024.
[44] A. Hannani, M. Rosenzweig, G. Staffilani, and M.-B. Tran. On the wave turbulence theory for a stochastic kdv type equation–generalization for the inhomogeneous kinetic limit. *arXiv preprint arXiv:2210.17445*, 2022.
[45] P. C. Hohenberg. Existence of long-range order in one and two dimensions. *Physical Review*, 158(2):383, 1967.
[46] T. R. Kirkpatrick and J. R. Dorfman. Transport theory for a weakly interacting condensed Bose gas. *Phys. Rev. A (3)*, 28(4):2576–2579, 1983.
[47] T. R. Kirkpatrick and J. R. Dorfman. Transport coefficients in a dilute but condensed bose gas. *J. Low Temp. Phys.*, 58:399–415, 1985.
[48] T. R. Kirkpatrick and J. R. Dorfman. Transport in a dilute but condensed nonideal bose gas: Kinetic equations. *J. Low Temp. Phys.*, 58:301–331, 1985.
[49] J. Lukkarinen and H. Spohn. Weakly nonlinear Schrödinger equation with random initial data. *Invent. Math.*, 183(1):79–188, 2011.
[50] X. Ma. Almost sharp wave kinetic theory of multidimensional kdv type equations with d >= 3. *arXiv preprint arXiv:2204.06148*, 2022.
[51] A. Menegaki. L2-stability near equilibrium for the 4 waves kinetic equation. *Kinetic and Related Models*, 17(4):514–532, 2024.
[52] D. R. Nelson and J. M. Kosterlitz. Universal jump in the superfluid density of two-dimensional superfluids. *Physical Review Letters*, 39(19):1201, 1977.
[53] T. T. Nguyen and M.-B. Tran. On the Kinetic Equation in Zakharov's Wave Turbulence Theory for Capillary Waves. *SIAM J. Math. Anal.*, 50(2):2020–2047, 2018.
[54] N. Pavlović, M. Tasković, and L. Velasco. Inhomogeneous six-wave kinetic equation in exponentially weighted linfty spaces. *arXiv preprint arXiv:2501.10565*, 2025.
[55] O. Penrose and L. Onsager. Bose-einstein condensation and liquid helium. *Physical Review*, 104(3):576, 1956.
[56] Y. Pomeau and M.-B. Tran. Statistical physics of non equilibrium quantum phenomena. *Lecture Notes in Physics, Springer*, 2019.
[57] L. E. Reichl. *A Modern Course in Statistical Physics.* John Wiley & Sons, 2016.
[58] L. E. Reichl and M.-B. Tran. A kinetic equation for ultra-low temperature bose–einstein condensates. *Journal of Physics A: Mathematical and Theoretical*, 52(6):063001, 2019.
[59] W. Rudin. Real and complex analysis. 1974.
[60] B. Rumpf, A. Soffer, and M.-B. Tran. On the wave turbulence theory: ergodicity for the elastic beam wave equation. *arXiv preprint arXiv:2108.13223*, 2021.
[61] D. Rychtarik, B. Engeser, H.-C. Nägerl, and R. Grimm. Two-dimensional bose-einstein condensate in an optical surface trap. *Physical review letters*, 92(17):173003, 2004.
[62] A. Soffer and M.-B. Tran. On the dynamics of finite temperature trapped bose gases. *Advances in Mathematics*, 325:533–607, 2018.
[63] A. Soffer and M.-B. Tran. On the energy cascade of 3-wave kinetic equations: beyond kolmogorov–zakharov solutions. *Communications in Mathematical Physics*, 376(3):2229–2276, 2020.
[64] G. Staffilani and M.-B. Tran. On the wave turbulence theory for a stochastic KdV type equation. *arXiv preprint arXiv:2106.09819*, 2021.

[65] G. Staffilani and M.-B. Tran. Condensation and non-condensation times for 4-wave kinetic equations. *arXiv preprint arXiv:2407.18533*, 2024.

[66] G. Staffilani and M.-B. Tran. On the energy transfer towards large values of wavenumbers for solutions of 4-wave kinetic equations. *arXiv preprint arXiv:2407.18508*, 2024.

[67] M.-B. Tran, G. Craciun, L. M. Smith, and S. Boldyrev. A reaction network approach to the theory of acoustic wave turbulence. *Journal of Differential Equations*, 269(5):4332–4352, 2020.

[68] M.-B. Tran and Y. Pomeau. Boltzmann-type collision operators for Bogoliubov excitations of Bose-Einstein condensates: A unified framework. *Physical Review E 101 (3), 032119*, 2020.

[69] M.-B. Tran and Y. Pomeau. On a thermal cloud-bose–einstein condensate coupling system. *The European Physical Journal Plus*, 136(5):1–11, 2021.

[70] S. Walton and M.-B. Tran. A numerical scheme for wave turbulence: 3-wave kinetic equations. *SIAM Journal on Scientific Computing*, 45(4):B467–B492, 2023.

[71] S. Walton and M.-B. Tran. Numerical schemes for 3-wave kinetic equations: A complete treatment of the collision operator. *arXiv preprint arXiv:2402.17481*, 2024.

[72] S. Walton, M.-B. Tran, and A. Bensoussan. A deep learning approximation of non-stationary solutions to wave kinetic equations. *Applied Numerical Mathematics*, 2022.

[73] S. M'etens Y. Pomeau, M.A. Brachet and S. Rica. Théorie cinétique d'un gaz de bose dilué avec condensat. *C. R. Acad. Sci. Paris S'er. IIb M'ec. Phys. Astr.*, 327:791–798, 1999.

[74] Nikuni T. Griffin A. Zaremba, E. Dynamics of trapped bose gases at finite temperatures. *J. Low Temp. Phys.*, 116:277–345, 1999.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Email address: gigliola@math.mit.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

Email address: minhbinh@tamu.edu