Math 421 - Practice Problems

The exercises below are practice problems for the final exam.

1 Topics to cover

Lectures 6-12, HW4-10

2 SPIVAK BOOK

1. Exercises 3, 4, 5, 6, 7, 8 Chapter 6
2. Exercises 6, 8, 10 Chapter 7
3. Exercise 3 Chapter 8
4. Exercises 10, 16, 17, 19 Chapter 9
5. Exercises 6, 16, 22, 29 Chapter 10

3 Problems

1. The Fibonacci sequence is defined thus: $f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$. So the first terms are: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34,... Prove that for $n \geq 2$: $f_1 + f_2 + ... + f_n = f_{n+2} - 1$.
2. Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.
3. $S_n = \{1, 2, ..., n\}$. Prove by induction that there are exactly 2^n subsets of S_n.
4. Prove that $\sqrt{2} + 3^{1/5}$ is irrational.
5. Prove that $\lim_{x \to a} \frac{1}{x-a}$ does not exist.
6. Show that $a + b\sqrt{2}$ is irrational for all $a, b \in \mathbb{Z}, a, b \neq 0$.
7. Let $c > 0$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy

   $$|f(x) - f(y)| \leq c|x - y|$$

   for all $x, y \in \mathbb{R}$. Show that f is continuous. Deduce that cosine is a continuous function.
8. Prove that if g is continuous on $[0, 1]$ and $g(x)$ is an integer for each $x \in [0, 1]$, then g is constant on $[0, 1]$.
9. Prove that $\lim_{x \to 0} \sqrt[3]{x} = 0$ using the definition of the limit and not a theorem.

10. Prove that $\lim_{x\to 5} \sqrt{x+4} = 3$ directly from the definition, and not by using a theorem.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) = 0$ for $x \in \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

12. Prove that $log_5 2$ is irrational.

13. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

14. Suppose that $f'(0)$ exists and $f(x+y) = f(x)f(y)$ for all x and y.
Prove that $f'(x)$ exists for all x. Find $f'(x)$ for all x.

15. Use the Product Rule and that $\frac{d}{dx}x^n = nx^{n-1}$, $n \in \mathbb{Z}^+$ to show that $f : \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = x^{n+1/2}$, $n \in \mathbb{Z}^+$ is differentiable, and find $f'(x)$.

16. Prove or disprove: if f and g are continuous on $[a, b]$ and differentiable on (a, b), then $\max(f, g)$ is continuous on $[a, b]$ and differentiable on (a, b).