On the dynamics of finite temperature trapped Bose gases

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A R T I C L E   I N F O

A B S T R A C T

The system that describes the dynamics of a Bose–Einstein Condensate (BEC) and the thermal cloud at finite temperature consists of a nonlinear Schrodinger (NLS) and a quantum Boltzmann (QB) equations. In such a system of trapped Bose gases at finite temperature, the QB equation corresponds to the evolution of the density distribution function of the thermal cloud and the NLS is the equation of the condensate. The quantum Boltzmann collision operator in this temperature regime is the sum of two operators C_{12} and C_{22}, which describe collisions of the condensate and the non-condensate atoms and collisions between non-condensate atoms. Above the BEC critical temperature, the system is reduced to an equation containing only a collision operator similar to C_{22}, which possesses a blow-up positive radial solution with respect to the L^∞ norm (cf. [29]). On the other hand, at the very low temperature regime (only a portion of the transition temperature T_{BEC}), the system can be simplified into an equation of C_{12}, with a different (much higher order) transition probability, which has a unique global classical positive radial solution with weighted L^1 norm (cf. [3]). In our model, we first decouple the QB, which contains $C_{12} + C_{22}$, and the NLS equations, then show a global existence and uniqueness result for classical positive radial solutions to the spatially homogeneous kinetic system. Different from the case considered in [29], due to the presence of the BEC, the collision integrals are associated to sophisticated energy manifolds rather than spheres, since the

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particle energy is approximated by the Bogoliubov dispersion law. Moreover, the mass of the full system is not conserved while it is conserved for the case considered in [29]. A new theory is then supplied.

1. Introduction

The study of kinetic equations has a very long history, starting with the classical Boltzmann equation, which provides a description of the dynamics of dilute monoatomic gases (cf. [21–23,42,84]). As an attempt to extend the Boltzmann equation to deal with quantum gases, the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation was introduced [69, 83]. However, the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation fails to describe a Bose gas at temperatures which are close to and below the Bose–Einstein Condensate (BEC) critical temperature, due to the fact that its steady-state solution is a Bose–Einstein distribution in particle energies. Below the critical temperature, many-body effects modify the equilibrium distribution so that this distribution depends on quasiparticle energies. These are accounted for by mean fields which break the unperturbed Hamiltonian $U(1)$ gauge symmetry. Therefore, a new description in terms of quasiparticles is required. Such a quantum kinetic theory was initiated by Kirkpatrick and Dorfman [56,57], based on the rich body of research carried out in the period 1940–67 by Bogoliubov, Lee and Yang, Beliaev, Pitaevskii, Hugenholtz and Pines, Hohenberg and Martin, Gavoret and Nozi'eres, Kane and Kadanoff and many others. After the production of the first BECs, that later led Cornell, Wieman, and Ketterle to the 2001 Nobel Prize of Physics [4,5,12], there has been an explosion of research on the kinetic theory associated to BECs. Based on Kirkpatrick–Dorfman's works, Zaremba, Nikuni and Griffin successfully formulated a self-consistent Gross–Pitaevskii–Boltzmann model, which is nowadays known as the 'ZNG' theory (cf. [15,89]). Independent of the mentioned authors, Pomeau et al. [70] also proposed a similar model for the kinetics of BECs. Later, Gardinier, Zoller and collaborators derived a Master Quantum Kinetic Equation (MQKE) for BECs, which returns to the ZNG model at the limits, and introduced the terminology "Quantum Kinetic Theory" in the series of papers [33–37,52,53]. The ZNG theory also gave the first quantitative predictions of vortex nucleation at finite temperatures [86]. Many other experiments have also confirmed the validity of the model (cf. [73]). We refer to the review paper [6] for discussions on the condensate growth problem concerning the MQKE model and the books [43,51,72] for more theoretical and experimental justifications of the ZNG model, as well as the tutorial article [73] for an easy introduction. Let us mention that besides the ZNG theory, there have been other works describing the kinetics of BECs as well (see [1,50,55,77,78,80,82], and references therein).

Let us first recall the ZNG model for finite temperature trapped Bose gases, i.e. the temperature T of the gas is below the transition temperature T_{BEC} but above absolute

zero. Denote $f(t, r, p)$ to be the density function of the Bose gas at time t, position r and momentum p and $\Phi(t, r)$ to be the wave function of the BEC. Employing the short-handed notation $f_i = f(t, r, p_i)$, $i = 1, 2, 3, 4$. The Schrödinger (or the Gross–Pitaevski) equation for the condensates reads (cf. [15]):

$$
\begin{aligned}
i\hbar\partial_t\Phi(t, r) &= \Big(-\frac{\hbar^2\Delta_r}{2m} + g[N_c(t, r) + 2n_n(t, r)] - i\Lambda_{12}[f](t, r) + V(r)\Big) \\
&\quad \times \Phi(t, r), \quad (t, r) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\Lambda_{12}[f](t, r) &= \frac{\hbar}{2N_c}\Gamma_{12}[f](t, r), \\
\Gamma_{12}[f](t, r) &= \int\limits_{\mathbb{R}^3} C_{12}[f](t, r, p)\frac{dp}{(2\pi\hbar)^3}, \\
n_n(t, r) &= \int\limits_{\mathbb{R}^3} f(t, r, p)dp, \\
\Phi(0, r) &= \Phi_0(r), \forall r \in \mathbb{R}^3,
\end{aligned}
\tag{1.1}
$$

where $N_c(t, r) = |\Phi|^2(t, r)$ is the condensate density, $\hbar$ is the Planck constant, g is the interaction coupling constant proportional to the s-wave scattering length a, $V(r)$ is the confinement potential, and the operator C_{12} can be found in the quantum Boltzmann equation for the non-condensate atoms (cf. [15]), written below:

$$
\partial_t f(t, r, p) + \frac{p}{m} \cdot \nabla_r f(t, r, p) - \nabla_r U(t, r) \cdot \nabla_p f(t, r, p) \tag{1.2}
$$

$$
= Q[f](t, r, p) := C_{12}[f](t, r, p) + C_{22}[f](t, r, p), (t, r, p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3,
$$

$$
\begin{aligned}
C_{12}[f](t, r, p_1) := \lambda_1 N_c(t, r) &\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} K^{12}(p_1, p_2, p_3)\delta(mv_c + p_1 - p_2 - p_3) \\
&\times \delta(\mathcal{E}_c + \mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\times [(1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3)]dp_2 dp_3 \\
-2\lambda_1 n_c(t, r) &\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \delta(mv_c + p_2 - p_1 - p_3)\delta(\mathcal{E}_c + \mathcal{E}_{p_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_3}) \\
&\times [(1 + f_2)f_1 f_3 - f_2(1 + f_1)(1 + f_3)]dp_2 dp_3,
\end{aligned}
\tag{1.3}
$$

$$
\begin{aligned}
C_{22}[f](t, r, p_1) := \lambda_2 &\iiint\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4)\delta(p_1 + p_2 - p_3 - p_4) \\
&\times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\
&\times [(1 + f_1)(1 + f_2)f_3 f_4 - f_1 f_2(1 + f_3)(1 + f_4)]dp_2 dp_3 dp_4, \\
f(0, r, p) = f_0(r, p), (r, p) &\in \mathbb{R}^3 \times \mathbb{R}^3,
\end{aligned}
\tag{1.4}
$$

where $\lambda_1 = \frac{2g^2}{(2\pi)^2\hbar^4}$, $\lambda_2 = \frac{2g^2}{(2\pi)^5\hbar^7}$, m is the mass of the particles, and $\mathcal{E}_p$ is the Hartree–Fock energy [43]

$$\mathcal{E}_p = \frac{|p|^2}{2m} + U(t, r). \tag{1.5}$$

Notice that C_{22} is the Boltzmann–Nordheim (Uehling–Ulenbeck) quantum Boltzmann collision operator. If one writes

$$\Phi = |\Phi(t, r)| e^{i\phi(r, t)}, \tag{1.6}$$

the condensate velocity can be defined as

$$v_c(t, r) = \frac{\hbar}{m} \nabla \phi(t, r), \tag{1.7}$$

and the condensate chemical potential is then

$$\mu_c = \frac{1}{\sqrt{n_c}} \left(-\frac{\hbar^2 \Delta_r}{2m} + V + g[2n_n + N_c] \right) \sqrt{N_c}. \tag{1.8}$$

The potential U and the condensate energy $\mathcal{E}_c$ are written as follows

$$U(t, r) = V(r) + 2g[N_c(t, r) + n_n(t, r)], \tag{1.9}$$

and

$$\mathcal{E}_c(t, r) = \mu_c(t, r) + \frac{m v_c^2(t, r)}{2}. \tag{1.10}$$

Notice that (1.15) describes collisions of the condensate and the non-condensate atoms (condensate growth term), (1.16) describes collisions between non-condensate atoms, and (1.1) is the defocusing nonlinear Schrodinger equation of the condensate (see Fig. 1). For the sake of simplicity, we denote $\lambda_1 = \frac{2g^2 N_c}{(2\pi)^2\hbar^4}$ and $\lambda_2 = \frac{2g^2}{(2\pi)^5\hbar^7}$.

The transition probability kernel

$$K^{12}(p_1, p_2, p_3) = |A^{12}(|p_1|, |p_2|, |p_3|)|^2$$

of C_{12} is given by the scattering amplitude (cf. [26,47,48,56,57,74])

$$A^{12}(|p_1|, |p_2|, |p_3|) :=$$
$$:= (u_{p_3} - v_{p_3})(u_{p_1}u_{p_2} + v_{p_1}v_{p_2}) + (u_{p_2} - v_{p_2})(u_{p_1}u_{p_3} + v_{p_1}v_{p_3}) \tag{1.11}$$
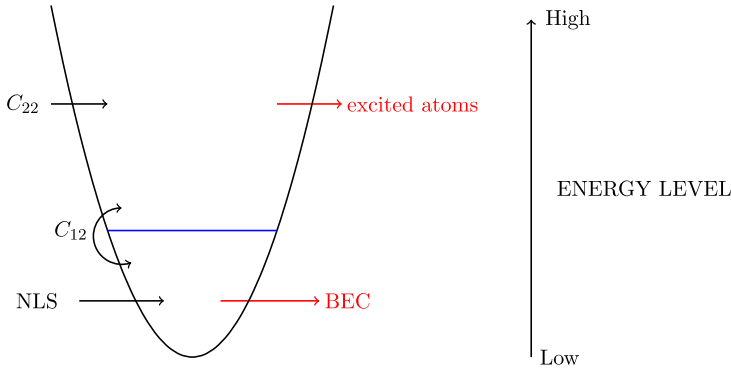$$- (u_{p_1} - v_{p_1})(u_{p_2}v_{p_3} + v_{p_2}u_{p_3}),$$

where

Fig. 1. The Bose–Einstein Condensate (BEC) and the excited atoms.

$$u_p^2 = \frac{\frac{p^2}{2m} + gn_c + \mathcal{E}_p}{2\mathcal{E}_p}, \quad u_p^2 - v_p^2 = 1.$$

The transition probability kernel

$$K^{22}(p_1, p_2, p_3, p_4) = |A^{22}(p_1, p_2, p_3, p_4)|^2$$

of C_{22} is given by the scattering amplitude (cf. (cf. [26,47,48,56,57,74]))

$$A^{22}(|p_1|, |p_2|, |p_3|, |p_4|) :=$$
$$:= u_{p_1} u_{p_2} u_{p_3} u_{p_4} + u_{p_1} v_{p_2} u_{p_3} v_{p_4} + u_{p_1} v_{p_2} v_{p_3} u_{p_4} + v_{p_1} u_{p_2} u_{p_3} v_{p_4} \qquad (1.12)$$
$$+ v_{p_1} u_{p_2} v_{p_3} u_{p_4} + v_{p_1} v_{p_2} v_{p_3} v_{p_4}.$$

When the temperature of the system is very low $T < 0.5 T_{BEC}$ (cf. [43]), the Hartree–Fock energy is no longer valid. A more general energy is used instead: the Bogoliubov dispersion law (cf. [26,56,57])

$$\mathcal{E}_p = \mathcal{E}(p) = \sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4}, \qquad \kappa_1 = \frac{gN_c}{m} > 0, \quad \kappa_2 = \frac{1}{4m^2} > 0. \qquad (1.13)$$

Notice that the first rigorous proof of BEC in a physically realistic, continuum model was given in 2002 by Lieb and Seiringer (cf. [58]). Besides the kinetic theory point of view, there are other approaches, valid with different physical assumptions, of understanding the dynamics of BECs and their thermal clouds, for instance, the works [11,13,25,44–46, 64,76], and cited references.

The toy model

Since the system (1.1)–(1.2) is too complicated, it is impossible to study all its properties in one single paper. As the first step to understand (1.1)–(1.2), we impose a few simplifications. We suppose that the equation (1.2) is homogeneous in space and the

condensate density distribution function $N_c = |\Phi|^2$ can be considered as a constant n_c. The system is then reduced to the following toy model:

$$\frac{\partial f}{\partial t} = Q[f] := C_{12}[f] + C_{22}[f], (t, p) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{1.14}$$

$$f(0, p) = f_0(p), p \in \mathbb{R}^3,$$

in which we rewrite C_{12} and C_{22} following the KD style [56,57], since it is simpler

$$C_{12}[f](t, r, p_1) := n_c \lambda_1 \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \tag{1.15}$$

$$\times [(1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3)] dp_2 dp_3$$

$$- 2n_c \lambda_1 \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3) \delta(p_2 - p_1 - p_3) \delta(\mathcal{E}_{p_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_3})$$

$$\times [(1 + f_2)f_1 f_3 - f_2(1 + f_1)(1 + f_3)] dp_2 dp_3,$$

$$C_{22}[f](t, r, p_1) := \lambda_2 \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \tag{1.16}$$

$$\times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times$$

$$\times [(1 + f_1)(1 + f_2)f_3 f_4 - f_1 f_2(1 + f_3)(1 + f_4)] dp_2 dp_3 dp_4,$$

where $\mathcal{E}_p$ is the Bogoliubov dispersion relation (1.13) and is valid when $T < 0.5T_{BEC}$ (cf. [26]). The form of the Bogoliubov dispersion relation makes the study of (1.14) much more difficult than the classical Boltzmann equation and it is the goal of our paper to develop techniques that can resolve this difficulty.

Let us mention that in real physical situations, the system of bosons is normally not spacial dependent. Moreover, the toy model (1.14) does not really describe the full dynamics of the thermal cloud-condensate system since the equation does not conserve the total number of atoms. However, it is interesting to begin studying the ZNG model by understanding the spatially homogeneous kinetic equation with the Bogoliubov dispersion relation, in order to gain some insights into the full model. Such toy models are indeed useful and have been used in the physics community (cf. [27,47–49]), under the assumption that the temperature of the system is very low $T < 0.01T_{BEC}$. In such a low temperature system, the portion of excited atoms outside the condensate is very small, in comparison with the number of condensed atoms; and therefore N_c can be regarded as a constant. For instance, in [48], the authors consider only the kinetic part of the system, which is spatially homogeneous, as what we are assuming in our paper (equation (10) in [48])

$$\partial_t f = C_{12}[f] + C_{22}[f] + C_{13}[f].$$

Indeed, the contribution of [48] is that they could derive a new collision operator $C_{13}[f]$ which complements the ZNG theory in some cases. We refer to [47–49] for the discussion about the reason why the number of the excitations (bogolon number) is not conserved. In those works, the solutions of the kinetic equations are shown to converge to equilibrium and explicit convergence rates are computed. The theoretical findings are shown to be in perfect agreements with experimental results and with the famous Lee–Yang theory. A similar model has also been used in [27]. We refer to the physics paper [75] for more theoretical and experimental discussions on spacial homogeneous models describing the dynamics of low temperature dilute Bose gases.

Notice that the two ways of writing (1.3)–(1.4) and (1.15)–(1.16) are equivalent as explained in [89]. Indeed, in (1.15)–(1.16), p_i represent the quasiparticle momentum in the local rest frame.

Let us mention a difficulty, pointed out by Eckern (cf. [26]), that arises from the form of the transition probability A^{22}. Define the characteristic momentum for the crossover between the linear and the quadratic part of the spectrum to be $p_0 = 2mn_c g$, in which g is the repulsive point interaction. If all momenta are much smaller than p_0 i.e. $|p_1|, |p_2|, |p_3| << p_0$, we obtain the following unphysical asymptotic behavior (cf. [26])

$$|A^{22}(|p_1|, |p_2|, |p_3|, |p_4|)|^2 \approx |p_1|^{-1}|p_2|^{-1}|p_3|^{-1}|p_4|^{-1}.$$

This question is still open in the physics community, as discussed in [14,26,66,71]. In a mathematical point of view, there are many kinetic equations with singular kernels for which it is possible to prove existence of solutions. However, such singularities in our case lead to the loss in moments of the solutions of the equation. An investigation of this sophisticated question will be the scope of our forthcoming paper. As a consequence, to avoid this singular behavior, the following transition probability is chosen for mathematical convenience

$$K^{22}(p_1, p_2, p_3, p_4) = |A^{22}(|p_1|, |p_2|, |p_3|, |p_4|)|^2 \chi_{\{|p_1|, |p_2|, |p_3|, |p_4| \geq p_0\}}, \tag{1.17}$$

where $\chi_{\{|p_1|, |p_2|, |p_3|, |p_4| \geq p_0\}}$ is the characteristic function of the set $\{|p_1|, |p_2|, |p_3|, |p_4| \geq p_0\}$: namely, it turns out that the dominant collision process of non-condensate atoms in the low temperature region with small momenta is the interactions between non-condensate atoms and the condensate; in this region, we there suppose that the effect of C_{22} is much smaller than C_{12}. In other words, we assume that when p is small C_{12} is dominant and when p is large C_{22} is dominant, since C_{12} describes the collisions between excited atoms at low quantum levels and atoms in the ground state, and C_{22} describes collisions between excited atoms at high quantum levels. As a consequence, it is reasonable to impose a cut-off on the kernel of C_{22} and not on C_{12}. With this truncated transition probability, there exists a positive constant Γ depending on p_0, such that

$$K^{22}(p_1, p_2, p_3, p_4) < \Gamma. \tag{1.18}$$

Note that some other mathematical results for quantum kinetic equations have been obtained in [3,24,28,32,40,54,67,68,75,79]. Quantum kinetic equations have very similar formulations with the so-called wave turbulence kinetic equations. We refer to [18,19,30, 31,38,39,41,62,65,81,87,88] for more recent advances on the theory.

In the current work, we restrict our attention to spatial homogeneous and radial solutions of (1.14)

$$f(0, p) = f_0(|p|), \quad f(t, p) = f(t, |p|).$$

Note that in [29], the authors consider the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation

$$\partial_t f_1 = \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4)\delta(|p_1|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2) \times$$
$$\times [(1 + f_1)(1 + f_2)f_3 f_4 - f_1 f_2(1 + f_3)(1 + f_4)]dp_2 dp_3 dp_4,$$

which, by the radially symmetry assumption of f, can be reduced to the equivalent form

$$\partial_t f_1 = \iiint\limits_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} \frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|}{|p_1|^2}$$
$$\times \delta(|p_1|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2)[f_3 f_4(1 + f_1 + f_2)$$
$$- f_1 f_2(1 + f_3 + f_4)]d|p_2|d|p_3|d|p_4|.$$

By the same argument as in [29], C_{22} also has the following form

$$
\begin{aligned}
C_{22}[f] = {} & \kappa_3 \iiint\limits_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} K^{22}(p_1, p_2, p_3, p_4)\frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|}{|p_1|^2} \\
& \times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \\
& \times [f_3 f_4(1 + f_1 + f_2) - f_1 f_2(1 + f_3 + f_4)]d|p_2|d|p_3|d|p_4|,
\end{aligned}
\tag{1.19}
$$

where κ_3 is some positive constant.

Equation (1.14) can be simplified as follows

$$\frac{\partial f}{\partial t} = Q[f] = C_{12}[f] + C_{22}[f], \quad f(0, p) = f_0(|p|), \forall p \in \mathbb{R}^3. \tag{1.20}$$

We are interested in the existence and uniqueness of strong, classical and radial solutions of (1.20).

Definition 1.1. A function f is defined to be a strong radial solution in $C([0, T], X) \cap C^1((0, T), Y)$, for some function spaces X, Y, to (1.20), where C_{22} is of the form (1.19),

K_{22} is of the form (1.18), $\mathcal{E}_p$ is the Bogoliubov dispersion law and N_c is assume to the constant n_c, if and only if f satisfies

$$\frac{\partial f}{\partial t} = Q[f] = C_{12}[f] + C_{22}[f], \quad f(0,p) = f_0(|p|), \text{ for a.e. } p \in \mathbb{R}^3.$$

Moreover $f(t,p) = f(t,|p|)$ for all $(t,p) \in [0,T) \times \mathbb{R}^3$.

Bosons are sensitive to temperature. When the temperature is below the transition temperature $T < T_{BEC}$, the BEC is formed. When we lower the temperature T, the behavior of the quasi-particles change. This can be seen clearly through the Bogoliubov dispersion relation, which depends on the density of the condensate and the temperature since g depends on T. In the lower temperature range when T is only a portion of T_{BEC}, sometimes, we can suppose (cf. [26,27]) that the interaction between bosons, i.e. the C_{22} collision operator, is negligible, and the BEC is very stable. In this case, the system can be reduced to a kinetic equation involving the C_{12} collision operator only:

$$\frac{\partial f}{\partial t} = C_{12}[f], \quad f(0,p) = f_0(p), \forall p \in \mathbb{R}^3. \tag{1.21}$$

In this regime, the transition probability takes the form $\mathcal{C}_K|p_1||p_2||p_3|$, which is unbounded, while (1.11) is bounded. In the series of beautiful works [8–10], the study of (1.21) has been done for the first time. In [3] it has been proved that (1.21) has a unique positive radial solution, based on an argument of propagation of polynomial and exponential moments. We will see later that, unlike (1.21), polynomial and exponential moments of solutions of (1.14) are not propagating on the time interval $[0,\infty)$, due to the presence of the collision operator C_{22}. In [67], it is prove that the solution of (1.21) is bounded from below by a Gaussian. In other words, the operator C_{12} is "strongly" positive.

Above the BEC critical temperature, the density of the condensate n_c is 0, then $C_{12} = 0$. Equation (1.20) is reduced to the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation

$$\frac{\partial f}{\partial t} = C_{22}[f], \quad f(0,p) = f_0(|p|), \forall p \in \mathbb{R}^3, \tag{1.22}$$

which has a blow-up positive radial solution in the L^∞ norm if the mass of the initial data is too concentrated around the origin (cf. [29]). Note that in this temperature regime, the transition probability is $K^{22} = 1$ (cf. [47,49]), which is different from the regime considered in this paper. The existence of a global weak and measure solution for the equation was treated in [59–61]. In [17], local existence and uniqueness results, with respect to the L^∞ norm, were obtained for the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation. Let us mention that when the temperature is above the BEC critical temperature, the energy is of the form $\frac{p^2}{2m}$. The collision of two microscopic boxes of particles with momenta p_1 and p_2 changes the momenta into p_3 and p_4; and the conservation laws read:

$$|p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2, \quad p_1 + p_2 = p_3 + p_4.$$

Since p_1, p_2, p_3, p_4 belong to the sphere centered at $\frac{p_1+p_2}{2}$ with radius $\frac{|p_1-p_2|}{2}$, the collision operator C_{22} can be expressed as a integration on a sphere, following the strategy represented in [20,84] for the classical Boltzmann operator.

In our case $\mathcal{E}_p$ is approximated by the Bogoliubov dispersion law (1.13), the collision operators are integrals on much more complicated manifolds. Classical techniques used for the classical Boltzmann equation cannot be applied. New estimates on energy manifolds, such as

$$\mathcal{E}_{p_1} = \mathcal{E}_{p_2} + \mathcal{E}_{p_3}, \quad p_1 = p_2 + p_3,$$

are then required.

Moreover, (1.22) conserves the mass of the solution, while the full equation (1.20) does not. As a consequence, estimating the mass of the solution to (1.20) is a crucial task.

Let us emphasize that due to the presence of the C_{12} term, which is much more complicated than the classical Boltzmann collision operators due to its non-symmetry structure, (1.20) is much more complicated than the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation, as it has already been noticed in a series of beautiful works [8–10], where the study of C_{12} has been done for the first time. In order to study C_{12}, the authors of [3] have developed special techniques, based on the ideas of propagation and creation of exponential and polynomial moments for only the collision operator C_{12}. In our case, as it is shown later, the mass of the solution of (1.20) is not conserve and the presence of $C_{12} + C_{22}$ makes the problem different from the case considered in [3]. Indeed, in the works [8–10], the authors impose a cut-off on C_{12} for small momentums. As a consequence, the authors approximate the Bogoliubov dispersion relation by a simplified one, which significantly simplifies the analysis. However, we do not need to impose this cut-off on C_{12}. Since C_{12} is used in its most general form in our case, it is very important that the Bogoliubov dispersion relation is kept.

Moreover, we tried the strategy of [7] used to prove the existence and uniqueness of classical solutions to the classical Boltzmann equation, but it does not seem applicable. The main reason is that for (1.20), we can only establish bounds on weighted L^1 norms of the solution. Bounds on weighted L^p norms is still an open question and the H-theorem is not useful in this case. As a consequence, the Dunford–Pettis theorem cannot be used and the strategy of [7], even though very powerful, cannot be directly applied. We then have to develop new ideas to show that (1.20) indeed has a global and classical solution in weighted L^1 spaces. Our result is different from the results considered in [59–61] about the existence and uniqueness of global weak solution for the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation. Note that our method also works for the collision kernel K^{12} of the more complicated form $C|p_1|^\varrho |p_2|^\varrho |p_3|^\varrho$, $(\varrho \geq 0)$.

Let us define

$$L^1_m(\mathbb{R}^3) = \left\{ \ f \ \Big| \ \|f\|_{L^1_m} := \int\limits_{\mathbb{R}^3} |p|^m |f(p)| dp < \infty \right\}, \tag{1.23}$$

$$\mathcal{L}^1_m(\mathbb{R}^3) = \left\{ \ f \ \Big| \ \|f\|_{\mathcal{L}^1_m} := \int\limits_{\mathbb{R}^3} |f(p)| \mathcal{E}_p^{m/2} dp < \infty \right\}, \tag{1.24}$$

$$\mathbb{L}^1_m(\mathbb{R}^3) = \left\{ \ f \ \Big| \ \|f\|_{\mathbb{L}^1_m} := \int\limits_{\mathbb{R}^3} |f(p)| \left(1 + \mathcal{E}_p^{m/2}\right) dp < \infty \right\}. \tag{1.25}$$

Our main result is the following theorem.

Theorem 1.1. *Suppose that $f_0(p) = f_0(|p|) \geq 0$, and*

$$\int\limits_{\mathbb{R}^3} (1 + \mathcal{E}_p) f_0(p) dp < \infty.$$

For any time interval $[0,T]$, let n, n^* be two positive integers, $n > 1$, n_* is an odd number, $n^* > n + 4$. For any positive number $\mathcal{R}$, there exists $\mathfrak{c}_{n_*}$ depending on $\mathcal{R}$ and T satisfying $\mathfrak{c}_{n_*}(\mathcal{R},T)$ tends to infinity as T or $\mathcal{R}$ tends to infinity, such that if

$$\int\limits_{\mathbb{R}^3} \mathcal{E}_p^{n^*} f_0(p) < \mathfrak{c}_{n_*}(\mathcal{R},0), \int\limits_{\mathbb{R}^3} f_0(p) < \mathcal{R},$$

then there exists a unique classical positive radial solution

$$f(t,p) = f(t,|p|) \in C^0([0,T], \mathbb{L}^1_{2n}(\mathbb{R}^3)) \cap C^1((0,T), \mathbb{L}^1_{2n}(\mathbb{R}^3))$$

of (1.20) where C_{22} is of the form (1.19), K_{22} is of the form (1.18), $\mathcal{E}_p$ is the Bogoliubov dispersion law and N_c is assume to the constant n_c.

One of the key ingredients of the proof of Theorem 1.1 is the following theorem about the existence and unique of solutions to ODEs on Banach spaces. The theorem has an inspiration from [2,3,16,63]. Notice that different from the previous cases considered in [2,3], we do not have the propagation of polynomial and exponential moments of the solution, as a consequence, we introduce new ideas to deal with this difficulty. Those ideas are discussed in Remarks 1.1, 1.2 and 1.3.

Theorem 1.2. *Let $[0,T]$ be a time interval, $E := (E, \|\cdot\|)$ be a Banach space, $\mathcal{S}_T$ be a bounded, convex and closed subset of E, and $Q : \mathcal{S}_T \to E$ be an operator satisfying the following properties:*

($\mathfrak{A}$) *Let* $\| \cdot \|_*$ *be a different norm of* E, *satisfying* $\| \cdot \|_* \leq C_E \| \cdot \|$ *for some universal constant* C_E, *and the function*

$$| \cdot |_* : E \longrightarrow \mathbb{R}$$
$$u \longrightarrow |u|_*,$$

satisfying

$$|u + v|_* \leq |u|_* + |v|_*, \ \text{and} \quad |\alpha u|_* = \alpha |u|_*$$

for all u, v *in* E *and* $\alpha \in \mathbb{R}_+$.
Moreover,

$$|u|_* = \|u\|_*, \forall u \in \mathcal{S}_T, \quad |u|_* \leq \|u\|_* \leq C_E \|u\|, \forall u \in E,$$
$$|Q(u)|_* \leq C_*(1 + |u|_*), \forall u \in \mathcal{S}_T,$$

and

$$\mathcal{S}_T \subset \overline{B_*\Big(O, (2R_* + 1)e^{(C_*+1)T}\Big)} := \overline{\Big\{u \in E \Big| \|u\|_* \leq (2R_* + 1)e^{(C_*+1)T}\Big\}},$$

for some positive constant $R_* \geq 1$.
($\mathfrak{B}$) *Sub-tangent condition*

$$\liminf_{h \to 0^+} h^{-1} dist\big(u + hQ[u], \mathcal{S}_T\big) = 0, \qquad \forall u \in \mathcal{S}_T \cap B_*\Big(O, (2R_* + 1)e^{(C_*+1)T}\Big),$$

($\mathfrak{C}$) *Hölder continuity condition*

$$\big\|Q[u] - Q[v]\big\| \leq C\|u - v\|^\beta, \quad \beta \in (0, 1), \quad \forall u, v \in \mathcal{S}_T,$$

($\mathfrak{D}$) *one-side Lipschitz condition*

$$\big[Q[u] - Q[v], u - v\big] \leq C\|u - v\|, \qquad \forall u, v \in \mathcal{S}_T,$$

where

$$\big[\varphi, \phi\big] := \lim_{h \to 0^-} h^{-1}\big(\|\phi + h\varphi\| - \|\phi\|\big).$$

Then the equation

$$\partial_t u = Q[u] \ \text{on} \ [0, T] \times E, \qquad u(0) = u_0 \in \mathcal{S}_T \cap B_*(O, R_*) \tag{1.26}$$

has a unique solution in $C^1((0, T), E) \cap C([0, T], \mathcal{S}_T)$.

Remark 1.1. Note that for (1.20), the mass is not conserved. We indeed prove that it grows exponentially in Section 2.1.3. As a consequence, in Theorem 1.2, besides the norm $\|\cdot\|$ of the Banach space E, we also need the second norm $\|\cdot\|_*$ and the ball

$$B_*\Big(O, (2R_* + 1)e^{(C_*+1)T}\Big),$$

which take the crucial role in controlling the mass of the solution on the time interval $[0, T]$.

Thanks to the control on the mass, we can later prove that the collision operator Q in (1.20) is indeed Holder continuous, which means Condition $(\mathfrak{C})$ of Theorem 1.2 is satisfied.

Remark 1.2. In Theorem 1.2, $|\cdot|_*$ is a function from E to $\mathbb{R}$, that coincides with the second norm in $\|\cdot\|_*$ in the set $\mathcal{S}_T$. This is due to the fact that, we will choose $\mathcal{S}_T$ to be a subset of the positive cone of $E = \mathbb{L}^1_{2n}(\mathbb{R}^3)$.

Remark 1.3. In Condition $(\mathfrak{B})$ of Theorem 1.2, we do not consider the boundary case where

$$\|u\|_* = (2R_* + 1)e^{(C_*+1)T}.$$

Our idea of the proof is to start with an initial condition $u(0)$ in the intersection of $\mathcal{S}_T$ and the ball $B_*(O, R_*)$, and make $u(t)$ evolve as long as

$$\|u(t)\|_* < (2R_* + 1)e^{(C_*+1)T}.$$

This idea is realized, in a discrete way, in Part 2 of the proof of Theorem 1.2.

The plan of the paper is as follows:

- Section 2 is devoted to the proof of Theorem 1.1. This proof is divided into several steps:
  - In Section 2.1, basic properties of Equation (1.20) are presented. We prove that solutions of (1.20) conserve momentum and energy in Section 2.1.1. However, different from the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation (1.22), the mass is not conserved for the full equation. Therefore, estimating the mass is a crucial task. Notice that different from previous studies (cf. [29]), where the energy is

  $$\mathcal{E}_p = \frac{p^2}{2m};$$

  in our case, due to the presence of the condensate, the energy is approximated by the Bogoliubov dispersion law (1.13). This requires new estimates on the energy

surfaces. Section 2.1.2 is devoted to such estimates. Based on these estimates, in Section 2.1.3, we provide a bound of the mass of solutions to Equation (1.20) on a finite time interval $[0, T]$.

– As a key ingredient of the proof of Theorem 1.1, we show in Section 2.2 that polynomial moments with arbitrary high orders of solutions of (1.20) are bounded on a finite time interval $[0, T]$, which is the content of Proposition 2.4. Note that different from the very low temperature regimes considered in [3], in our regimes, polynomial moments are not propagating an created on $[0, \infty)$. The strategy of the proof of the proposition is to estimate moments of the collision operators C_{12} and C_{22}, which are done in Sections 2.2.1 and 2.2.2 using results on energy surfaces of Section 2.1.2. Based on these estimates, we obtain a differential inequality for finite time moments of high orders in Section 2.2.3, which leads to the desired results of Proposition 2.4.

– In Section 2.3, we prove that the collision operators C_{12} and C_{22} are Holder continuous, thanks to Proposition 2.4. In order to do this, we decompose C_{22} as the sum of two operators C_{22}^1 and C_{22}^2, where the first one is of second order and the second one is of third order. The operators C_{12}, C_{22}^1 and C_{22}^2 are proven to be Holder continuous in Sections 2.3.1, 2.3.2 and 2.3.3, respectively, on any time interval $[0, T]$.

– Using Theorem 1.2, we prove in Section 2.4 that Equation (1.20) has a unique positive, radial solution on any time interval $[0, T]$.

• The proof of Theorem 1.2 is given in Section 3.

2. The quantum Boltzmann equation

2.1. Mass, momentum and energy of solutions of the kinetic equation

We will make use of the following notation

$$m_k[f] = \int_{\mathbb{R}^3} \mathcal{E}^k(p_1) f(p_1) dp_1. \tag{2.1}$$

For convenience, we introduce

$$C_{12}[f] = C_{12}^1[f] + C_{12}^2[f] \tag{2.2}$$

with

$$C_{12}^1[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{K}^{12}(p_1, p_2, p_3) \Big[f(p_2) f(p_3) - f(p_1)(f(p_2) + f(p_3) + 1) \Big] dp_2 dp_3$$

$$C_{12}^2[f] := -2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{K}^{12}(p_2, p_1, p_3) \Big[f(p_1) f(p_3) - f(p_2)(f(p_1) + f(p_3) + 1) \Big] dp_2 dp_3,$$

where the collision kernel is defined by

$$\mathcal{K}^{12}(p_1, p_2, p_3) = \lambda_1 n_c K^{12}(p_1, p_2, p_3)\Big(\delta(\mathcal{E}(p_1) - \mathcal{E}(p_2) - \mathcal{E}(p_3))\delta(p_1 - p_2 - p_3)\Big).$$

We also define the energy surfaces/resonance manifolds

$$S_p^0 := \Big\{p_* \in \mathbb{R}^3 \ : \ \mathcal{E}(p - p_*) + \mathcal{E}(p_*) = \mathcal{E}(p)\Big\}$$

$$S_p^1 := \Big\{p_* \in \mathbb{R}^3 \ : \ \mathcal{E}(p + p_*) = \mathcal{E}(p) + \mathcal{E}(p_*)\Big\} \tag{2.3}$$

$$S_p^2 := \Big\{p_* \in \mathbb{R}^3 \ : \ \mathcal{E}(p_*) = \mathcal{E}(p) + \mathcal{E}(p_* - p)\Big\}$$

for all $p \in \mathbb{R}^3 \setminus \{0\}$ and the functions

$$H_0^p(x) := \mathcal{E}(p - x) + \mathcal{E}(x) - \mathcal{E}(p),$$

$$H_1^p(x) := \mathcal{E}(p + x) - \mathcal{E}(p) - \mathcal{E}(x), \tag{2.4}$$

$$H_2^p(x) := \mathcal{E}(x) - \mathcal{E}(p) - \mathcal{E}(x - p).$$

Set

$$\bar{K}^{12}(p_1, p_2, p_3) = \lambda_1 n_c K^{12}(p_1, p_2, p_3),$$

by the nature of the Dirac delta function, the collision operators can be expressed under the form of the following surface integrals

$$C_{12}^1[f]$$
$$:= \int\limits_{S_{p_1}^0} \bar{K}_0^{12}(p_1, p_1 - p_3, p_3)\Big[f(p_1 - p_3)f(p_3) - f(p_1)(f(p_1 - p_3) + f(p_3) + 1)\Big]\, d\sigma(p_3)$$

$$C_{12}^2[f]$$
$$:= 2\int\limits_{S_{p_1}^1} \bar{K}_1^{12}(p_1 + p_3, p_1, p_3)\Big[f(p_1 + p_3)(f(p_1) + f(p_3) + 1) - f(p_1)f(p_3)\Big]\, d\sigma(p_3),$$

where

$$\bar{K}_0^{12}(p_1, p_1 - p_3, p_3) = \frac{\bar{K}^{12}(p_1, p_1 - p_3, p_3)}{|\nabla H_p^0(p_3)|}, \quad \bar{K}_1^{12}(p_1 + p_3, p_1, p_3) = \frac{\bar{K}^{12}(p_1 + p_3, p_1, p_3)}{|\nabla H_p^1(p_3)|}.$$

We also split $C_{12}[f]$ as the sum of gain and loss terms:

$$C_{12}[f] = C_{12}^{\text{gain}}[f] - C_{12}^{\text{loss}}[f] \tag{2.5}$$

with

$$C_{12}^{\text{gain}}[f] := \int_{S_{p_1}^0} \bar{K}_0^{12}(p_1, p_1 - p_3, p_3) f(p_1 - p_3) f(p_3)\, d\sigma(p_3)$$

$$+ 2 \int_{S_{p_1}^1} \bar{K}_1^{12}(p_1 + p_3, p_1, p_3) f(p_1 + p_3) \Big(f(p_1) + f(p_3) + 1 \Big)\, d\sigma(p_3),$$

$$C_{12}^{\text{loss}}[f] := f C_{12}^-[f],$$

$$C_{12}^-[f] := \int_{S_{p_1}^0} \bar{K}_0^{12}(p_1, p_1 - p_3, p_3) \Big(f(p_1 - p_3) + f(p_3) + 1 \Big)\, d\sigma(p_3)$$

$$+ 2 \int_{S_{p_1}^1} \bar{K}_1^{12}(p_1 + p_3, p_1, p_3) f(p_3)\, d\sigma(p_3).$$

Similar as for C_{12}, we also split C_{22} into gain and loss operators, as follows

$$C_{22}[f] = C_{22}^{\text{gain}}[f] - C_{22}^{\text{loss}}[f], \qquad (2.6)$$

where

$$C_{22}^{\text{gain}}[f] := \lambda_2 \iiint_{\mathbb{R}^{3 \times 3}} \mathcal{K}^{22}(p_1, p_2, p_3, p_4)(1 + f(p_1))(1 + f(p_2)) f(p_3) f(p_4)\, dp_2 dp_3 dp_4,$$

$$C_{22}^{\text{loss}}[f] := f C_{22}^-[f],$$

$$C_{22}^-[f] := \lambda_2 \iiint_{\mathbb{R}^{3 \times 3}} \mathcal{K}^{22}(p_1, p_2, p_3, p_4) f(p_2)(1 + f(p_3))(1 + f(p_4))\, dp_2 dp_3 dp_4,$$

and

$$\mathcal{K}^{22}(p_1, p_2, p_3, p_4) = \lambda_2 K^{22}(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}).$$

We also split Q into the sum of a gain and a loss operators

$$Q[f] = Q^{\text{gain}}[f] - Q^{\text{loss}}[f], \qquad (2.7)$$

where

$$Q^{\text{gain}}[f] = C_{12}^{\text{gain}}[f] + C_{22}^{\text{gain}}[f],$$
$$Q^{\text{loss}}[f] = C_{12}^{\text{loss}}[f] + C_{22}^{\text{loss}}[f],$$

and

$$Q^{\text{loss}}[f] = fQ^-[f],$$

with

$$Q^-[f] = C_{12}^-[f] + C_{22}^-[f].$$

2.1.1. Conservation of momentum and energy and the H-Theorem

In this section, we obtain the basic properties of smooth solutions of (1.20).

Lemma 2.1. *There holds*

$$\int_{\mathbb{R}^3} Q[f](p_1)\varphi(p_1)dp_1$$

$$= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} R_{12}[f](p_1,p_2,p_3)\Big(\varphi(p_1) - \varphi(p_2) - \varphi(p_3)\Big)\, dp_1 dp_2 dp_3$$

$$+ \frac{1}{2} \iiiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} R_{22}[f](p_1,p_2,p_3,p_4)\Big(\varphi(p_1) + \varphi(p_2) - \varphi(p_3) - \varphi(p_4)\Big)\, dp_1 dp_2 dp_3 dp_4,$$

for any smooth test function φ, where

$$R_{12}[f](p_1,p_2,p_3) = \lambda_1 n_c K^{12}(p_1,p_2,p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$$
$$\times [(1 + f(p_1))f(p_2)f(p_3) - f(p_1)(1 + f(p_2))(1 + f(p_3))],$$
$$R_{22}[f](p_1,p_2,p_3,p_4) = \lambda_2 K^{22}(p_1,p_2,p_3,p_4)\delta(p_1 + p_2 - p_3 - p_4)\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$
$$\times [(1 + f(p_1))(1 + f(p_2))f(p_3)f(p_4)$$
$$- f(p_1)f(p_2)(1 + f(p_3))(1 + f(p_4))].$$

Proof. By a view of (1.20), we have

$$\int_{\mathbb{R}^3} C_{12}[f](p_1)\varphi(p_1)dp_1 + \int_{\mathbb{R}^3} C_{22}[f](p_1)\varphi(p_1)dp_1 = I_1 + I_2,$$

where

$$I_1 := \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \Big(R_{12}[f](p_1,p_2,p_3) - R_{12}[f](p_2,p_1,p_3) - R_{12}[f](p_3,p_2,p_1)\Big)\varphi(p_1)\, dp_1 dp_2 dp_3,$$

$$I_2 := \iiiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} R_{22}[f](p_1,p_2,p_3,p_4)\varphi(p_1)\, dp_1 dp_2 dp_3 dp_4.$$

By switching the variables $p_1 \leftrightarrow p_2$, $p_1 \leftrightarrow p_3$ in the integrals of I_1 and $(p_1, p_2) \leftrightarrow (p_2, p_1)$, $(p_1, p_2) \leftrightarrow (p_3, p_4)$ in the integrals of I_2, respectively, as in [3,29,67], the lemma follows at once. $\quad\square$

As a consequence, we obtain the following two corollaries.

Corollary 2.1 *(Conservation of momentum and energy). Smooth solutions $f(t,p)$ of* (1.20) *satisfy*

$$\int_{\mathbb{R}^3} f(t,p)p\,dp = \int_{\mathbb{R}^3} f_0(p)p\,dp \tag{2.8}$$

$$\int_{\mathbb{R}^3} f(t,p)\mathcal{E}(p)\,dp = \int_{\mathbb{R}^3} f_0(p)\mathcal{E}(p)\,dp \tag{2.9}$$

for all $t \geq 0$.

Proof. This follows from Lemma 2.1 by taking $\varphi(p) = p$ or $\mathcal{E}(p)$. □

Corollary 2.2 *(H-Theorem). Smooth solutions $f(t,p)$ of* (1.20) *satisfy*

$$\frac{d}{dt}\int_{\mathbb{R}^3} \left[f(t,p)\log f(t,p) - (1 + f(t,p))\log(1 + f(t,p)) \right] dp \leq 0$$

A radial symmetric equilibrium of the equation has the following form

$$f_\infty(p) = \frac{1}{e^{c\mathcal{E}(p)} - 1} \tag{2.10}$$

where c is some positive constant.

Proof. Observe that

$$\partial_t \int_{\mathbb{R}^3} \left[f(t,p)\log f(t,p) - (1 + f(t,p))\log(1 + f(t,p)) \right] dp$$

$$= \int_{\mathbb{R}^3} \partial_t f(t,p)\log\left(\frac{f(t,p)}{f(t,p) + 1} \right) dp,$$

and

$$\int_{\mathbb{R}^3} Q[f](t,p)\varphi(t,p)\,dp$$

$$= \lambda_1 n_c \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1,p_2,p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$$

$$\times (1 + f(t,p_1))(1 + f(t,p_2))(1 + f(t,p_3))\left(\frac{f(t,p_2)}{f(t,p_2) + 1}\frac{f(t,p_3)}{f(t,p_3) + 1} - \frac{f(t,p_1)}{f(t,p_1) + 1} \right)$$

$$\times \, [\varphi(p_1) - \varphi(p_2) - \varphi(p_3)] dp_1 dp_2 dp_3$$

$$+ \frac{\lambda_2}{2} \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{22}(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$

$$\times \, (1 + f(t, p_1))(1 + f(t, p_2))(1 + f(t, p_3))(1 + f(t, p_4)) \times$$

$$\times \left(\frac{f(t, p_3)}{f(t, p_3) + 1} \frac{f(t, p_4)}{f(t, p_4) + 1} - \frac{f(t, p_1)}{f(t, p_1) + 1} \frac{f(t, p_2)}{f(t, p_2) + 1} \right)$$

$$\times \, [\varphi(p_1) + \varphi(p_2) - \varphi(p_3) - \varphi(p_4)] dp_1 dp_2 dp_3 dp_4.$$

Notice that

$$(\alpha - \beta) \log \left(\frac{\alpha}{\beta} \right) \geq 0.$$

In the above inequality, the equality holds if and only if $\alpha = \beta$. Now suppose that $f_\infty(p)$ is a radial symmetric equilibrium. By Lemma 2.1 with $\varphi(p) = \log \left(\frac{f_\infty(p)}{f_\infty(p) + 1} \right)$, we obtain

$$\int\limits_{\mathbb{R}^3} Q[f_\infty](p) \varphi(p) dp \leq 0.$$

This yields the inequalities in the H-theorem:

$$\frac{f_\infty(p_2)}{f_\infty(p_2) + 1} \frac{f_\infty(p_3)}{f_\infty(p_3) + 1} - \frac{f_\infty(p_1)}{f_\infty(p_1) + 1} = 0,$$

$$\frac{f_\infty(p_2')}{f_\infty(p_2') + 1} \frac{f_\infty(p_1')}{f_\infty(p_1') + 1} - \frac{f_\infty(p_4')}{f_\infty(p_4') + 1} \frac{f_\infty(p_3')}{f_\infty(p_3') + 1} = 0.$$

Setting $h(p) = \log \left(\frac{f_\infty(p)}{f_\infty(p) + 1} \right)$, with the notice that h is radial symmetric, we get the following set of equations

$$h(p_2) + h(p_3) = h(p_1), \tag{2.11}$$

and

$$h(p_3') + h(p_4') = h(p_2') + h(p_1'). \tag{2.12}$$

Let us consider (2.11). In particular, by the conservation law

$$p_1 = p_2 + p_3,$$

the function $h(p)$ possesses the following property

$$h(p_2 + p_3) = h(p_2) + h(p_3),$$

for all $(p_2, p_3) \in \mathbb{R}^6$ satisfying

$$\mathcal{E}(p_2 + p_3) = \mathcal{E}(p_2) + \mathcal{E}(p_3).$$

As a consequence, since h is radial symmetric,

$$h \circ \mathcal{E}^{-1}(\alpha + \beta) = h \circ \mathcal{E}^{-1}(\alpha) + h \circ \mathcal{E}^{-1}(\beta),$$

where $p_2 = \mathcal{E}^{-1}(\alpha)$ and $p_3 = \mathcal{E}^{-1}(\beta)$. Notice that α, β can take arbitrary values in $\mathbb{R}_+$, which implies $h \circ \mathcal{E}^{-1}(\alpha) = -c\alpha$ for some positive constant c and for all $\alpha \geq 0$. Hence $h(p) = -c\mathcal{E}(p)$, for all $p \in \mathbb{R}^3$. Identity (2.10) is proved. $\quad\square$

2.1.2. Resonance manifolds/energy surfaces

We establish the following estimates on the energy surface integrals on S_p^1 and S_p^2 following the strategy proposed in [67].

Lemma 2.2. *Let S_p^0 be defined as in (2.3). The following estimate holds*

$$\int\limits_{S_p^0} \frac{K^{12}(p, w, p - w)|w|^{k_1}|p - w|^{k_2}}{|\nabla H_0^p(w)|} d\sigma(w) \geq c_1 |p|^{k_1 + k_2 + 1} \ \min\{1, |p|\}^{k_1 + k_2 + 7}, \quad (2.13)$$

where k_1, k_2 is are non-negative constants.

Moreover, for any function $F(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ which is radial and positive

$$F(u) = F(|u|),$$

we have

$$\int\limits_{S_p^0} \frac{F(|w|)}{|\nabla H_0^p(|w|)|} d\sigma(w) \leq c_2 \int\limits_0^{|p|} |u| F(|u|) \, d|u|, \quad (2.14)$$

for some positive constant c_2 independent of p.

Proof. By definition S_p^0 is the surface containing all w satisfying

$$\mathcal{E}(p - w) + \mathcal{E}(w) = \mathcal{E}(p).$$

For $w = 0$ and p, the above identity is automatically satisfied, hence $\{0, p\} \subset S_p^0$. If we consider $\mathcal{E}(\varrho)$ as a function of $|\varrho|$: $\mathcal{E}(\varrho) = \mathcal{E}(|\varrho|)$, then

$$\mathcal{E}'(|\varrho|) = \frac{\kappa_1 + 2\kappa_2 |\varrho|^2}{\sqrt{\kappa_1 + \kappa_2 |\varrho|^2}} > 0,$$

which means that $\mathcal{E}(|\varrho|)$ is strictly increasing. Since for all $w \in S_p^0 \backslash \{0, p\}$, $\mathcal{E}(|p - w|) < \mathcal{E}(|p|)$ and $\mathcal{E}(|w|) < \mathcal{E}(|p|)$, by the monotonicity of $\mathcal{E}(|\varrho|)$, we have $|w| < |p|$ and $|p - w| < |p|$, for all $w \in S_p \backslash \{0, p\}$. As a consequence, the energy surface S_p^0 is a subset of $\overline{B(0, |p|)} \cap \overline{B(p, |p|)}$. Now, recall

$$H_0^p(w) := \mathcal{E}(p - w) + \mathcal{E}(w) - \mathcal{E}(p).$$

The directional derivative of H_0^p in the direction of w can be computed as

$$\nabla_w H_0^p = \frac{w - p}{|p - w|} \mathcal{E}'(|p - w|) + \frac{w}{|w|} \mathcal{E}'(|w|). \tag{2.15}$$

For w of the form $w = \gamma p + q e_0, \gamma, q \in \mathbb{R}_+, e_0 \cdot p = 0$, the derivative of H with respect to q is

$$\partial_q H_0^p = \partial_q w \cdot \nabla_w H_0^p = e_0 \cdot \nabla_w H = q |e_0|^2 \left[\frac{\mathcal{E}'(p - w)}{|p - w|} + \frac{\mathcal{E}'(w)}{|w|} \right] > 0, \tag{2.16}$$

which means that $H(w)$ is strictly increasing with respect to q.

For $q = 0$ and $\gamma \in (0, 1)$, we will show that

$$H_0^p(w) = H_0^p(\gamma p) < 0. \tag{2.17}$$

Let us start by the following true fact

$$\sqrt{(\kappa_1 + \kappa_2 \gamma^2 |p|^2)(\kappa_1 + \kappa_2 (1 - \gamma)^2 |p|^2)} < \kappa_1 + \kappa_2 (\gamma^2 - \gamma + 2)|p|^2 \quad \text{for } p \neq 0.$$

Multiplying both sides of the above inequality with $2\gamma(1 - \gamma)|p|^2$ yields

$$2\sqrt{(\kappa_1 \gamma^2 + \kappa_2 \gamma^4 |p|^2)(\kappa_1 (1 - \gamma)^2 + \kappa_2 (1 - \gamma)^4 |p|^2)}|p|^2$$
$$< 2\kappa_1 \gamma (1 - \gamma)|p|^2 + 2\kappa_2 \gamma (1 - \gamma)(\gamma^2 - \gamma + 2)|p|^4.$$

Adding $\kappa_1 \gamma^2 |p|^2 + \kappa_2 \gamma^4 |p|^4 + \kappa_1 (1 - \gamma)^2 |p|^2 + \kappa_2 (1 - \gamma)^4 |p|^4$ to both sides of the above inequality, we obtain

$$\kappa_1 \gamma^2 |p|^2 + \kappa_2 \gamma^4 |p|^4 + \kappa_1 (1 - \gamma)^2 |p|^2 + \kappa_2 (1 - \gamma)^4 |p|^4$$
$$+ 2\sqrt{(\kappa_1 \gamma^2 + \kappa_2 \gamma^4 |p|^2)(\kappa_1 (1 - \gamma)^2 + \kappa_2 (1 - \gamma)^4 |p|^2)}|p|^2$$
$$< \kappa_1 |p|^2 + \kappa_2 |p|^4.$$

Rearranging the terms in the above inequality and taking the square root gives

$$\sqrt{\kappa_1 \gamma^2 |p|^2 + \kappa_2 \gamma^4 |p|^4} + \sqrt{\kappa_1 (1 - \gamma)^2 |p|^2 + \kappa_2 (1 - \gamma)^4 |p|^4} < \sqrt{\kappa_1 |p|^2 + \kappa_2 |p|^4},$$

and (2.17) is proved.

As a consequence, for a unit vector e_0 which is orthogonal to p, the surface S_p and the set $\mathcal{P}_\gamma = \{\gamma p + q e_0, q \in \mathbb{R}_+\}$ intersect at only one point, for each $\gamma \in (0,1)$. Define the intersection by $W_\gamma = \gamma p + q_\gamma e_0$. Since

$$\mathcal{E}(p - W_\gamma) + \mathcal{E}(W_\gamma) = \mathcal{E}(p),$$

then $\mathcal{E}(W_\gamma) < \mathcal{E}(p)$; there holds

$$|W_\gamma| = \sqrt{\gamma^2 |p|^2 + |q_\gamma|^2} < |p|, \quad |W_\gamma - p| = \sqrt{(1-\gamma)^2 |p|^2 + |q_{1-\gamma}|^2} < |p|$$

which implies

$$|q_\gamma| < |p|, \tag{2.18}$$

and

$$\gamma |p| < |W_\gamma| < |p|, \quad (1-\gamma)|p| < |p - W_\gamma| < |p|. \tag{2.19}$$

Taking the derivative with respect to γ of the identity

$$H_0^p(W_\gamma) = 0$$

yields:

$$
\begin{aligned}
0 = \partial_\gamma W_\gamma \cdot \nabla_w H_0^p &= \partial_\gamma W_\gamma \cdot \left(\frac{W_\gamma - p}{|p - W_\gamma|} \mathcal{E}'(|p - W_\gamma|) + \frac{W_\gamma}{|W_\gamma|} \mathcal{E}'(|W_\gamma|) \right) \\
&= \frac{1}{2} \partial_\gamma |W_\gamma|^2 \left[\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right] - |p|^2 \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} \\
&= \frac{1}{2} \partial_\gamma |q_\gamma|^2 \left[\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right] + \gamma |p|^2 \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} - (1-\gamma)|p|^2 \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|}
\end{aligned}
\tag{2.20}
$$

where the identities $\partial_\gamma W_\gamma = p$, $|W_\gamma|^2 = \gamma^2 |p|^2 + |q_\gamma|^2$ have been used.

With the notice that $\mathcal{E}'(|W_\gamma|) > 0$, the above identity yields

$$\frac{1}{2} \partial_\gamma |q_\gamma|^2 \le (1-\gamma)|p|^2 \tag{2.21}$$

for all p and all $\gamma \in (0,1)$.

We now provide an estimate on q_γ. In order to do this, let us consider two cases $|p| \ge 1$ and $|p| < 1$.

- Case 1: $|p| \ge 1$. Observe that at $\gamma = \frac{1}{2}$, due to the symmetry of the geometry

$$|W_{1/2}| = |W_{1/2} - p|,$$

which implies

$$2\mathcal{E}(W_{1/2}) = \mathcal{E}(p).$$

Noting that $|W_{1/2}|^2 = \frac{1}{4}|p|^2 + |q_{1/2}|^2$, yields

$$4\left[\kappa_1\left(\frac{1}{4}|p|^2 + |q_{1/2}|^2\right) + \kappa_2\left(\frac{1}{4}|p|^2 + |q_{1/2}|^2\right)^2\right] = \kappa_1|p|^2 + \kappa_2|p|^4,$$

then

$$\kappa_2\left(\frac{1}{4}|p|^2 + |q_{1/2}|^2\right)^2 + \kappa_1|q_{1/2}|^2 = \frac{\kappa_2}{4}|p|^4,$$

which implies

$$c_0|p|^2 = c_0|p|^2 \min\left\{1, |p|^2\right\} \leq |q_{1/2}|^2 \leq C_0|p|^2 \min\left\{1, |p|^2\right\} = C_0|p|^2 \qquad (2.22)$$

for some constants c_0, C_0, independent of $|p|$.
Combining (2.21), (2.22) and the fact that

$$|q_\gamma|^2 = |q_{1/2}|^2 - \int\limits_\gamma^{\frac{1}{2}} \partial_{\gamma'}|q_{\gamma'}|^2 \, d\gamma'$$

yields

$$|q_\gamma|^2 \geq c_0|p|^2 - 2\left|\gamma - \frac{1}{2}\right| |p|^2 \geq \frac{1}{2}c_0|p|^2 \qquad (2.23)$$

for all γ satisfying $\left|\gamma - \frac{1}{2}\right| \leq \frac{c_0}{4}$.

• Case 2: $|p|$ is small. Recall that

$$\left(\mathcal{E}(w) + \mathcal{E}(p-w)\right)^2 - \mathcal{E}(p)^2$$
$$= \kappa_1(|p-w|^2 + |w|^2 - |p|^2) + \kappa_2(|p-w|^4 + |w|^4 - |p|^4) + 2\mathcal{E}(w)\mathcal{E}(p-w)$$
$$= 2\kappa_1 w \cdot (w-p) + 2\kappa_2 w \cdot (w-p)\left(|w|^2 + |w-p|^2 + |p|^2\right)$$
$$- 2\kappa_2|w|^2|p-w|^2 + 2\mathcal{E}(w)\mathcal{E}(p-w), \qquad (2.24)$$

which leads to

$$-w \cdot (w-p)\left(\kappa_1 + \kappa_2|w|^2 + \kappa_2|w-p|^2 + \kappa_2|p|^2\right) = \mathcal{E}(w)\mathcal{E}(p-w) - \kappa_2|w|^2|p-w|^2 \quad (2.25)$$

for all $w \in S_p$, in which the right hand side can be computed explicitly as

$$\mathcal{E}(w)\mathcal{E}(p-w) - \kappa_2 |w|^2 |p-w|^2$$
$$= |w||p-w|\sqrt{(\kappa_1 + \kappa_2|w|^2)(\kappa_1 + \kappa_2|w-p|^2)} - \kappa_2|w|^2|p-w|^2$$
$$= |w||p-w|\frac{\kappa_1\left(\kappa_1 + \kappa_2|w|^2 + \kappa_2|w-p|^2\right)}{\sqrt{(\kappa_1 + \kappa_2|w|^2)(\kappa_1 + \kappa_2|w-p|^2)} + \kappa_2|w||p-w|}.$$

We will develop an asymptotic expansion of the above expression in term of $|p|$. In order to do this, we observe that

$$\sqrt{\left(1 + \frac{\kappa_2}{\kappa_1}|w|^2\right)\left(1 + \frac{\kappa_2}{\kappa_1}|w-p|^2\right)} = 1 + \frac{\kappa_2}{2\kappa_1}(|w|^2 + |w-p|^2) + \mathcal{O}(|p|^4),$$

which leads to

$$\mathcal{E}(w)\mathcal{E}(p-w) - \kappa_2|w|^2|p-w|^2$$
$$= |w||p-w|\left(\kappa_1 + \kappa_2|w|^2 + \kappa_2|w-p|^2\right)$$
$$\times \left(1 - \frac{1}{2}\frac{\kappa_2}{\kappa_1}(|w|^2 + |w-p|^2) - \frac{\kappa_2}{\kappa_1}|w||w-p| + \mathcal{O}(|p|^4)\right)$$
$$= |w||p-w|\left(\kappa_1 + \frac{1}{2}\kappa_2|w|^2 + \frac{1}{2}\kappa_2|w-p|^2 - \kappa_2|w||w-p| + \mathcal{O}(|p|^4)\right) \quad (2.26)$$
$$= |w||p-w|\left(\kappa_1 + \kappa_2|w|^2 + \kappa_2|w-p|^2 + \kappa_2|p|^2\right) -$$
$$- \frac{\kappa_2}{2}|w||w-p|\left(|w|^2 + |w-p|^2 + 2|w||w-p| + 2|p|^2\right)\left(1 + \mathcal{O}(|p|^2)\right).$$

Define ρ_γ be the angle between W_γ and $W_\gamma - p$, then $W_\gamma \cdot (W_\gamma - p) = |W_\gamma||W_\gamma - p|\cos\rho_\gamma$, which, together with (2.25)–(2.26), leads to

$$1 + \cos\rho_\gamma = \frac{\kappa_2}{2}\frac{\left(|W_\gamma|^2 + |W_\gamma - p|^2 + 2|W_\gamma||W_\gamma - p| + 2|p|^2\right)\left(1 + \mathcal{O}(|p|^2)\right)}{\kappa_1 + \kappa_2|W_\gamma|^2 + \kappa_2|W_\gamma - p|^2 + \kappa_2|p|^2}$$
$$= \mathcal{O}(|p|^2).$$

Hence $\sin\rho_\gamma = \mathcal{O}(|p|)$. The area of the parallelogram formed by W_γ and $W_\gamma - p$ can be computed as

$$2|p||q_\gamma| = |W_\gamma \times (W_\gamma - p)| = |W_\gamma||W_\gamma - p|\sin\rho_\gamma,$$

which, together with (2.19), implies that there exist universal constants c_2, c_3 satisfying

$$c_3\gamma(1-\gamma)|p|^2 \le |q_\gamma| \le c_2|p|^2 \quad (2.27)$$

for all $\gamma \in (0,1)$.

The two inequalities (2.23) and (2.27) are the two estimates we need to obtain (2.13). To continue, we parametrize the surface S_p^0 as follows: We choose $p^\perp$ to be a vector in $\mathcal{P}_0 = \{p \cdot q = 0\}$ and e_θ to be the unit vector in $\mathcal{P}_0$ so that the angle between $p^\perp$ and e_θ is θ. The surface S_p can be represented as

$$S_p^0 = \Big\{ W(\gamma, \theta) = \gamma p + |q_\gamma| e_\theta \; : \; \theta \in [0, 2\pi], \; \gamma \in [0, 1] \Big\}.$$

Notice that the vector $\partial_\theta e_\theta$ is orthogonal to both vectors p and e_θ, the surface area can be computed as

$$
\begin{aligned}
d\sigma(w) &= |\partial_\gamma W_\gamma \times \partial_\theta W_\gamma| d\gamma d\theta = \Big| (p + \partial_\gamma |q_\gamma| e_\theta) \times |q_\gamma| \partial_\theta e_\theta \Big| d\gamma d\theta \\
&= \Big| (|q_\gamma| p + \frac{1}{2} \partial_\gamma |q_\gamma|^2 e_\theta) \times \partial_\theta e_\theta \Big| d\gamma d\theta \\
&= \sqrt{|p|^2 |q_\gamma|^2 + \frac{1}{4} |\partial_\gamma (|q_\gamma|^2)|^2} d\gamma d\theta.
\end{aligned}
\tag{2.28}
$$

It is straightforward from the identity (2.20) that

$$
\partial_\gamma |q_\gamma|^2 = 2|p|^2 \frac{\gamma \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} + (\gamma - 1) \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|}}{\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|}}.
\tag{2.29}
$$

Now, let us compute $|\nabla H_0^p|$ under the new parametrization

$$
\begin{aligned}
|\nabla H_0^p|^2 &= |p|^2 \left[\gamma \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} + (\gamma - 1) \frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} \right]^2 \\
&\quad + |q_\gamma|^2 \left[\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right]^2,
\end{aligned}
$$

which, in companion with (2.29), implies

$$
\begin{aligned}
|\nabla H_0^p|^2 &= \frac{\left| \partial_\gamma |q_\gamma|^2 \right|^2}{4|p|^2} \left[\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right]^2 \\
&\quad + |q_\gamma|^2 \left[\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right]^2.
\end{aligned}
\tag{2.30}
$$

Using the fact that $\mathcal{E}'(x) \geq cx$ for all $x \in \mathbb{R}_+$, we get the following lower bound on $|\nabla H_0^p|$

$$
|\nabla H_0^p| = \frac{\sqrt{\frac{|\partial_\gamma |q_\gamma|^2|^2}{4} + |q_\gamma|^2 |p|^2}}{|p|} \left[\frac{\mathcal{E}'(|p - W_\gamma|)}{|p - W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right].
\tag{2.31}
$$

With (2.23) and (2.27), we are now able to estimate the integral

$$Z := \int_{S_p^0} \bar{K}_0^{12}(p, w, p - w) |w|^{k_1} |p - w|^{k_2} d\sigma(w).$$

Notice that

$$K^{12}(p, w, p - w) \geq C(|p| \wedge p_0)(|p - w| \wedge p_0)(|w| \wedge p_0),$$

where C is some positive constant varying from line to line. As a result, Z can be bounded from below by CZ', where Z' is defined as

$$Z' := \int_{S_p^0} (|p| \wedge p_0)(|w| \wedge p_0)(|p - w| \wedge p_0) |w|^{k_1} |p - w|^{k_2} d\sigma(w).$$

By (2.28), Z' can be rewritten as

$$\int_0^{2\pi} \int_0^1 \frac{|p|(|p| \wedge p_0)(|w| \wedge p_0)(|p - w| \wedge p_0) |w|^{k_1} |p - w|^{k_2}}{\frac{\mathcal{E}'(|p-w|)}{|p-w|} + \frac{\mathcal{E}'(|w|)}{|w|}} d\gamma d\theta.$$

Due to (2.23), for $|p|$ large, and $\gamma \in \left[\frac{2-c_0}{4}, \frac{2+c_0}{4} \right]$,

$$|W_\gamma|^2 \geq |q_\gamma|^2 \geq \frac{1}{2} c_0 |p|^2$$

and

$$|p - W_\gamma|^2 \geq |q_\gamma|^2 \geq \frac{1}{2} c_0 |p|^2.$$

We therefore can bound

$$\frac{|p|}{\frac{\mathcal{E}'(|p-w|)}{|p-w|} + \frac{\mathcal{E}'(|w|)}{|w|}} \geq \frac{|p|}{2 \frac{\mathcal{E}'(c_0|p|)}{c_0|p|}} \geq \frac{c_0 |p|^2}{2 \frac{\kappa_1 + 2\kappa_2 |c_0 p|^2}{\sqrt{\kappa_1 + \kappa_2 |c_0 p|^2}}},$$

where we have use the fact that $\frac{\mathcal{E}'(|\varrho|)}{|\varrho|}$ is decreasing with respect to $|\varrho|$. Since $|p|$ is large,

$$\frac{|p|}{\frac{\mathcal{E}'(|p-w|)}{|p-w|} + \frac{\mathcal{E}'(|w|)}{|w|}} \geq C|p|,$$

for some positive constant $C > 0$.

Therefore, Z' can be estimated as follows

$$Z' \geq C \int\limits_{0}^{2\pi} \int\limits_{\frac{1-c_0}{2}}^{\frac{1+c_0}{2}} (|p| \wedge p_0) \left(\left| \sqrt{\frac{c_0}{2}} |p| \right| \wedge p_0 \right)^2 \left| \sqrt{\frac{c_0}{2}} |p| \right|^{k_1+k_2} |p| d\gamma d\theta$$

$$\geq C(|p| \wedge 1)^3 |p|^{k_1+k_2+1}$$

$$\geq C|p|^{k_1+k_2+1},$$

where C is some positive constant varying from line to line.

Thanks to (2.27), for p small, on the interval $\gamma \in \left[\frac{1}{3}, \frac{1}{2} \right]$,

$$|W_\gamma|^2 \geq |q_\gamma|^2 \geq c_1 |p|^4$$

and

$$|p - W_\gamma|^2 \geq |q_\gamma|^2 \geq c_1 |p|^4.$$

We therefore can bound

$$\frac{|p|}{\frac{\mathcal{E}'(|p-w|)}{|p-w|} + \frac{\mathcal{E}'(|w|)}{|w|}} \geq \frac{|p|}{2\frac{\mathcal{E}'(c_1|p|^2)}{c_1|p|^2}} \geq \frac{c_1|p|^3}{2\frac{\kappa_1 + 2\kappa_2 c_1^2 |p|^4}{\sqrt{\kappa_1 + \kappa_2 c_1^2 |p|^4}}},$$

where we have use the fact that $\frac{\mathcal{E}'(|\varrho|)}{|\varrho|}$ is decreasing with respect to $|\varrho|$. Since $|p|$ is small,

$$\frac{|p|}{\frac{\mathcal{E}'(|p-w|)}{|p-w|} + \frac{\mathcal{E}'(|w|)}{|w|}} \geq C|p|^3,$$

for some positive constant $C > 0$.

Therefore, Z' can be estimated as follows

$$Z' \geq \int\limits_{0}^{2\pi} \int\limits_{\frac{1}{3}}^{\frac{1}{2}} (|p| \wedge p_0) \left(\left| \sqrt{c_1} |p|^2 \right| \wedge p_0 \right)^2 \left| \sqrt{c_1} |p| \right|^{2k_1+2k_2} C|p|^3 d\gamma d\theta$$

$$\geq C(|p| \wedge 1)^5 |p|^{2k_1+2k_2+3}$$

$$\geq C|p|^{2k_1+2k_2+8}.$$

The above shows that (2.13) holds true.

As for the surface integral of a radial function $\mathcal{G}(|w|)$, we introduce the radial variable $u = |W_\alpha| = \sqrt{\alpha^2 |p|^2 + |q_\alpha|^2}$. We compute $2u du = \partial_\alpha |W_\alpha|^2 d\alpha$ and hence

$$\frac{1}{|\nabla H_0^p|} d\sigma(w) = \frac{|p|}{2 \left[\frac{\mathcal{E}'(|p-W_\gamma|)}{|p-W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right] \partial_\alpha |W_\alpha|^2} u du d\theta.$$

Using (2.20), we compute

$$\frac{|p|}{2 \left[\frac{\mathcal{E}'(|p-W_\gamma|)}{|p-W_\gamma|} + \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right] \partial_\alpha |W_\alpha|^2} = \frac{1}{4|p| \frac{\mathcal{E}'(|p-W_\gamma|)}{|p-W_\gamma|}} \leq \frac{1}{4\mathcal{E}'(|p|)},$$

where we have used the fact that $\frac{\mathcal{E}'(|p-W_\gamma|)}{|p-W_\gamma|} \geq \frac{\mathcal{E}'(|p|)}{|p|}$. This, together with the bound

$$\mathcal{E}'(|p|) \geq C,$$

for some positive constant C, proves that

$$d\sigma(w) \leq C u du d\theta, \tag{2.32}$$

for some positive constants C. This yields the upper bound on the surface integral. $\quad\square$

Lemma 2.3. *Let S_p^1 be defined as in (2.3) and F be an arbitrary function in $L_1^1(\mathbb{R}_+)$. There is a positive constant C_0 so that*

$$\int_{S_p^1} \frac{F(|w|)}{|\nabla H_1^{p_1}(|w|)|} d\sigma(w) \leq \frac{C_0}{|p|} \|uF(\cdot)\|_{L^1(\mathbb{R}_+)}$$

uniformly in $p \in \mathbb{R}^3$.

Proof. Let us recall that S_p^1 is the surface consisting of w satisfying

$$\mathcal{E}(p+w) = \mathcal{E}(w) + \mathcal{E}(p).$$

First, we compute

$$\mathcal{E}(p+w)^2 - \left(\mathcal{E}(p) + \mathcal{E}(w) \right)^2$$
$$= \kappa_1 |p+w|^2 + \kappa_2 |p+w|^4 - \kappa_1(|p|^2 + |w|^2) - \kappa_2(|p|^4 + |w|^4) - 2\mathcal{E}(p)\mathcal{E}(w) \tag{2.33}$$
$$= 2\kappa_1 w \cdot p + 2\kappa_2 w \cdot p(|p|^2 + |w|^2 + |p+w|^2) + 2\kappa_2 |p|^2 |w|^2 - 2\mathcal{E}(p)\mathcal{E}(w).$$

Now, since $\kappa_1 \neq 0$, it follows that $\kappa_2 |p|^2 |w|^2 < \mathcal{E}(p)\mathcal{E}(w)$. This, in combination with (2.33), proves that if $w \in S_p^1 \setminus \{0\}$, then $w \cdot p > 0$. Let us calculate the derivative of $H_1^{p_1}$

$$\nabla_w H_1^p = \frac{p+w}{|p+w|} \mathcal{E}'(|p+w|) - \frac{w}{|w|} \mathcal{E}'(|w|),$$

where

$$\mathcal{E}'(|\varrho|) = \frac{2\kappa_1 + 4\kappa_2|\varrho|^2}{\sqrt{\kappa_1 + \kappa_2|\varrho|^2}}.$$

The derivative at $w = \gamma p$ with $\gamma \in \mathbb{R}_+$ can be determined using the previous formulation

$$\partial_\gamma H_1^p = \partial_\gamma w \cdot \nabla_w H_1^p|_{w=\gamma p} = |p|\mathcal{E}'((1+\gamma)p) - |p|\mathcal{E}'(\gamma p).$$

By the monotonicity of $\mathcal{E}(p)$ with respect to the length $|p|$, it follows that $\mathcal{E}'((1+\gamma)p) > \mathcal{E}'(\gamma p)$, and hence $\partial_\gamma H_1^p > 0$ for all $\gamma > 0$. Since $H_1^p(0) = 0$, $H_1^p(\gamma p) > 0$ for all positive γ.

Now, let us consider all the points $W_\gamma = \gamma p + q$, with $q \cdot p = 0$, for each fixed $\gamma > 0$. The directional derivative of H_1^p at $W_\gamma = \gamma p + q$ in the direction of $q \neq 0$ satisfies

$$q \cdot \nabla_w H_1^p = |q|^2 \left[\frac{\mathcal{E}'(p + W_\gamma)}{|p + W_\gamma|} - \frac{\mathcal{E}'(W_\gamma)}{|W_\gamma|} \right] < 0$$

in which the fact that $\mathcal{E}'(p)/|p|$ is strictly decreasing in $|p|$ has been used. By a view of (2.33), the sign of $H_1^p(w)$, with $W_\gamma = \gamma p + q$, is the same with the quantity

$$\gamma |p|^2 \Big(\kappa_1 + \kappa_2(|p|^2 + |W_\gamma|^2 + |p + W_\gamma|^2) \Big) + \kappa_2|p|^2|W_\gamma|^2 - \mathcal{E}(p)\mathcal{E}(W_\gamma)$$

$$= \gamma |p|^2 \Big(\kappa_1 + 2\kappa_2(|p|^2 + \gamma|p|^2 + |W_\gamma|^2) \Big)$$

$$- \frac{(\kappa_1|p|^2 + \kappa_2|p|^4)(\kappa_1|W_\gamma|^2 + \kappa_2|W_\gamma|^4) - \kappa_2^2|p|^4|W_\gamma|^4}{\sqrt{\kappa_1|p|^2 + \kappa_2|p|^4}\sqrt{\kappa_1|W_\gamma|^2 + \kappa_2|W_\gamma|^4} + \kappa_2|p|^2|W_\gamma|^2}$$

$$= \gamma |p|^2 \Big(\kappa_1 + 2\kappa_2(|p|^2 + \gamma|p|^2 + |W_\gamma|^2) \Big)$$

$$- \frac{\kappa_1^2|W_\gamma|^2|p|^2 + \kappa_1\kappa_2|W_\gamma|^2|p|^4 + \kappa_1\kappa_2|p|^2|W_\gamma|^4}{\sqrt{\kappa_1|p|^2 + \kappa_2|p|^4}\sqrt{\kappa_1|W_\gamma|^2 + \kappa_2|W_\gamma|^4} + \kappa_2|p|^2|W_\gamma|^2}.$$

This yields that $H_1^p(\gamma p + q) < 0$ as long as

$$\gamma < \frac{\kappa_1(\kappa_1 + \kappa_2|p|^2) + \kappa_1\kappa_2|W_\gamma|^2}{\sqrt{\kappa_1|p|^2 + \kappa_2|p|^4}\sqrt{\frac{\kappa_1}{|W_\gamma|^2} + \kappa_2} + \kappa_2|p|^2} \frac{1}{\Big(\kappa_1 + 2\kappa_2(|p|^2 + \gamma|p|^2 + |W_\gamma|^2) \Big)}.$$

Since $H_1^p(\gamma p) > 0$ and $q \cdot \nabla_w H_1^p < 0$, for a given direction q, there exists q_γ, such that $q_\gamma \cdot q > 0$ and q_γ is parallel with q, if an only if

$$\lim_{q \to \infty} H_1^p(\gamma p + q) < 0 \tag{2.34}$$

Taking $q \to \infty$ (and so $|W_\gamma| \to \infty$), we obtain (2.34) if and only if

$$\gamma < \gamma_p := \frac{1}{2} \frac{\kappa_1}{\kappa_2|p|^2 + 2\sqrt{\kappa_2}\sqrt{\kappa_1|p|^2 + \kappa_2|p|^4}}. \tag{2.35}$$

In particular, we note that

$$\gamma_p |p|(1+|p|) \le C_0, \qquad \forall\, p \in \mathbb{R}^3 \tag{2.36}$$

for some positive constant C_0.

Hence, for positive values of γ satisfying (2.35), there is a unique $|q_v|$ so that $\overline{G}(\gamma p + q) = 0$, for all $|q| = |q_\gamma|$. Moreover, from the continuity of $H_1^p(W_\gamma)$, $|q_\gamma|$ is continuously differentiable with respect to γ. For $\gamma \ge \gamma_p$, $H_1^p(\gamma p + q) > 0$, for all q so that $q \cdot p = 0$.

Now, we can parametrize the surface S_p^1 as follows:

$$S_p^1 = \left\{ w(\gamma, \theta) = \gamma p + |q_\gamma| e_\theta \; : \; \gamma \in [0, \gamma_p), \; \theta \in [0, 2\pi] \right\}, \tag{2.37}$$

in which γ_p and $|q_\gamma|$ are defined as above and e_θ is the unit vector rotating around p and on the orthogonal plane to p. As in (2.28), we have

$$d\sigma(w) = \sqrt{|p|^2 |q_\gamma|^2 + \frac{1}{4}|\partial_\gamma(|q_\gamma|^2)|^2}\, d\gamma d\theta$$

and hence, the surface integral is estimated by

$$\int\limits_{S_p^1} \frac{F(|w|)}{|\nabla H_1^p(w)|}\, d\sigma(w) = \iint\limits_{[0,2\pi] \times [0,\gamma_p]} \frac{F(|\gamma p + q_\gamma|)}{|\nabla H_1^p(|\gamma p + q_\gamma|)|} \sqrt{|p|^2|q_\gamma|^2 + \frac{1}{4}|\partial_\gamma(|q_\gamma|^2)|^2}\, d\gamma d\theta.$$

Let us introduce the variable $u = |W_\gamma| = \sqrt{\gamma^2 |p|^2 + |q_\gamma|^2}$. We compute

$$2u du = \partial_\gamma |W_\gamma|^2 d\gamma$$

and hence

$$\int\limits_{S_p^1} \frac{F(|w|)}{|\nabla H_1^p(w)|}\, d\sigma(w) \le 2\pi \int\limits_0^\infty F(u) \frac{\sqrt{|p|^2|q_\gamma|^2 + \frac{1}{4}|\partial_\gamma(|q_\gamma|^2)|^2}}{2\partial_\gamma |W_\gamma|^2 |\nabla H_1^p(|\gamma p + q_\gamma|)|} u\, du. \tag{2.38}$$

We recall that $H_1^p(W_\gamma) = 0$ and hence

$$0 = \partial_\gamma W_\gamma \cdot \nabla_w H_1^p = \frac{1}{2}\partial_\gamma |W_\gamma|^2 \left[\frac{\mathcal{E}'(p+w_\gamma)}{|p+W_\gamma|} - \frac{\mathcal{E}'(W_\gamma)}{|w_\gamma|} \right] + |p|^2 \frac{\mathcal{E}'(p+w_\gamma)}{|p+W_\gamma|}$$

which leads to

$$|p|^2 \frac{\mathcal{E}'(p+W_\gamma)}{|p+W_\gamma|} = \frac{1}{2}\partial_\gamma |W_\gamma|^2 \left[\frac{\mathcal{E}'(W_\gamma)}{|W_\gamma|} - \frac{\mathcal{E}'(p+W_\gamma)}{|p+W_\gamma|} \right], \tag{2.39}$$

and

$$\partial_\gamma |q_\gamma|^2 = 2 \frac{-\gamma |p|^2 \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} + (1+\gamma)|p|^2 \frac{\mathcal{E}'(|p+W_\gamma|)}{|p+W_\gamma|}}{\left[\frac{\mathcal{E}'(W_\gamma)}{|W_\gamma|} - \frac{\mathcal{E}'(p+W_\gamma)}{|p+W_\gamma|} \right]}. \tag{2.40}$$

We deduce from (2.40) that

$$
\begin{aligned}
|\nabla H_1^p|^2 &= |p|^2 \left[-\gamma \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} + (\gamma+1)\frac{\mathcal{E}'(|p+W_\gamma|)}{|p+W_\gamma|} \right]^2 \\
&\quad + |q_\gamma|^2 \left[\frac{\mathcal{E}'(|p+W_\gamma|)}{|p+W_\gamma|} - \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right]^2 \\
&= \frac{\left| \partial_\gamma |q_\gamma|^2 \right|^2}{4|p|^2} \left[\frac{\mathcal{E}'(|p+W_\gamma|)}{|p+W_\gamma|} - \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right]^2 \\
&\quad + |q_\gamma|^2 \left[\frac{\mathcal{E}'(|p+W_\gamma|)}{|p+W_\gamma|} - \frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} \right]^2,
\end{aligned}
$$

which implies

$$\int\limits_{S_p^1} \frac{F(|w|)}{|\nabla H_1^p(w)|} \, d\sigma(w) \leq \pi \int\limits_0^\infty \frac{F(u)|p|}{\partial_\gamma |W_\gamma|^2 \left[\frac{\mathcal{E}'(|W_\gamma|)}{|W_\gamma|} - \frac{\mathcal{E}'(|p+W_\gamma|)}{|p+W_\gamma|} \right]} u \, du.$$

The above and (2.39) yield

$$\int\limits_{S_p^1} \frac{F(|w|)}{|\nabla H_1^p(w)|} \, d\sigma(w) \leq \pi \int\limits_0^\infty \frac{F(u)}{2|p| \frac{\mathcal{E}'(|p+w|)}{|p+w|}} u \, du.$$

Using the fact that

$$\frac{\mathcal{E}'(|p+w|)}{|p+w|} \geq C$$

for some positive constant C, we obtain

$$\int\limits_{S_p^1} \frac{F(|w|)}{|\nabla H_1^p(w)|} \, d\sigma(w) \leq \frac{C}{|p|} \int\limits_0^\infty F(u) u \, du. \quad \square$$

Lemma 2.4. *Let S_p^2 be defined as in* (2.3) *and F be an arbitrary function in $L_1^1(\mathbb{R}_+)$. There are positive constants C_0 so that*

$$\int\limits_{S_p^2} \frac{F(|w|)}{|\nabla H_2^p(|w|)|}\, d\sigma(w) \le \frac{C_0}{|p|}\|uF(\cdot)\|_{L^1(\mathbb{R}_+)}$$

uniformly in $p \in \mathbb{R}^3$.

Proof. We observe that

$$
\begin{aligned}
S_p^2 &= \{p_* \ | \ \mathcal{E}(p_*) = \mathcal{E}(p) + \mathcal{E}(p_* - p)\} \\
&= \{p_* + p \ | \ \mathcal{E}(p_* + p) = \mathcal{E}(p) + \mathcal{E}(p_*)\} \\
&= p + S_p^1.
\end{aligned}
$$

The above identity means that the same argument of Lemma 2.3 could be applied and the conclusion of the lemma follows. □

2.1.3. Boundedness of the total mass for the kinetic equation

Proposition 2.1. *Suppose that the positive radial initial condition* $f_0(p) = f_0(|p|)$ *satisfies*

$$\int\limits_{\mathbb{R}^3} f_0(p_1)dp_1 < \infty, \int\limits_{\mathbb{R}^3} f_0(p_1)\mathcal{E}(p_1)dp_1 < \infty.$$

There exist universal positive constants $\mathcal{C}_1$, $\mathcal{C}_2$ *such that the mass of the positive radial solution* $f(t,p) = f(t,|p|)$ *of* (1.20) *could be bounded as*

$$\int\limits_{\mathbb{R}^3} f(t,p_1)dp_1 \le \mathcal{C}_1 e^{\mathcal{C}_2 t}.$$

Proof. First, observe that the constant function 1 can be used as the test function for (1.20), to get

$$\frac{d}{dt}\int\limits_{\mathbb{R}^3} f(p_1)dp_1 \;=\; \int\limits_{\mathbb{R}^3} C_{12}[f](p_1)dp_1 + \int\limits_{\mathbb{R}^3} C_{22}[f](p_1)dp_1, \tag{2.41}$$

with the notice that

$$\int\limits_{\mathbb{R}^3} C_{22}[f](p_1)dp_1 = 0,$$

and

$$\int\limits_{\mathbb{R}^3} C_{12}[f](p_1)dp_1 = \lambda_1 n_c \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$$

$$\times [f(p_1) + 2f(p_1)f(p_2) - f(p_2)f(p_3)]dp_1 dp_2 dp_3.$$

From the above computations, we can see that the control of the total mass really comes from estimating the collision operator C_{12}, since the integral of C_{22} is already 0. Set

$$J_1 = \lambda_1 n_c \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})f(p_1)dp_1 dp_2 dp_3$$

and

$$J_2 = 2\lambda_1 n_c \iiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})f(p_1)f(p_2)dp_1 dp_2 dp_3,$$

to get

$$\frac{d}{dt}\int_{\mathbb{R}^3} f(p_1)dp_1 = \int_{\mathbb{R}^3} Q[f](p_1)dp_1 \le J_1 + J_2, \tag{2.42}$$

note that in the above inequality, we have dropped the negative term containing $f(p_2)f(p_3)$.

Now, J_1 can be estimated the following way, by using the definition of the Dirac functions $\delta(p_1 - p_2 - p_3)$, $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$ and the boundedness of $K^{12}(p_1, p_2, p_1 - p_2)$

$$
\begin{aligned}
J_1 &= \lambda_1 n_c \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_1 - p_2)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1 - p_2})f(p_1)dp_1 dp_2 \\
&\le C\int_{\mathbb{R}^3} f(p_1)\left(\int_{S^0_{p_1}} \frac{1}{|\nabla H^{p_1}_0|}d\sigma(p_2)\right)dp_1,
\end{aligned}
$$

which, by Lemma 2.2, can be bounded as

$$J_1 \le C\int_{\mathbb{R}^3} f(p_1)|p_1|^2 dp_1.$$

Using the fact that $|p_1|^2$ is dominated by $\mathcal{E}_{p_1}$ up to a constant, yields

$$J_1 \le C\int_{\mathbb{R}^3} f(p_1)\mathcal{E}_{p_1} dp_1 \le C, \tag{2.43}$$

where C is a constant varying from line to line and the last inequality follows from the conservation of energy (2.9).

It remains to estimate J_2. By a straightforward use of the definition of the Dirac functions $\delta(p_1 - p_2 - p_3)$ and $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$

$$J_2 = 2\lambda_1 n_c \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} K^{12}(p_1, p_2, p_1 - p_2)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1 - p_2})f(p_1)f(p_2)dp_1 dp_2$$

$$= 2\lambda n_c \int\limits_{\mathbb{R}^3} f(p_2)\left(\int\limits_{S^2_{p_2}} K_2^{12}(p_1, p_2, p_1 - p_2)f(p_1)d\sigma(p_1)\right)dp_2,$$

which, by Lemma 2.4, can be bounded as

$$J_2 \le C \int\limits_{\mathbb{R}^3} f(p_2)\left(\int\limits_{S^2_{p_2}} K_2^{12}(p_1, p_2, p_1 - p_2)f(p_1)d\sigma(p_1)\right)dp_2$$

$$\le C \int\limits_{\mathbb{R}^3} f(p_2)\left(\int\limits_{\mathbb{R}^3} \frac{K^{12}(p_1, p_2, p_1 - p_2)}{|p_1||p_2|}f(p_1)dp_1\right)dp_2.$$

Since $\frac{K^{12}(p_1,p_2,p_1-p_2)}{|p_1||p_2|}$ is bounded by $|p_1 - p_2|$, up to a constant, J_2 is dominated by

$$J_2 \le C \int\limits_{\mathbb{R}^3} f(p_2)\left(\int\limits_{\mathbb{R}^3} f(p_1)|p_1 - p_2|dp_1\right)dp_2$$

$$\le C \int\limits_{\mathbb{R}^3} f(p_2)\left(\int\limits_{\mathbb{R}^3} f(p_1)(|p_1| + |p_2|)dp_1\right)dp_2$$

$$\le C \left(\int\limits_{\mathbb{R}^3} f(p_2)\mathcal{E}(p_2)dp_2\right)\left(\int\limits_{\mathbb{R}^3} f(p_1)dp_1\right),$$

notice that C is a positive constant varying from line to line and we have just used the fact that $|p|$ is bounded by $\mathcal{E}(p)$ up to a constant, which by the conservation of energy (2.9), implies

$$J_2 \le C \left(\int\limits_{\mathbb{R}^3} f(p_1)dp_1\right). \tag{2.44}$$

Combining (2.42), (2.43) and (2.44) leads to

$$\frac{d}{dt}\int\limits_{\mathbb{R}^3} f(p_1)dp_1 = \int\limits_{\mathbb{R}^3} Q[f](p_1)dp_1 \le C^* \left(1 + \int\limits_{\mathbb{R}^3} f(p_1)dp_1\right), \tag{2.45}$$

for some positive constant C^*, which implies the conclusion of the Proposition. $\quad\square$

2.2. Finite time moment estimates of the solution to the kinetic equation

2.2.1. Estimating C_{12}

Proposition 2.2. *For any positive, radial function $f(p) = f(|p|)$, for any $n \in \mathbb{N}$, there exists a universal positive constant $\mathcal{C}$ depending on n, such that the following bound on the collision operator C_{12} holds true*

$$
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1
$$
$$
\leq \sum_{k=1}^{n-1} \mathcal{C}(m_k[f] + m_{k-1}[f])(m_{n-k-1}[f] + m_{n-k}[f]) - \mathcal{C}m_{n+1}[f] + \mathcal{C}m_1[f].
$$
(2.46)

Proof. For the sake of simplicity, we denote $m_k[f]$ by m_k. By a view of Lemma 2.1,

$$
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 =
$$
$$
= n_c\lambda_1 \iiint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
$$
$$
\times [f(p_2)f(p_3) - f(p_1) - 2f(p_1)f(p_2)][\mathcal{E}_{p_1}^n - \mathcal{E}_{p_2}^n - \mathcal{E}_{p_3}^n]dp_1 dp_2 dp_3.
$$
(2.47)

By the definition of $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$, the term $\mathcal{E}_{p_1}^n - \mathcal{E}_{p_2}^n - \mathcal{E}_{p_3}^n$ could be rewritten as

$$
(\mathcal{E}_{p_2} + \mathcal{E}_{p_3})^n - \mathcal{E}_{p_2}^n - \mathcal{E}_{p_3}^n = \sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_3}^{n-k},
$$

which yields

$$
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 =
$$
$$
= n_c\lambda_1 \iiint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_3)\delta(p_1 - p_2 - p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
$$
$$
\times [f(p_2)f(p_3) - f(p_1) - 2f(p_1)f(p_2)] \left[\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_3}^{n-k}\right] dp_1 dp_2 dp_3.
$$

Dropping the term containing $-2f(p_1)f(p_2)$, the above quantity could be bounded as

$$
\int_{\mathbb{R}^3} C_{12}[f](p_1)\mathcal{E}^n(p_1)dp_1 \leq L_1 + L_2,
$$
(2.48)

where

$$
L_1 := \quad n_c \lambda_1 \iiint_{\mathbb{R}^{3\times3}} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
$$

$$
\times f(p_2) f(p_3) \left[\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_3}^{n-k} \right] dp_1 dp_2 dp_3
$$

$$
L_2 := \quad -n_c \lambda_1 \iiint_{\mathbb{R}^{3\times3}} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
$$

$$
\times f(p_1) \left[\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_3}^{n-k} \right] dp_1 dp_2 dp_3.
$$

Let us first look at L_1. By the definition of $\delta(p_1 - p_2 - p_3)$,

$$
L_1 = \quad n_c \lambda_1 \iint_{\mathbb{R}^{3\times2}} K^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
$$

$$
\times f(p_2) f(p_3) \left[\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_3}^{n-k} \right] dp_2 dp_3,
$$

which by the boundedness of K^{12}, could be bounded as

$$
L_1 \le C \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \times
$$

$$
\times f(p_2) f(p_3) \left[\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_3}^{n-k} \right] dp_2 dp_3
$$

$$
\le \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_2) \mathcal{E}_{p_2}^k \left[\int_{S_{p_2}^1} f(p_3) \frac{\mathcal{E}_{p_3}^{n-k}}{|\nabla H_2^{p_2}|} d\sigma(p_3) \right] dp_2.
$$

Applying Lemma 2.3 to the above inequality leads to

$$
L_1 \le \quad \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_2) \frac{\mathcal{E}_{p_2}^k}{|p_2|} \left[\int_{\mathbb{R}^3} f(p_3) \frac{\mathcal{E}_{p_3}^{n-k}}{|p_3|} dp_3 \right] dp_2,
$$

where C is some constant varying from line to line.

Observe that

$$
\frac{\mathcal{E}_{p_3}^{n-k}}{|p_3|} \le C \left(\mathcal{E}_{p_3}^{n-k-1} + \mathcal{E}_{p_3}^{n-k} \right), \frac{\mathcal{E}_{p_2}^k}{|p_2|} \le C \left(\mathcal{E}_{p_2}^{k-1} + \mathcal{E}_{p_2}^k \right)
$$

which implies

$$
\begin{aligned}
L_1 &\leq \sum_{k=1}^{n-1} C \left[\int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^k dp_1 \right] \left[\int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^{n-k-1} dp_1 + \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^{n-k} dp_1 \right] \\
&\leq \sum_{k=1}^{n-1} C[m_k + m_{k-1}][m_{n-k-1} + m_{n-k}].
\end{aligned}
\tag{2.49}
$$

Now, by the definition of $\delta(p_1 - p_2 - p_3)$ and $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1-p_2})$, the second term L_2 can be rewritten as

$$
\begin{aligned}
L_2 = \ & - n_c \lambda_1 \iint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_1 - p_2) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1-p_2}) \times \\
& \times f(p_1) \left[\sum_{k=1}^{n-1} \binom{n}{k} \mathcal{E}_{p_2}^k \mathcal{E}_{p_1-p_2}^{n-k} \right] dp_1 dp_2 \\
\leq \ & - \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_1) \left[\int_{S_{p_1}^0} K_0^{12}(p_1, p_2, p_1 - p_2) \mathcal{E}_{p_2}^k \mathcal{E}_{p_1-p_2}^{n-k} d\sigma(p_2) \right] dp_1.
\end{aligned}
$$

Since

$$
\mathcal{E}_{p_2}^k \mathcal{E}_{p_1-p_2}^{n-k} \geq C \left[|p_2|^k |p_1 - p_2|^{n-k} + |p_2|^{2k} |p_1 - p_2|^{2(n-k)} \right],
$$

where C is some positive constant varying from line to line, L_2 can be estimated as follows

$$
\begin{aligned}
L_2 \leq \ & \\
\leq \ & - \sum_{k=1}^{n-1} C \int_{\mathbb{R}^3} f(p_1) \left[\int_{S_{p_1}^0} K_0^{12}(p_1, p_2, p_1 - p_2) \right. \\
& \left. \times \left(|p_2|^k |p_1 - p_2|^{n-k} + |p_2|^{2k} |p_1 - p_2|^{2(n-k)} \right) d\sigma(p_2) \right] dp_1,
\end{aligned}
$$

which, due to Lemma 2.2, can be bounded by

$$
L_2 \ \leq \ - C \int_{\mathbb{R}^3} f(p_1) \left((|p_1| \wedge 1)^{n+7} |p_1|^{n+1} + (|p_1| \wedge 1)^{2n+7} |p_1|^{2n+1} \right) dp_1.
$$

Splitting the integral on $\mathbb{R}^3$ into two integrals on $|p_1| > 1$ and $|p_1| \leq 1$ yields

$$
\begin{aligned}
L_2 \quad \leq \quad & - C \int\limits_{|p_1|>1} f(p_1) \left(|p_1|^{n+2} + |p_1|^{2n+1} \right) dp_1 \\
& - C \int\limits_{|p_1|\leq 1} f(p_1) \left(|p_1|^{2n+7} + |p_1|^{4n+7} \right) dp_1 \\
\leq \quad & - C \int\limits_{|p_1|>1} f(p_1) \left(|p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1,
\end{aligned}
$$

where C is some positive constant varying from line to line and we have used the in-equality $-|p_1|^{n+1} > -|p_1|^{n+2}$ for $|p_1| > 1$. Adding and subtracting the right hand side of the above inequality with an integral on the domain $|p_1| \leq 1$, we obtain

$$
\begin{aligned}
L_2 \quad \leq \quad & - C \left[\int\limits_{\mathbb{R}^3} f(p_1) \left(|p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 - \int\limits_{|p_1|\leq 1} f(p_1) \left(|p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 \right] \\
\leq \quad & - C \left[\int\limits_{\mathbb{R}^3} f(p_1) \left(|p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 - \int\limits_{|p_1|\leq 1} |p_1| f(p_1) dp_1 \right],
\end{aligned}
$$

where the last inequality is due to the fact that we are integrating on $|p_1| \leq 1$. Bounding the integral on $|p_1| \leq 1$ by the integral on the full space $\mathbb{R}^3$, we get

$$
L_2 \quad \leq \quad - C \int\limits_{\mathbb{R}^3} f(p_1) \left(|p_1|^{n+1} + |p_1|^{2n+2} \right) dp_1 + C \int\limits_{\mathbb{R}^3} |p_1| f(p_1) dp_1.
$$

By the inequality

$$
|p_1|^{n+1} + |p_1|^{2n+2} \geq C \mathcal{E}_{p_1}^{n+1},
$$

we obtain the following estimate on L_2

$$
L_2 \leq \quad - C m_{n+1} + C m_1. \tag{2.50}
$$

Combining (2.48), (2.49) and (2.50), we get the conclusion of the Proposition. $\quad\square$

2.2.2. Estimating C_{22}

Proposition 2.3. *For any positive, radial function $f(p) = f(|p|)$, for any $n \in \mathbb{N}$, $n > 2$, n is odd, there exists a universal positive constant C depending on n, such that the following bound on the collision operator C_{22} holds true*

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}^n_{p_1} dp_1 \leq$$

$$\leq C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} m_{i+s} \left(m_{j+k-s} + m_{j+k-s+1/2}\right) + \qquad (2.51)$$

$$+ C \sum_{0 \leq i,j,k < n; \ i+j+k=n: \ j,k > 0} m_i \left(m_{j-1} + m_{j-1/2}\right) \left(m_{k-1} + m_{k-1/2}\right).$$

Proof. For the sake of simplicity, we denote $m_k[f]$ by m_k. We first observe that, by a spherical change of variables

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}^n_{p_1} dp_1 = C \int_{\mathbb{R}_+} C_{22}[f](p_1)|p_1|^2 \mathcal{E}^n_{p_1} d|p_1|,$$

where C is some universal constant varying from line to line, and

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}^n_{p_1} dp_1 =$$

$$= \kappa_3 \int_{\mathbb{R}^4_+} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$

$$\times [f(p_3)f(p_4)(1 + f(p_1) + f(p_2)) - f(p_1)f(p_2)(1 + f(p_3) + f(p_4))]$$

$$\times \mathcal{E}^n_{p_1} d|p_1| d|p_2| d|p_3| d|p_4|.$$

By the classical change of variables $(p_1, p_2) \leftrightarrow (p_2, p_1)$, $(p_1, p_2) \leftrightarrow (p_3, p_4)$ (cf. [84]), the above equation could be expressed in the following way

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}^n_{p_1} dp_1 =$$

$$= C \int_{\mathbb{R}^4_+} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$

$$\times f(p_1)f(p_2)(1 + f(p_3) + f(p_4))\left[\mathcal{E}^n_{p_4} + \mathcal{E}^n_{p_3} - \mathcal{E}^n_{p_2} - \mathcal{E}^n_{p_1}\right] d|p_1| d|p_2| d|p_3| d|p_4|,$$

where C is some universal constant varying from line to line.

Taking into account the fact that p_3 and p_4 are symmetric, and using the definition of the Dirac function to get $\mathcal{E}_{p_4} = \mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3}$, one obtains

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}_{p_1}^n dp_1 =$$

$$= C\int_{\mathbb{R}_+^4} K^{22}(p_1,p_2,p_3,p_4)\min\{|p_1|,|p_2|,|p_3|,|p_4|\}|p_1||p_2||p_3||p_4|\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})$$

$$\times f(p_1)f(p_2)(1+2f(p_3))\Big[(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3})^n+\mathcal{E}_{p_3}^n-\mathcal{E}_{p_2}^n-\mathcal{E}_{p_1}^n\Big]d|p_1|d|p_2|d|p_3|d|p_4|.$$

$$(2.52)$$

Notice that for

$$\mathcal{E}(|p|) = \sqrt{\kappa_1|p|^2+\kappa_2|p|^4},$$

its derivative is bounded from below as

$$\mathcal{E}'(|p|) = \frac{\kappa_1+2\kappa_2|p|^2}{\sqrt{\kappa_1+\kappa_2|p|^2}} \geq C|p|, \tag{2.53}$$

where C is some universal constant varying from line to line, which means $C|p_4|d|p_4|$ can be bounded by $d\mathcal{E}_{p_4}$. As a consequence, the following estimate on the right hand side of (2.52) follows

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}_{p_1}^n dp_1 =$$

$$\leq C\int_{\mathbb{R}_+^4} K^{22}(p_1,p_2,p_3,p_4)\min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3|\delta(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3}-\mathcal{E}_{p_4})$$

$$\times f(p_1)f(p_2)(1+2f(p_3))\Big[(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3})^n+\mathcal{E}_{p_3}^n-\mathcal{E}_{p_2}^n-\mathcal{E}_{p_1}^n\Big]d|p_1|d|p_2|d|p_3|d\mathcal{E}_{p_4},$$

$$(2.54)$$

where, we have used the fact that

$$\min\{|p_1|,|p_2|,|p_3|,|p_4|\} \leq \min\{|p_1|,|p_2|,|p_3|\}.$$

Since n is an odd number, applying Newton formula to the term $(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3})^n+\mathcal{E}_{p_3}^n-\mathcal{E}_{p_2}^n-\mathcal{E}_{p_1}^n$ yields

$$(\mathcal{E}_{p_1}+\mathcal{E}_{p_2}-\mathcal{E}_{p_3})^n+\mathcal{E}_{p_3}^n-\mathcal{E}_{p_2}^n-\mathcal{E}_{p_1}^n = \sum_{0\leq i,j,k<n;\ i+j+k=n} C_{i,j,k,n}\mathcal{E}_{p_1}^i\mathcal{E}_{p_2}^j\mathcal{E}_{p_3}^k. \tag{2.55}$$

Plugging (2.55) into (2.54), integrating with respect to $d\mathcal{E}_4$ and using the bound (1.18) leads to

$$\int\limits_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}_{p_1}^n dp_1 \le$$

$$\le C \int\limits_{\{\mathcal{E}_{p_1}+\mathcal{E}_{p_2}\ge\mathcal{E}_{p_3}\}} \min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3|f(p_1)f(p_2)(1+2f(p_3))$$

$$\times \left[\sum_{0\le i,j,k<n;\ i+j+k=n} |C_{i,j,k,n}|\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k\right] d|p_1|d|p_2|d|p_3| \tag{2.56}$$

$$\le C \sum_{0\le i,j,k<n;\ i+j+k=n} \int\limits_{\{\mathcal{E}_{p_1}+\mathcal{E}_{p_2}\ge\mathcal{E}_{p_3}\}} \min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3|$$

$$\times f(p_1)f(p_2)(1+2f(p_3))\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1|d|p_2|d|p_3|.$$

In order to estimate the right hand side of (2.56), we estimate each term containing $f(p_1)f(p_2)$ and $2f(p_1)f(p_2)f(p_3)$ separately.

Let us first look at the term containing $f(p_1)f(p_2)$

$$L_1 := C \sum_{0\le i,j,k<n;\ i+j+k=n} \int\limits_{\{\mathcal{E}_{p_1}+\mathcal{E}_{p_2}\ge\mathcal{E}_{p_3}\}} \min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3|$$

$$\times f(p_1)f(p_2)\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1|d|p_2|d|p_3|$$

$$\le C \sum_{0\le i,j,k<n;\ i+j+k=n} \int\limits_{\{\mathcal{E}_{p_1}+\mathcal{E}_{p_2}\ge\mathcal{E}_{p_3}\}} \min\{|p_1|,|p_2|\}|p_1||p_2| \tag{2.57}$$

$$\times f(p_1)f(p_2)\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1|d|p_2|d\mathcal{E}_{p_3},$$

where we have used (2.53) to get $|p_3|dp_3 \le Cd\mathcal{E}_{p_3}$ and the fact that

$$\min\{|p_1|,|p_2|,|p_3|\} \le \min\{|p_1|,|p_2|\}.$$

In (2.57), integrating with respect to $d\mathcal{E}_{p_3}$ leads to

$$L_1 \le C \sum_{0\le i,j,k<n;\ i+j+k=n} \int\limits_{\mathbb{R}_+^2} \min\{|p_1|,|p_2|\}|p_1||p_2|f(p_1)f(p_2)$$

$$\times \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \frac{(\mathcal{E}_{p_1}+\mathcal{E}_{p_2})^{k+1}}{k+1} d|p_1|d|p_2|, \tag{2.58}$$

where C is some universal constant varying from line to line.

Again, by Newton formula

$$(\mathcal{E}_{p_1}+\mathcal{E}_{p_2})^{k+1} = \sum_0^{k+1} \binom{k+1}{s} \mathcal{E}_{p_1}^s \mathcal{E}_{p_2}^{k+1-s}, \tag{2.59}$$

which, together with (2.57) leads to

$$L_1 \leq$$

$$\leq C \sum_{0 \leq i,j,k < n;\ i+j+k=n} \sum_{s=0}^{k+1} \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1| |p_2| f(p_1) f(p_2) \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{j+k+1-s} d|p_1| d|p_2|$$

$$\leq C \sum_{0 \leq i,j,k < n;\ i+j+k=n} \sum_{s=0}^{k+1} \int_{\mathbb{R}_+^2} |p_1|^2 |p_2| f(p_1) f(p_2) \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{j+k+1-s} d|p_1| d|p_2|. \qquad (2.60)$$

Note that integrals of $d|p_1|$ and $d|p_2|$ in (2.60) are separated and it is straightforward that the integral of $d|p_1|$ can be computed, by a spherical coordinate change of variables, as

$$\int_{\mathbb{R}_+} |p_1|^2 f(p_1) \mathcal{E}_{p_1}^{i+s} d|p_1| = \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^{i+s} dp_1 = m_{i+s}. \qquad (2.61)$$

Now, for the second integral concerning $d|p_2|$, by the inequality

$$\mathcal{E}_{p_2} \leq C(|p_2| + |p_2|^2),$$

for some positive constant C, one gets

$$\int_{\mathbb{R}_+} |p_2| f(p_2) \mathcal{E}_{p_2}^{j+k+1-s} d|p_2| \leq C \int_{\mathbb{R}_+} (|p_2|^2 + |p_2|^3) f(p_2) \mathcal{E}_{p_2}^{j+k-s} d|p_2|$$

$$\leq C \int_{\mathbb{R}^3} (1 + |p_2|) f(p_2) \mathcal{E}_{p_2}^{j+k-s} dp_2,$$

which, by the inequality

$$\mathcal{E}_{p_2}^{1/2} \geq C|p_2|,$$

implies that

$$\int_{\mathbb{R}_+} |p_2| f(p_2) \mathcal{E}_{p_2}^{j+k+1-s} d|p_2| \leq C \int_{\mathbb{R}^3} \left(1 + \mathcal{E}_{p_2}^{1/2}\right) f(p_2) \mathcal{E}_{p_2}^{j+k-s} dp_2$$

$$\leq C \left(m_{j+k-s} + m_{j+k-s+1/2}\right). \qquad (2.62)$$

Combining (2.60), (2.61) and (2.62) lead to

$$L_1 \leq C \sum_{0 \leq i,j,k < n;\ i+j+k=n} \sum_{s=0}^{k+1} m_{i+s} \left(m_{j+k-s} + m_{j+k-s+1/2}\right). \qquad (2.63)$$

Now, for the term containing $2f(p_1)f(p_2)f(p_3)$, by bounding the integral on $\{\mathcal{E}_{p_1} + \mathcal{E}_{p_2} \geq \mathcal{E}_{p_3}\}$ by the integral on $\mathbb{R}_+^3$, we get

$$
\begin{aligned}
L_2 := \ & C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \int_{\{\mathcal{E}_{p_1} + \mathcal{E}_{p_2} \geq \mathcal{E}_{p_3}\}} \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| \\
& \times 2f(p_1)f(p_2)f(p_3) \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1| d|p_2| d|p_3| \\
\leq \ & C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \int_{\mathbb{R}_+^3} \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| \\
& \times f(p_1)f(p_2)f(p_3) \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1| d|p_2| d|p_3|,
\end{aligned}
\tag{2.64}
$$

where C is some universal constant varying from line to line.

Notice that there are only two cases: $i, j, k > 0$ and one of i, j, k is 0. Indeed, due to the condition that $i + j + k = n$ and $0 \leq i, j, k < n$, the case where two of the index i, j, k are 0 will not happen. Therefore, we can suppose without loss of generality that $i \geq 0$ and $j, k > 0$.

The terms on the right hand side of (2.64) can be estimated as

$$
\begin{aligned}
& \int_{\mathbb{R}_+^3} \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| f(p_1)f(p_2)f(p_3) \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1| d|p_2| d|p_3| \\
& \leq \int_{\mathbb{R}_+} |p_1|^2 \mathcal{E}_{p_1}^i f(p_1) d|p_1| \int_{\mathbb{R}_+} |p_2| \mathcal{E}_{p_2}^j f(p_2) d|p_2| \int_{\mathbb{R}_+} |p_3| \mathcal{E}_{p_3}^k f(p_3) d|p_3| \ .
\end{aligned}
\tag{2.65}
$$

For each term on the right hand side of (2.65), one can write, by the spherical coordinate change of variables

$$
\int_{\mathbb{R}_+} |p_1|^2 \mathcal{E}_{p_1}^i f(p_1) d|p_1| = \int_{\mathbb{R}^3} \mathcal{E}_{p_1}^i f(p_1) dp_1 = m_i,
\tag{2.66}
$$

$$
\int_{\mathbb{R}_+} |p_2| \mathcal{E}_{p_2}^j f(p_2) d|p_2| \leq C \left(m_{j-1} + m_{j-1/2} \right),
\tag{2.67}
$$

$$
\int_{\mathbb{R}_+} |p_3| \mathcal{E}_{p_3}^k f(p_3) d|p_3| \leq C \left(m_{k-1} + m_{k-1/2} \right),
\tag{2.68}
$$

where (2.67) and (2.68) are obtained by exactly the same manner as (2.62).

Combining (2.65), (2.66), (2.67) and (2.68) yields

$$
\begin{aligned}
& \int_{\mathbb{R}_+^3} \min\{|p_1|, |p_2|, |p_3|\} |p_1||p_2||p_3| f(p_1)f(p_2)f(p_3) \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k d|p_1| d|p_2| d|p_3| \\
& \leq C m_i \left(m_{j-1} + m_{j-1/2} \right) \left(m_{k-1} + m_{k-1/2} \right) \ .
\end{aligned}
\tag{2.69}
$$

The two inequalities (2.64) and (2.69) yield

$$L_2 \leq C \sum_{\substack{0 \leq i,j,k < n; \ i+j+k=n: \ j,k>0}} m_i \left(m_{j-1} + m_{j-1/2} \right) \left(m_{k-1} + m_{k-1/2} \right), \qquad (2.70)$$

where C is some universal constant varying from line to line.

From (2.56), (2.63) and (2.70), we get

$$\int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}_{p_1}^n dp_1 \leq$$

$$\leq C \sum_{\substack{0 \leq i,j,k < n; \ i+j+k=n}} \sum_{s=0}^{k+1} m_{i+s} \left(m_{j+k-s} + m_{j+k-s+1/2} \right) +$$

$$+ C \sum_{\substack{0 \leq i,j,k < n; \ i+j+k=n; \ j,k>0}} m_i \left(m_{j-1} + m_{j-1/2} \right) \left(m_{k-1} + m_{k-1/2} \right). \quad \square$$

2.2.3. Finite time moment estimates

Proposition 2.4. *Suppose that $f_0(p) = f_0(|p|)$ is a positive radial initial condition and*

$$\int_{\mathbb{R}^3} f_0(p) \mathcal{E}_p dp < \infty, \quad \int_{\mathbb{R}^3} f_0(p) dp < \infty,$$

then for any finite time interval $[0, T]$, and for any $n \geq 1$, the positive radial solution $f(t,p) = f(t,|p|)$ of (1.20) satisfies

$$\sup_{t \in [\tau, T]} \int_{\mathbb{R}^3} f(t,p) \mathcal{E}_p^n dp < C_\tau, \quad \forall \, 0 < \tau \leq T,$$

where C_τ is a constant depending on τ.

If

$$\int_{\mathbb{R}^3} f_0(p) \mathcal{E}_p^n dp < \infty,$$

then

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3} f(t,p) \mathcal{E}_p^n dp < \infty.$$

In order to prove Proposition 2.4, we would need the following Holder inequality.

Lemma 2.5. *Let f be a function in $L^1(\mathbb{R}^3) \cap L_n^1(\mathbb{R}^3)$, then*

$$\|f\|_{L_k^1} \leq C \|f\|_{L_n^1}^{\frac{k}{n}},$$

where C is a constant depending on $\|f\|_{L^1}$, k and n.

Proof. By Holder inequality, we have

$$\int_{\mathbb{R}^3} |p|^k f(p) dp \leq \left(\int_{\mathbb{R}^3} |f(p)| dp \right)^{\frac{n-k}{n}} \left(\int_{\mathbb{R}^3} |p|^n |f(p)| dp \right)^{\frac{k}{n}}$$

$$\leq C\left(\|f\|_{L^1}, k, n\right) \left(\int_{\mathbb{R}^3} |p|^n f(p) dp \right)^{\frac{k}{n}}. \quad \square$$

Proof of Proposition 2.4. Fix a time interval $[0, T]$. It is sufficient to prove Proposition 2.4 for $n \in \mathbb{N}$, n odd. Using $\mathcal{E}_{p_1}^n$ as a test function in (1.20), as a view of Lemma 2.1, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(p_1) \mathcal{E}_{p_1}^n dp_1 = \int_{\mathbb{R}^3} C_{12}[f](p_1) \mathcal{E}_{p_1}^n dp_1 + \int_{\mathbb{R}^3} C_{22}[f](p_1) \mathcal{E}_{p_1}^n dp_1. \tag{2.71}$$

For the sake of simplicity, we denote $m_k[f(t)]$ as $m_k(t)$. First, let us consider the C_{12} collision operator. By Proposition 2.2

$$\int_{\mathbb{R}^3} C_{12}[f](p_1) \mathcal{E}^n(p_1) dp_1$$

$$\leq \sum_{k=1}^{n-1} C(m_k(t) + m_{k-1}(t))(m_{n-k-1}(t) + m_{n-k}(t)) - Cm_{n+1}(t) + Cm_1(t).$$

Since, according to Proposition 2.1, $m_0(t)$ is bounded by a constant C on $[0, T]$, we deduce from Lemma 2.5 that

$$m_k(t) \leq Cm_n(t)^{\frac{k}{n}}, \qquad m_{k-1}(t) \leq Cm_n(t)^{\frac{k-1}{n}}, \qquad m_{n-k-1}(t) \leq Cm_n(t)^{\frac{n-k-1}{n}},$$
$$m_{n-k}(t) \leq Cm_n(t)^{\frac{n-k}{n}}, \qquad Cm_{n+1}(t) \geq m_n(t)^{\frac{n+1}{n}}, \qquad Cm_1(t) \leq m_n(t)^{\frac{1}{n}},$$

where C depends on n, k, and the bound of the mass on $[0, T]$ in Proposition 2.1. As a consequence, we obtain the following estimate for C_{12}

$$\int_{\mathbb{R}^3} C_{12}[f](p_1) \mathcal{E}^n(p_1) dp_1$$

$$\leq Cm_n(t) + Cm_n(t)^{\frac{n-1}{n}} + Cm_n(t)^{\frac{n-2}{n}} + Cm_n(t)^{\frac{1}{n}} - Cm_n(t)^{\frac{n+1}{n}}. \tag{2.72}$$

Now, for the C_{22} collision operator, according to Proposition 2.3,

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}_{p_1}^n \, dp_1 \leq$$

$$\leq C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} \left(m_{i+s}(t) + m_{j+k-s}(t) + m_{j+k-s+1/2}(t) \right) +$$

$$+ C \sum_{0 \leq i,j,k < n; \ i+j+k=n: \ j,k>0} m_i(t) \left(m_{j-1}(t) + m_{j-1/2}(t) \right) \left(m_{k-1}(t) + m_{k-1/2}(t) \right).$$

Again, by Proposition 2.1, and Lemma 2.5

$$m_{i+s}(t) \leq C m_n(t)^{\frac{i+s}{n}}, \qquad m_{j+k-s}(t) \leq C m_n(t)^{\frac{j+k-s}{n}},$$

$$m_{j+k-s+1/2}(t) \leq C m_n(t)^{\frac{j+k-s+1/2}{n}}, \qquad m_i(t) \leq C m_n(t)^{\frac{i}{n}},$$

$$m_{j-1}(t) \leq C m_n(t)^{\frac{j-1}{n}}, \qquad m_{j-1/2}(t) \leq C m_n(t)^{\frac{j-1/2}{n}},$$

$$m_{k-1}(t) \leq C m_n(t)^{\frac{k-1}{n}}, \qquad m_{k-1/2}(t) \leq C m_n(t)^{\frac{k-1/2}{n}},$$

we obtain

$$\int_{\mathbb{R}^3} C_{22}[f](p_1)\mathcal{E}_{p_1}^n \, dp_1 \leq$$

$$\leq C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} m_n(t)^{\frac{i+s}{n}} \left(m_n(t)^{\frac{j+k-s}{n}} + m_n(t)^{\frac{j+k-s+1/2}{n}} \right) + \tag{2.73}$$

$$+ C \sum_{0 \leq i,j,k < n; \ i+j+k=n: \ j,k>0} m_n(t)^{\frac{i}{n}} \left(m_n(t)^{\frac{j-1}{n}} + m_n(t)^{\frac{j-1/2}{n}} \right) \times$$

$$\times \left(m_n(t)^{\frac{k-1}{n}} + m_n(t)^{\frac{k-1/2}{n}} \right).$$

Combining (2.71), (2.72) and (2.73) yields

$$\frac{d}{dt} m_n(t)$$

$$\leq C m_n(t) + C m_n(t)^{\frac{n-1}{n}} + C m_n(t)^{\frac{n-2}{n}} + C m_n(t)^{\frac{1}{n}} - C m_n^{\frac{n+1}{n}}$$

$$+ C \sum_{0 \leq i,j,k < n; \ i+j+k=n} \sum_{s=0}^{k+1} m_n(t)^{\frac{i+s}{n}} \left(m_n(t)^{\frac{j+k-s}{n}} + m_n(t)^{\frac{j+k-s+1/2}{n}} \right) + \tag{2.74}$$

$$+ C \sum_{0 \leq i,j,k < n; \ i+j+k=n; \ j,k>0} m_n(t)^{\frac{i}{n}} \left(m_n(t)^{\frac{j-1}{n}} + m_n(t)^{\frac{j-1/2}{n}} \right) \times$$

$$\times \left(m_n(t)^{\frac{k-1}{n}} + m_n(t)^{\frac{k-1/2}{n}} \right),$$

where C depends on n, k, and the bound of the mass on $[0, T]$ in Proposition 2.1. Notice that $-Cm_n(t)^{\frac{n+1}{n}}$ has the highest order on the right hand side of (2.74). By the same argument as in [85], the conclusion of the theorem then follows. □

2.3. Holder estimates for the collision operators

In this section, we will provide Holder estimates for the two collision operators C_{12} and C_{22}. For C_{22}, we split it into two operators

$$C_{22}^1[f](p_1) = \kappa_3 \iiint_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} K^{22}(p_1, p_2, p_3, p_4) \frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|}{|p_1|^2}$$
$$\times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})[f(p_3)f(p_4) - f(p_1)f(p_2)]d|p_2|d|p_3|d|p_4|, \tag{2.75}$$

and

$$C_{22}^2[f](p_2) = \kappa_3 \iiint_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+} K^{22}(p_1, p_2, p_3, p_4) \frac{\min\{|p_1|, |p_2|, |p_3|, |p_4|\}|p_1||p_2||p_3||p_4|}{|p_1|^2}$$
$$\times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})[f(p_3)f(p_4)(f(p_1) + f(p_2)) -$$
$$- f(p_1)f(p_2)(f(p_3) + f(p_4))]d|p_2|d|p_3|d|p_4|. \tag{2.76}$$

We will show in Proposition 2.2, Proposition 2.6 and Proposition 2.7 that C_{12}, C_{22}^1 and C_{22}^2 are Holder continuous.

2.3.1. Holder estimates for C_{12}

Proposition 2.5. *Let f and g be two functions in $L_{n+3}^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $n \in \mathbb{R}_+$, n can be 0; then there exists a constant $\mathcal{C}$ depending on $\|f\|_{L_{n+3}^1}, \|f\|_{L^1}, \|g\|_{L_{n+3}^1}, \|g\|_{L^1}$ such that*

$$\|C_{12}[f] - C_{12}[g]\|_{L_n^1} \leq \mathcal{C}\left(\|f - g\|_{L_{n+3}^1} + \|f - g\|_{L^1}\right). \tag{2.77}$$

If $\|f\|_{L_{n+4}^1}, \|g\|_{L_{n+4}^1} < \mathcal{C}_0$, then

$$\|C_{12}[f] - C_{12}[g]\|_{L_n^1} \leq \mathcal{C}_1\left(\|f - g\|_{L^1}^{\frac{1}{n+4}} + \|f - g\|_{L^1}\right), \tag{2.78}$$

where $\mathcal{C}_1$ is a constant depending on $\mathcal{C}_0, \mathcal{C}$.

Proof. First, let us consider the L_n^1 norm of the difference $C_{12}[f] - C_{12}[g]$. As a view of Lemma 2.1

$$\begin{aligned}
\|C_{12}[f] - C_{12}[g]\|_{L_n^1} &= \int_{\mathbb{R}^3} |p_1|^n |C_{12}[f] - C_{12}[g]| dp_1 \\
&\leq n_c \lambda_1 \iiint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times |f(p_2)f(p_3) - 2f(p_3)f(p_1) - f(p_1) - g(p_2)g(p_3) \\
&\quad + 2g(p_3)g(p_1) + g(p_1)| \left[|p_1|^n + |p_2|^n + |p_3|^n\right] dp_1 dp_2 dp_3.
\end{aligned}$$

(2.79)

The above identity implies that $\|C_{12}[f] - C_{12}[g]\|_{L_n^1}$ can be bounded by the sum of the following three terms

$$\begin{aligned}
N_1 &= n_c \lambda_1 \iiint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times |f(p_2)f(p_3) - g(p_2)g(p_3)| \left[|p_1|^n + |p_2|^n + |p_3|^n\right] dp_1 dp_2 dp_3, \\
N_2 &= 2n_c \lambda_1 \iiint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times |f(p_3)f(p_1) - g(p_3)g(p_1)| \left[|p_1|^n + |p_2|^n + |p_3|^n\right] dp_1 dp_2 dp_3,
\end{aligned}$$

and

$$\begin{aligned}
N_3 &= n_c \lambda_1 \iiint_{\mathbb{R}^{3\times 3}} K^{12}(p_1, p_2, p_3) \delta(p_1 - p_2 - p_3) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times |f(p_1) - g(p_1)| \left[|p_1|^n + |p_2|^n + |p_3|^n\right] dp_1 dp_2 dp_3.
\end{aligned}$$

In the sequel, we will estimate N_1, N_2, N_3 in three steps.

Step 1: Estimating N_1.

By the definition of $\delta(p_1 - p_2 - p_3)$, N_1 can be rewritten as:

$$\begin{aligned}
N_1 &= n_c \lambda_1 \iint_{\mathbb{R}^{3\times 2}} K^{12}(p_2 + p_3, p_2, p_3) \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times |f(p_2)f(p_3) - g(p_2)g(p_3)| \left[|p_2 + p_3|^n + |p_2|^n + |p_3|^n\right] dp_2 dp_3.
\end{aligned}$$

By the triangle inequality,

$$|f(p_2)f(p_3) - g(p_2)g(p_3)| \leq |f(p_2) - g(p_2)||f(p_3)| + |f(p_3) - g(p_3)||g(p_2)|,$$

the term N_1 can be bounded as

$$N_1 \leq n_c\lambda_1 \iint_{\mathbb{R}^{3\times 2}} K^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$$

$$\times |f(p_2) - g(p_2)||f(p_3)|\left[|p_2 + p_3|^n + |p_2|^n + |p_3|^n\right]dp_2 dp_3$$

$$+ n_c\lambda_1 \iint_{\mathbb{R}^{3\times 2}} K^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$$

$$\times |f(p_3) - g(p_3)||g(p_2)|\left[|p_2 + p_3|^n + |p_2|^n + |p_3|^n\right]dp_2 dp_3.$$

Again, by the triangle inequality

$$|p_2 + p_3|^n \leq (|p_2| + |p_3|)^n \leq 2^{n-1}(|p_2|^n + |p_3|^n),$$

one can estimate N_1 as

$$N_1 \leq C \iint_{\mathbb{R}^{3\times 2}} \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})K^{12}(p_2 + p_3, p_2, p_3)\times$$

$$\times |f(p_2) - g(p_2)||f(p_3)|\left[|p_2|^n + |p_3|^n\right]dp_2 dp_3$$

$$+ C \iint_{\mathbb{R}^{3\times 2}} \delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})K^{12}(p_2 + p_3, p_2, p_3)\times$$

$$\times |f(p_3) - g(p_3)||g(p_2)|\left[|p_2|^n + |p_3|^n\right]dp_2 dp_3,$$

where C is a constant varying from line to line. The above estimate can be rewritten, taking into account the definition of $\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})$, as

$$N_1 \leq C \int_{\mathbb{R}^3}\int_{S_{p_3}^1} K_1^{12}(p_2 + p_3, p_2, p_3)|f(p_2) - g(p_2)||f(p_3)|\left[|p_2|^n + |p_3|^n\right]d\sigma(p_3)dp_2$$

$$+ C \int_{\mathbb{R}^3}\int_{S_{p_2}^1} K_1^{12}(p_2 + p_3, p_2, p_3)|f(p_3) - g(p_3)||g(p_2)|\left[|p_2|^n + |p_3|^n\right]d\sigma(p_2)dp_3.$$

By Lemma 2.3, one can estimate N_1 as follows

$$N_1 \leq C \iint_{\mathbb{R}^{3\times 2}} |f(p_2) - g(p_2)||f(p_3)|\frac{K^{12}(p_2 + p_3, p_2, p_3)}{|p_2||p_3|}\left[|p_2|^n + |p_3|^n\right]dp_3 dp_2$$

$$+ C \iint_{\mathbb{R}^{3\times 2}} |f(p_3) - g(p_3)||g(p_2)|\frac{K^{12}(p_2 + p_3, p_2, p_3)}{|p_2||p_3|}\left[|p_2|^n + |p_3|^n\right]dp_2 dp_3.$$

Since $\frac{K^{12}(p_2+p_3,p_2,p_3)}{|p_2||p_3|}$ and $\frac{K^{12}(p_2+p_3,p_2,p_3)}{|p_2||p_3|}$ are bounded, N_1 is bounded as

$$
\begin{aligned}
N_1 \ \leq \ & C \iint\limits_{\mathbb{R}^{3\times 2}} |f(p_2) - g(p_2)||f(p_3)| \left[|p_2|^n + |p_3|^n \right] dp_3 dp_2 \\
& + C \iint\limits_{\mathbb{R}^{3\times 2}} |f(p_3) - g(p_3)||g(p_2)| \left[|p_2|^n + |p_3|^n \right] dp_2 dp_3,
\end{aligned}
$$

which leads to the following straightforward estimates on N_1

$$
\begin{aligned}
N_1 \leq \ & C \int\limits_{\mathbb{R}^3} |f(p_2) - g(p_2)||p_2|^n dp_2 \int\limits_{\mathbb{R}^3} |f(p_3)| dp_3 \\
& + C \int\limits_{\mathbb{R}^3} |f(p_2) - g(p_2)| dp_2 \int\limits_{\mathbb{R}^3} |f(p_3)||p_3|^n dp_3 \\
& + C \int\limits_{\mathbb{R}^3} |f(p_3) - g(p_3)||p_3|^n dp_3 \int\limits_{\mathbb{R}^3} |f(p_2)| dp_2 \qquad (2.80) \\
& + C \int\limits_{\mathbb{R}^3} |f(p_3) - g(p_3)| dp_3 \int\limits_{\mathbb{R}^3} |f(p_2)||p_2|^n dp_2 \\
\leq \ & C \int\limits_{\mathbb{R}^3} |f(p_1) - g(p_1)||p_1|^n dp_1 + C \int\limits_{\mathbb{R}^3} |f(p_1) - g(p_1)| dp_1.
\end{aligned}
$$

Step 2: Estimating N_2.

By the definition of $\delta(p_1 - p_2 - p_3)$, N_2 can be rewritten as:

$$
\begin{aligned}
N_2 \ = \ & 2n_c\lambda_1 \iint\limits_{\mathbb{R}^{3\times 2}} K^{12}(p_1, p_1 - p_3, p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_1-p_3} - \mathcal{E}_{p_3}) \\
& \times |f(p_3)f(p_1) - g(p_3)g(p_1)| \left[|p_1|^n + |p_1 - p_3|^n + |p_3|^n \right] dp_1 dp_3,
\end{aligned}
$$

which, by the inequality,

$$
|p_1 - p_3|^n \leq (|p_1| + |p_3|)^n \leq 2^{n-1}(|p_1|^n + |p_3|^n),
$$

can be bounded as

$$N_2 \;\leq\; C \iint\limits_{\mathbb{R}^{3\times2}} K^{12}(p_1, p_1 - p_3, p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_1 - p_3} - \mathcal{E}_{p_3})$$

$$\times |f(p_3) - g(p_3)||f(p_1)| \left[|p_1|^n + |p_3|^n\right] dp_1 dp_3$$

$$+ C \iint\limits_{\mathbb{R}^{3\times2}} K^{12}(p_1, p_1 - p_3, p_3)\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_1 - p_3} - \mathcal{E}_{p_3})$$

$$\times |f(p_1) - g(p_1)||g(p_3)| \left[|p_1|^n + |p_3|^n\right] dp_1 dp_3.$$

Employing the definition of $\delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_1 - p_3} - \mathcal{E}_{p_3})$, one can estimate N_2 as

$$N_2 \;\leq\; C \int\limits_{\mathbb{R}^3} \int\limits_{S^0_{p_1}} K_0^{12}(p_1, p_1 - p_3, p_3)|f(p_3) - g(p_3)||f(p_1)| \left[|p_1|^n + |p_3|^n\right] d\sigma(p_3) dp_1$$

$$+ C \int\limits_{\mathbb{R}^3} \int\limits_{S^0_{p_1}} K_0^{12}(p_1, p_1 - p_3, p_3)|f(p_1) - g(p_1)||g(p_3)| \left[|p_1|^n + |p_3|^n\right] dd\sigma(p_3) dp_1,$$

which, by [Lemma 2.2](), yields

$$N_2 \;\leq\; \int\limits_{\mathbb{R}^3} \int\limits_0^{|p_1|} K^{12}(p_1, p_1 - p_3, p_3)|f(p_1) - g(p_1)||g(p_3)| \left[|p_1|^n + |p_3|^n\right] |p_3| d|p_3| dp_1$$

$$+ \int\limits_{\mathbb{R}^3} \int\limits_0^{|p_1|} K^{12}(p_1, p_1 - p_3, p_3)|f(p_3) - g(p_3)||f(p_1)| \left[|p_1|^n + |p_3|^n\right] |p_3| d|p_3| dp_1.$$

Bounding the integral from 0 to $|p_1|$ by an integral from 0 to ∞ implies

$$N_2 \;\leq\; \int\limits_{\mathbb{R}^3} \int\limits_0^{\infty} K^{12}(p_1, p_1 - p_3, p_3)|f(p_1) - g(p_1)||g(p_3)| \left[|p_1|^n + |p_3|^n\right] |p_3| d|p_3| dp_1$$

$$+ \int\limits_{\mathbb{R}^3} \int\limits_0^{\infty} K^{12}(p_1, p_1 - p_3, p_3)|f(p_3) - g(p_3)||f(p_1)| \left[|p_1|^n + |p_3|^n\right] |p_3| d|p_3| dp_1.$$

We now switch the integral from $d|p_3|$ to dp_3 from the above inequality to obtain

$$N_2 \;\leq\; \int\limits_{\mathbb{R}^{3\times2}} \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|}|f(p_1) - g(p_1)||g(p_3)| \left[|p_1|^n + |p_3|^n\right] dp_3 dp_1$$

$$+ \int\limits_{\mathbb{R}^{3\times2}} (1 + |p_1|)\frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|}|f(p_3) - g(p_3)||f(p_1)| \left[|p_1|^n + |p_3|^n\right] dp_3 dp_1.$$

Applying the inequality

$$|p_1|^n + |p_3|^n \leq C(1 + |p_1|^n + |p_3|^n)$$

to the above bound on N_2, we get

$$N_2 \leq$$
$$\leq C \int_{\mathbb{R}^{3 \times 2}} \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|} |f(p_3) - g(p_3)||f(p_1)| \left[1 + |p_1|^n + |p_3|^n \right] dp_3 dp_1$$
$$+ C \int_{\mathbb{R}^{3 \times 2}} \frac{K^{12}(p_1, p_1 - p_3, p_3)}{|p_3|} |f(p_1) - g(p_1)||g(p_3)| \left[1 + |p_1|^n + |p_3|^n \right] dp_3 dp_1.$$

The same argument as for (2.80) yields

$$N_2 \leq C \int_{\mathbb{R}^3} |f(p_1) - g(p_1)||p_1|^n dp_1 + C \int_{\mathbb{R}^3} |f(p_1) - g(p_1)| dp_1. \qquad (2.81)$$

Step 3: Estimating N_3.

By the definition of $\delta(p_1 - p_2 - p_3)$, N_3 can be rewritten as:

$$N_3 = n_c \lambda_1 \iint_{\mathbb{R}^{3 \times 2}} K^{12}(p_1, p_2, p_1 - p_2) \delta(\mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_1 - p_2})$$
$$\times |f(p_1) - g(p_1)| \left[|p_1|^n + |p_2|^n + |p_1 - p_2|^n \right] dp_1 dp_2,$$

which, by the inequality,

$$|p_1 - p_2|^n \leq (|p_1| + |p_2|)^n \leq 2^{n-1}(|p_1|^n + |p_2|^n),$$

can be bounded as

$$N_3 \leq C \iint_{\mathbb{R}^3} \int_{S_{p_1}} K^{12}(p_1, p_2, p_1 - p_2)|f(p_1) - g(p_1)| \left[|p_1|^n + |p_2|^n \right] d\sigma(p_2) dp_1.$$

Now, as an application of Lemma 2.2,

$$\int_{S_{p_1}} (|p_1|^n + |p_2|^n) d\sigma(p_2) \leq C \left(|p_1|^{n+2} + \int_{S_{p_1}} |p_2|^n d\sigma(p_2) \right)$$

$$\leq \ C\left(|p_1|^{n+2} + \int_0^{|p_1|} |p_2|^{n+1} d|p_2|\right)$$

$$\leq \ C\left(1 + |p_1|^{n+3}\right),$$

which together with the fact that $K^{12}(p_1, p_2, p_1 - p_2)$ is bounded, implies

$$N_3 \ \leq \ C\int_{\mathbb{R}^3} |f(p_1) - g(p_1)| \left[|p_1|^{n+3} + 1\right] dp_1. \tag{2.82}$$

Combining (2.80), (2.81), and (2.82) yields

$$\|C_{12}[f] - C_{12}[g]\|_{L_n^1} \ \leq \ C\int_{\mathbb{R}^3} |f(p_1) - g(p_1)| \left[|p_1|^{n+3} + |p_1|^{n+1} + |p_1|^n + 1\right] dp_1. \tag{2.83}$$

Since

$$|p|^n \leq C\left(|p|^{n+3} + 1\right), |p|^{n+1} \leq C\left(|p|^{n+3} + 1\right).$$

Inequality (2.77) follows from (2.83). Inequality (2.78) is a consequence of Inequality (2.77), Lemma 2.5 and

$$\|f - g\|_{L_{n+3}^1} \leq \|f - g\|_{L^1}^{\frac{1}{n+4}} \left(\|f\|_{L_{n+4}^1} + \|g\|_{L_{n+4}^1}\right)^{\frac{n+3}{n+4}}. \quad \square$$

2.3.2. Holder estimates for C_{22}^1

Proposition 2.6. *Let f and g be two functions in $L_n^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $n \in \mathbb{N}$, $n/2$ is an odd number, or $n = 0$, then there exists a constant $\mathcal{C}$ depending on $\|f\|_{L_{n+1}^1}$, $\|f\|_{L^1}$, $\|g\|_{L_{n+1}^1}$, $\|g\|_{L^1}$ such that*

$$\|C_{22}^1[f] - C_{22}^1[g]\|_{L_n^1} \leq \mathcal{C}\left(\|f - g\|_{L_{n+1}^1} + \|f - g\|_{L^1}\right). \tag{2.84}$$

If $\|f\|_{L_{n+2}^1}, \|g\|_{L_{n+2}^1} < \mathcal{C}_0$, then

$$\|C_{22}^1[f] - C_{22}^1[g]\|_{L_n^1} \leq \mathcal{C}_1\left(\|f - g\|_{L^1}^{\frac{1}{n+2}} + \|f - g\|_{L^1}\right), \tag{2.85}$$

where $\mathcal{C}_1$ is a constant depending on $\mathcal{C}_0, \mathcal{C}$.

Proof. Let us consider the L_n^1 norm of the difference $C_{22}^1[f] - C_{22}^1[g]$. As a view of Lemma 2.1

$$\int_{\mathbb{R}^3} \left| C_{22}^1[f](p_1) - C_{22}^1[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4| \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$

$$\times |f(p_1)f(p_2) - g(p_1)g(p_2)| \Big[|p_4|^n + |p_3|^n + |p_2|^n + |p_1|^n \Big] d|p_1|d|p_2|d|p_3|d|p_4|.$$

By the inequality

$$|p|^n \leq C\mathcal{E}_p^{n/2},$$

one gets

$$|p_4|^n + |p_3|^n + |p_2|^n + |p_1|^n \leq C\mathcal{E}_{p_4}^{n/2} + C\mathcal{E}_{p_3}^{n/2} + C\mathcal{E}_{p_2}^{n/2} + C\mathcal{E}_{p_1}^{n/2},$$

which implies

$$\int_{\mathbb{R}^3} \left| C_{22}^1[f](p_1) - C_{22}^1[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4| \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$

$$\times |f(p_1)f(p_2) - g(p_1)g(p_2)| \Big[\mathcal{E}_{p_4}^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \Big] d|p_1|d|p_2|d|p_3|d|p_4|.$$

Now, thanks to the Dirac function $\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$, one can write $\mathcal{E}_{p_4}$ as $\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3}$, which implies

$$\int_{\mathbb{R}^3} \left| C_{22}^1[f](p_1) - C_{22}^1[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4| \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$$

$$\times |f(p_1)f(p_2) - g(p_1)g(p_2)|$$

$$\times \Big[(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \Big] d|p_1|d|p_2|d|p_3|d|p_4|.$$

Similar as for (2.56), $|p_4|d|p_4|$ can be bounded by $Cd\mathcal{E}_{p_4}$ and $\min\{|p_1|, |p_2|, |p_3|, |p_4|\}$ can be bounded by $\min\{|p_1|, |p_2|, |p_3|\}$. Moreover, $K^{22}(p_1, p_2, p_3, p_4)$ is bounded by Γ due to (1.18). As a consequence,

$$\int\limits_{\mathbb{R}^3} \left| C_{22}^1[f](p_1) - C_{22}^1[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \int\limits_{\mathbb{R}_+^4} \min\{|p_1|, |p_2|, |p_3|\}|p_1||p_2||p_3|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})|f(p_1)f(p_2)-$$

$$- g(p_1)g(p_2)| \left[(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] d|p_1|d|p_2|d|p_3|d\mathcal{E}_{p_4}$$

$$\leq C \int\limits_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} \min\{|p_1|, |p_2|, |p_3|\}|p_1||p_2||p_3||f(p_1)f(p_2)-$$

$$- g(p_1)g(p_2)| \left[(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] d|p_1|d|p_2|d|p_3|,$$

where in the last inequality, we have taken the integration with respect to $d\mathcal{E}_{p_4}$.

Since $n/2$ is an odd number, by Newton formula

$$(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} = \sum_{0 \leq i,j,k \ ; \ i+j+k=n/2 \ ; \ k \neq n/2} B_{i,j,k,n} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k,$$

we obtain

$$\int\limits_{\mathbb{R}^3} \left| C_{22}^1[f](p_1) - C_{22}^1[g](p_1) \right| |p_1|^n dp_1 \leq X, \tag{2.86}$$

where

$$X := C \int\limits_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} \min\{|p_1|, |p_2|, |p_3|\}|p_1||p_2||p_3| \left| f(p_1)f(p_2) - g(p_1)g(p_2) \right| \times$$

$$\times \left[\sum_{0 \leq i,j,k \ ; \ i+j+k=n/2 \ ; \ k \neq n/2} |B_{i,j,k,n} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k| \right] d|p_1|d|p_2|d|p_3|.$$

The rest of the proof is devoted to estimates of X.

Similar as for (2.56), $|p_3|d|p_3|$ can be bounded by $Cd\mathcal{E}_{p_3}$ and $\min\{|p_1|, |p_2|, |p_3|\}$ can be bounded by $\min\{|p_1|, |p_2|\}$:

$$X_1 \leq C \int\limits_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} \min\{|p_1|, |p_2|\}|p_1||p_2| \left| f(p_1)f(p_2) - \right.$$

$$\left. - g(p_1)g(p_2) \right| \left[\sum_{0 \leq i,j,k \ ; \ i+j+k=n/2 \ ; \ k \neq n/2} \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k \right] d|p_1|d|p_2|\mathcal{E}_{p_3}.$$

Integrating with respect to $d\mathcal{E}_{p_3}$ the above integral and using Newton formula yields

$$
\begin{aligned}
X \le \ & C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1||p_2| |f(p_1)f(p_2)- \\
& - g(p_1)g(p_2)| \left[\sum_{0 \le i,j,k \ ; \ i+j+k=n/2 \ ; \ k \ne n/2} \sum_{s=0}^{k+1} \binom{k+1}{s} \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} \right] d|p_1|d|p_2| \\
\le \ & \sum_{0 \le i,j,k \ ; \ i+j+k=n/2 \ ; \ k \ne n/2} \sum_{s=0}^{k+1} C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1||p_2| |f(p_1)f(p_2)- \\
& - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1|d|p_2| \\
\le \ & \sum_{0 \le i,j,k \ ; \ i+j+k=n/2 \ ; \ k \ne n/2} \sum_{s=0;i+s\ne0}^{k+1} C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1||p_2| |f(p_1)f(p_2)- \\
& - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1|d|p_2| + \\
& + C \int_{\mathbb{R}_+^2} \min\{|p_1|, |p_2|\} |p_1||p_2| |f(p_1)f(p_2) - g(p_1)g(p_2)| \mathcal{E}_{p_2}^{n/2+1} d|p_1|d|p_2|.
\end{aligned}
$$

By the inequalities

$$
\min\{|p_1|, |p_2|\} |p_1||p_2| \le |p_1||p_2|^2,
$$

and

$$
\min\{|p_1|, |p_2|\} |p_1||p_2| \le |p_1|^2|p_2|,
$$

one deduce that

$$
X \le X_1 + X_2, \tag{2.87}
$$

where

$$
\begin{aligned}
X_1 := \ & \sum_{0 \le i,j,k \ ; \ i+j+k=n/2 \ ; \ k \ne n/2} \sum_{s=0;i+s\ne0}^{k+1} C \int_{\mathbb{R}_+^2} |p_1||p_2|^2 \Big| f(p_1)f(p_2)- \\
& - g(p_1)g(p_2) \Big| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1|d|p_2|; \\
X_2 := \ & C \int_{\mathbb{R}_+^2} |p_1|^2|p_2| \Big| f(p_1)f(p_2) - g(p_1)g(p_2) \Big| \mathcal{E}_{p_2}^{n/2+1} d|p_1|d|p_2|.
\end{aligned}
$$

Let us first estimate X_1 by looking at the terms inside the sum

$$\int_{\mathbb{R}_+^2} |p_1||p_2|^2 |f(p_1)f(p_2) - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1| d|p_2|$$

$$\leq \int_{\mathbb{R}_+^2} |p_1||p_2|^2 |f(p_1) - g(p_1)||g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1| d|p_2|$$

$$+ \int_{\mathbb{R}_+^2} |p_1||p_2|^2 |f(p_2) - g(p_2)||f(p_1)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1| d|p_2|,$$

where we have used the triangle inequality

$$|f(p_1)f(p_2) - g(p_1)g(p_2)| \leq |f(p_1) - g(p_1)||g(p_2)| + |f(p_2) - g(p_2)||f(p_1)|.$$

Since $0 < i + s \leq n/2 + 1$ and $0 \leq k + 1 + j - s \leq n/2 + 1$, we have

$$\mathcal{E}_{p_1}^{i+s} \leq C \left(|p_1| + |p_1|^{n+2} \right),$$

and

$$\mathcal{E}_{p_2}^{k+1+j-s} \leq C \left(1 + |p_2|^{n+2} \right),$$

which yields

$$\int_{\mathbb{R}_+^2} |p_1||p_2|^2 |f(p_1)f(p_2) - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1| d|p_2|$$

$$\leq C \int_{\mathbb{R}_+} |p_1| \left(|p_1| + |p_1|^{n+2} \right) |f(p_1) - g(p_1)| d|p_1| \int_{\mathbb{R}_+} |p_1|^2 \left(1 + |p_1|^{n+2} \right) |g(p_1)| d|p_1|$$

$$+ C \int_{\mathbb{R}_+} |p_1| \left(|p_1| + |p_1|^{n+2} \right) |f(p_1) - g(p_1)| d|p_1| \int_{\mathbb{R}_+} |p_1|^2 \left(1 + |p_1|^{n+2} \right) |f(p_1)| d|p_1|$$

$$\leq C \int_{\mathbb{R}^3} \left(1 + |p_1|^{n+1} \right) |f(p_1) - g(p_1)| dp_1 \int_{\mathbb{R}^3} \left(1 + |p_1|^{n+2} \right) |g(p_1)| dp_1$$

$$+ C \int_{\mathbb{R}^3} \left(1 + |p_1|^{n+1} \right) |f(p_1) - g(p_1)| dp_1 \int_{\mathbb{R}^3} \left(1 + |p_1|^{n+2} \right) |f(p_1)| dp_1,$$

where in the last inequality, we have switched the integration on $\mathbb{R}_+$ to $\mathbb{R}^3$, by a spherical change of variables. Now, by the boundedness of f and g in L^1 and L_{n+2}^1,

$$\int_{\mathbb{R}_+^2} |p_1||p_2|^2 |f(p_1)f(p_2) - g(p_1)g(p_2)| \mathcal{E}_{p_1}^{i+s} \mathcal{E}_{p_2}^{k+1+j-s} d|p_1|d|p_2|$$

$$\leq C \int_{\mathbb{R}^3} \left(1 + |p_1|^{n+1}\right) |f(p_1) - g(p_1)| dp_1,$$

which implies the following estimate on X_1

$$X_1 \leq C\|f - g\|_{L^1} + C\|f - g\|_{L^1_{n+1}}. \tag{2.88}$$

We now estimate X_2. As an application of the inequality

$$\mathcal{E}_{p_2}^{n/2+1} \leq C \left(|p_2| + |p_2|^{n+2}\right),$$

X_2 can be bounded as follows

$$
\begin{aligned}
X_2 \leq\ & C \int_{\mathbb{R}_+^2} |p_1|^2 |p_2| |f(p_1)f(p_2) - g(p_1)g(p_2)| \left(|p_2| + |p_2|^{n+2}\right) d|p_1|d|p_2| \\
\leq\ & C \int_{\mathbb{R}_+^2} |p_1|^2 |f(p_1) - g(p_1)||g(p_2)| \left(|p_2|^2 + |p_2|^{n+3}\right) d|p_1|d|p_2| \\
& + C \int_{\mathbb{R}_+^2} |p_1|^2 |f(p_2) - g(p_2)||f(p_1)| \left(|p_2|^2 + |p_2|^{n+3}\right) d|p_1|d|p_2|.
\end{aligned}
$$

The same argument as for (2.88) leads to

$$X_2 \leq C\|f - g\|_{L^1} + C\|f - g\|_{L^1_{n+1}}. \tag{2.89}$$

Combining (2.87), (2.88) and (2.89) yields

$$X \leq C \left(\|f - g\|_{L^1} + \|f - g\|_{L^1_{n+1}}\right). \tag{2.90}$$

The two inequalities (2.86) and (2.90) lead to

$$\int_{\mathbb{R}^3} \left|C_{22}^1[f](p_1) - C_{22}^1[g](p_1)\right| |p_1|^n dp_1 \leq C \left(\|f - g\|_{L^1} + \|f - g\|_{L^1_{n+1}}\right). \tag{2.91}$$

Inequality (2.85) is a consequence of Inequality (2.84), Lemma 2.5 and

$$\|f - g\|_{L^1_{n+1}} \leq \|f - g\|_{L^1}^{\frac{1}{n+2}} \left(\|f\|_{L^1_{n+2}} + \|g\|_{L^1_{n+2}}\right)^{\frac{n+1}{n+2}}. \qquad \square$$

2.3.3. Holder estimates for C_{22}^2

Proposition 2.7. *Let f and g be two functions in $L_n^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $n/2 \in \mathbb{N}$, n can be 0, then there exists a constant $\mathcal{C}$ depending on $\|f\|_{L_n^1}, \|f\|_{L^1}, \|g\|_{L_n^1}, \|g\|_{L^1}$, such that*

$$\|C_{22}^2[f] - C_{22}^2[g]\|_{L_n^1} \leq \mathcal{C}\left(\|f - g\|_{L_n^1} + \|f - g\|_{L^1}\right). \tag{2.92}$$

If $\|f\|_{L_{n+1}^1}, \|g\|_{L_{n+1}^1} < \mathcal{C}_0$, then

$$\|C_{22}^2[f] - C_{22}^2[g]\|_{L_n^1} \leq \mathcal{C}_1\left(\|f - g\|_{L^1}^{\frac{1}{n+1}} + \|f - g\|_{L^1}\right), \tag{2.93}$$

where $\mathcal{C}_1$ is a constant depending on $\mathcal{C}_0, \mathcal{C}$.

Proof. As a view of Lemma 2.1, the L_n^1 norm of the difference $C_{22}^2[f] - C_{22}^2[g]$ can be written as

$$\int_{\mathbb{R}^3} \left|C_{22}^2[f](p_1) - C_{22}^2[g](p_1)\right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4|\delta \times$$

$$\times (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})|f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| \times$$

$$\times \left[|p_4|^n + |p_3|^n + |p_2|^n + |p_1|^n\right] d|p_1|d|p_2|d|p_3|d|p_4|.$$

Similar as for Proposition 2.6, by the inequality

$$|p_4|^n + |p_3|^n + |p_2|^n + |p_1|^n \leq C\mathcal{E}_{p_4}^{n/2} + C\mathcal{E}_{p_3}^{n/2} + C\mathcal{E}_{p_2}^{n/2} + C\mathcal{E}_{p_1}^{n/2},$$

one has

$$\int_{\mathbb{R}^3} \left|C_{22}^2[f](p_1) - C_{22}^2[g](p_1)\right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1, p_2, p_3, p_4) \min\{|p_1|, |p_2|, |p_3|, |p_4|\} |p_1||p_2||p_3||p_4|\delta \times$$

$$\times (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})|f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| \times$$

$$\times \left[\mathcal{E}_{p_4}^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2}\right] d|p_1|d|p_2|d|p_3|d|p_4|.$$

By the Dirac function $\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})$, $\mathcal{E}_{p_4}$ can be written as $\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3}$, which implies

$$\int_{\mathbb{R}^3} \left| C_{22}^2[f](p_1) - C_{22}^2[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1,p_2,p_3,p_4)\min\{|p_1|,|p_2|,|p_3|,|p_4|\}|p_1||p_2||p_3||p_4|\delta \times$$

$$\times (\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4})|f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)| \times$$

$$\times \left[(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] d|p_1|d|p_2|d|p_3|d|p_4|.$$

Similar as for (2.56), $|p_4|d|p_4|$ can be bounded by $Cd\mathcal{E}_{p_4}$ and $\min\{|p_1|,|p_2|,|p_3|,|p_4|\}$ can be bounded by $\min\{|p_1|,|p_2|,|p_3|\}$, which leads to

$$\int_{\mathbb{R}^3} \left| C_{22}^2[f](p_1) - C_{22}^2[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \int_{\mathbb{R}_+^4} K^{22}(p_1,p_2,p_3,p_4)\min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3|\delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times$$

$$\times |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3))| \times$$

$$\times \left[(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] d|p_1|d|p_2|d|p_3|d\mathcal{E}_{p_4}$$

$$\leq C \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} K^{22}(p_1,p_2,p_3,p_4)\min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3| \times$$

$$\times |f(p_1)f(p_2)f(p_3)g(p_1)g(p_2)g(p_3)| \times$$

$$\times \left[(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} \right] d|p_1|d|p_2|d|p_3|,$$

where we have taken the integration with respect to $d\mathcal{E}_{p_4}$.

Since $n/2$ is a natural number, by Newton formula

$$(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3})^{n/2} + \mathcal{E}_{p_3}^{n/2} + \mathcal{E}_{p_2}^{n/2} + \mathcal{E}_{p_1}^{n/2} = \sum_{0 \leq i,j,k \ ; \ i+j+k=n/2} D_{i,j,k,n}\mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k,$$

where $D_{i,j,k,n}$ are positive constants. As an application of the above Newton formula, one has

$$\int_{\mathbb{R}^3} \left| C_{22}^2[f](p_1) - C_{22}^2[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \sum_{0 \leq i,j,k \ ; \ i+j+k=n/2} \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} K^{22}(p_1,p_2,p_3,p_4)\min\{|p_1|,|p_2|,|p_3|\}|p_1||p_2||p_3|$$

$$\times |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)|\mathcal{E}^i_{p_1}\mathcal{E}^j_{p_2}\mathcal{E}^k_{p_3}d|p_1|d|p_2|d|p_3|,$$

where C is a positive constant varying from line to line.

By using the fact that

$$K^{22}(p_1, p_2, p_3, p_4)\min\{|p_1|, |p_2|, |p_3|\}|p_1||p_2||p_3| \leq C|p_1|^2|p_2|^2|p_3|^2,$$

where C is a positive constant depending on p_* defined in (1.17), we get

$$\int_{\mathbb{R}^3} \left|C^2_{22}[f](p_1) - C^2_{22}[g](p_1)\right| |p_1|^n dp_1$$

$$\leq C \sum_{0 \leq i,j,k \ ; \ i+j+k=n/2} \int_{\mathcal{E}_{p_3} \leq \mathcal{E}_{p_1} + \mathcal{E}_{p_2}} |p_1|^2|p_2|^2|p_3|^2|f(p_1)f(p_2)f(p_3)-$$

$$- g(p_1)g(p_2)g(p_3)|\mathcal{E}^i_{p_1}\mathcal{E}^j_{p_2}\mathcal{E}^k_{p_3}d|p_1|d|p_2|d|p_3|$$

$$\leq C \sum_{0 \leq i,j,k \ ; \ i+j+k=n/2} \int_{\mathbb{R}^3_+} |p_1|^2|p_2|^2|p_3|^2|f(p_1)f(p_2)f(p_3)-$$

$$- g(p_1)g(p_2)g(p_3)|\mathcal{E}^i_{p_1}\mathcal{E}^j_{p_2}\mathcal{E}^k_{p_3}d|p_1|d|p_2|d|p_3|.$$

Changing from the radial integration on $\mathbb{R}_+$ to the integration on $\mathbb{R}^3$ in the above inequality, by a spherical coordinate change of variables, yields

$$\int_{\mathbb{R}^3} \left|C^2_{22}[f](p_1) - C^2_{22}[g](p_1)\right| |p_1|^n dp_1$$

$$\leq C \sum_{0 \leq i,j,k \ ; \ i+j+k=n/2} \int_{\mathbb{R}^{3\times3}} |f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)|\mathcal{E}^i_{p_1}\mathcal{E}^j_{p_2}\mathcal{E}^k_{p_3}dp_1dp_2dp_3.$$

Applying the triangle inequality

$$|f(p_1)f(p_2)f(p_3) - g(p_1)g(p_2)g(p_3)|$$

$$\leq |f(p_1) - g(p_1)||f(p_2)||f(p_3)| + |f(p_2) - g(p_2)||g(p_1)||f(p_3)|$$

$$+ |f(p_3) - g(p_3)||g(p_1)||g(p_3)|,$$

to the previous inequality gives

$$\int_{\mathbb{R}^3} \left| C_{22}^2[f](p_1) - C_{22}^2[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \sum_{0 \leq i,j,k \; ; \; i+j+k=n/2} \int_{\mathbb{R}^{3\times 3}} |f(p_1) - g(p_1)||f(p_2)||f(p_3)| \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k dp_1 dp_2 dp_3$$

$$+ C \sum_{0 \leq i,j,k \; ; \; i+j+k=n/2} \int_{\mathbb{R}^{3\times 3}} |f(p_2) - g(p_2)||g(p_1)||f(p_3)| \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k dp_1 dp_2 dp_3$$

$$+ C \sum_{0 \leq i,j,k \; ; \; i+j+k=n/2} \int_{\mathbb{R}^{3\times 3}} |f(p_3) - g(p_3)||g(p_1)||g(p_3)| \mathcal{E}_{p_1}^i \mathcal{E}_{p_2}^j \mathcal{E}_{p_3}^k dp_1 dp_2 dp_3.$$

Notice that we can estimate $\mathcal{E}_{p_1}^i$, $\mathcal{E}_{p_2}^j$ and $\mathcal{E}_{p_3}^k$ as

$$\mathcal{E}_p^i \leq C(1 + |p|^n), \quad \mathcal{E}_p^j \leq C(1 + |p|^n), \quad \mathcal{E}_p^k \leq C(1 + |p|^n),$$

which leads to the following estimate on the norm of $C_{22}^2[f] - C_{22}^2[g]$

$$\int_{\mathbb{R}^3} \left| C_{22}^2[f](p_1) - C_{22}^2[g](p_1) \right| |p_1|^n dp_1$$

$$\leq C \sum_{0 \leq i,j,k \; ; \; i+j+k=n/2} \int_{\mathbb{R}^{3\times 3}} |f(p_1) - g(p_1)||f(p_2)||f(p_3)|$$

$$\times (1 + |p_1|^n)(1 + |p_2|^n)(1 + |p_3|^n)dp_1 dp_2 dp_3$$

$$+ C \sum_{0 \leq i,j,k \; ; \; i+j+k=n/2} \int_{\mathbb{R}^{3\times 3}} |f(p_2) - g(p_2)||g(p_1)||f(p_3)|$$

$$\times (1 + |p_1|^n)(1 + |p_2|^n)(1 + |p_3|^n)dp_1 dp_2 dp_3$$

$$+ C \sum_{0 \leq i,j,k \; ; \; i+j+k=n/2} \int_{\mathbb{R}^{3\times 3}} |f(p_3) - g(p_3)||g(p_1)||g(p_3)|$$

$$\times (1 + |p_1|^n)(1 + |p_2|^n)(1 + |p_3|^n)dp_1 dp_2 dp_3.$$

Now, since

$$\int_{\mathbb{R}^3} |f(p)|(1 + |p|^n) = \|f\|_{L^1} + \|f\|_{L_n^1}, \quad \int_{\mathbb{R}^3} |g(p)|(1 + |p|^n) = \|g\|_{L^1} + \|g\|_{L_n^1},$$

we get from the above inequality that

$$\int_{\mathbb{R}^3} \left| C_{22}^2[f](p_1) - C_{22}^2[g](p_1) \right| |p_1|^n dp_1 \leq C \left(\|f - g\|_{L^1} + \|f - g\|_{L_n^1} \right).$$

Inequality (2.93) is a consequence of Inequality (2.92), Lemma 2.5 and

$$\|f - g\|_{L^1_n} \le \|f - g\|_{L^1}^{\frac{1}{n+1}} \left(\|f\|_{L^1_{n+1}} + \|g\|_{L^1_{n+1}} \right)^{\frac{n}{n+1}}. \quad \square$$

2.4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we will use Theorem 1.2. Choose $E = \mathbb{L}^1_{2n}(\mathbb{R}^3)$. We define the function $|\cdot|_*$ to be

$$|f|_* = \int_{\mathbb{R}^3} f(p) dp.$$

Set

$$\|f\|_* = \int_{\mathbb{R}^3} |f(p)| dp.$$

By (2.45), it is clear that for all $f \ge 0$, $f \in E$, the following inequality holds true

$$|Q[f]|_* \le C^* \left(1 + \|f\|_* \right), \tag{2.94}$$

where C^* depends on $\|f\|_{\mathcal{L}^1_2(\mathbb{R}^3)}$. We then choose C_* in Theorem 1.2 as C^*.

The set $\mathcal{S}_T$ is defined as follows:

$$\mathcal{S}_T := \left\{ f \in \mathbb{L}^1_{2n}(\mathbb{R}^3) \mid (S_1) \, f \ge 0, \quad f(p) = f(|p|), \ (S_2) \int_{\mathbb{R}^3} f(|p|) dp \le \mathfrak{c}_0, \right.$$
$$\left. (S_3) \int_{\mathbb{R}_+} f(|p|) \mathcal{E}_p dp = \mathfrak{c}_1, \ (S_4) \int_{\mathbb{R}_+} f(|p|) \mathcal{E}_p^{n^*} dp \le \mathfrak{c}_{n^*} \right\}, \tag{2.95}$$

where

$$\mathfrak{c}_0 := (2\mathcal{R} + 1) e^{(C^*+1)T}, \tag{2.96}$$

and

$$\mathfrak{c}_{n^*} = \frac{3\rho_{n_*}}{2}, \tag{2.97}$$

with ρ_{n_*} defined in (2.99). It is clear that $\mathcal{S}_T$ is a bounded, convex and closed subset of $\mathbb{L}^1_{2n}(\mathbb{R}^3)$. Moreover for all f in $\mathcal{S}_T$, it is straightforward that $|f|_* = \|f\|_*$.

In the four Sections 2.4.2, 2.4.1, 2.4.3, 2.4.4, we will verify the four conditions $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$ and $(\mathfrak{D})$ of Theorem 1.2. Then, Theorem 1.1 follows as an application of Theorem 1.2.

2.4.1. Checking condition ($\mathfrak{A}$)

We choose the constant R_* to be $\mathcal{R}+1$, then for all u in $\mathcal{S}_T$, $\|u\|_* \leq (2R_*+1)e^{(C^*+1)T}$. Condition ($\mathfrak{A}$) is satisfied.

2.4.2. Checking condition ($\mathfrak{B}$)

First, the same argument as for (2.74) gives

$$\int_{\mathbb{R}^3} Q[f]\mathcal{E}_p^{n^*}\,dp \leq \mathcal{P}\big[m_{n^*}(f)\big] :=$$

$$Cm_{n^*}(f) + Cm_{n^*}(f)^{\frac{n^*-1}{n^*}} + Cm_{n^*}(f)^{\frac{n^*-2}{n^*}} + Cm_{n^*}(f)^{\frac{1}{n^*}} - Cm_{n^*}(f)^{\frac{n^*+1}{n^*}}$$

$$+ C\sum_{0\leq i,j,k<n^*;\ i+j+k=n^*}\sum_{s=0}^{k+1} m_{n^*}(f)^{\frac{i+s}{n^*}}\left(m_{n^*}(f)^{\frac{j+k-s}{n^*}} + m_{n^*}(f)^{\frac{j+k-s+1/2}{n^*}}\right) + \quad (2.98)$$

$$+ C\sum_{0\leq i,j,k<n^*;\ i+j+k=n^*;\ j,k>0} m_{n^*}(f)^{\frac{i}{n^*}}\left(m_{n^*}(f)^{\frac{j-1}{n^*}} + m_{n^*}(f)^{\frac{j-1/2}{n^*}}\right)\times$$

$$\times\left(m_{n^*}(f)^{\frac{k-1}{n^*}} + m_{n^*}(f)^{\frac{k-1/2}{n^*}}\right), \quad \forall f\in\mathcal{S}_T,$$

where C is a positive constant depending on $\mathfrak{c}_0$.

Let ρ_{n^*} be the solution of $\mathcal{P}(\rho)=0$: if $0<\rho<\rho_{n^*}$, $\mathcal{P}(\rho)<0$; if $\rho>\rho_{n^*}$, $]\mathcal{P}(\rho)>0$.

$$(2.99)$$

Notice that ρ_{n^*} depends on $\mathfrak{c}_0$.

Let f be an arbitrary element of the set $\mathcal{S}_T\cap B_*\Big(O,(2R_*+1)e^{(C_*+1)T}\Big)$ and consider the element $f+hQ[f]$. We will show that for all $\epsilon>0$, there exists h_* depending on f and ϵ such that $B(f+hQ[f],h\epsilon)\cap\mathcal{S}_T$ is not empty for all $0<h<h_*$. Define $\chi_R(p)$ to be the characteristic function of the ball $B(O,R)$ centered at the origin with radius R. Set $f_R(p)=\chi_R(p)f(p)$ and $w_R=f+hQ[f_R]$. Since $Q[f_R]\in\mathbb{L}^1_{2n}(\mathbb{R}^3)$, we find that $w_R\in\mathbb{L}^1_{2n}(\mathbb{R}^3)$. We will prove that for h_* small enough and R large enough, w_R belongs to $\mathcal{S}_T$. We now verify the four conditions (S_1), (S_2), (S_3) and (S_4).

- Condition (S_1): Since f_R is compactly supported, it is clear that $Q^-[f_R]$, with Q^- defined in (2.7), is bounded by $C(f,R,\mathfrak{c}_0,\mathfrak{c}_{n^*})$, a positive constant depending on f, R, $\mathfrak{c}_0$, $\mathfrak{c}_{n^*}$, which implies

$$w_R \geq f - hf_R Q^-[f_R] \geq f(1-hQ^-[f_R]) \geq 0,$$

  for $h<C(f,R,\mathfrak{c}_0,\mathfrak{c}_{n^*})^{-1}$.

- Condition (S_2): Since

$$\|f\|_* < (2R_*+1)e^{(C_*+1)T},$$

and

$$\lim_{h \to 0} \|f - w_R\|_* = 0,$$

we can choose h_* small enough such that

$$\|w_R\|_* < (2R_* + 1)e^{(C_* + 1)T}.$$

- *Condition (S_3):* By the conservation of energy, we have

$$\int_{\mathbb{R}^3} w_R \mathcal{E}_p dp = \int_{\mathbb{R}^3} (f + hQ[f_R]) \mathcal{E}_p dp = \int_{\mathbb{R}^3} f \mathcal{E}_p dp = \mathfrak{c}_1.$$

- *Condition (S_4):* Now, we claim that R and h_* can be chosen, such that

$$\int_{\mathbb{R}^3} w_R \mathcal{E}_p^{n^*} dp < \frac{3\rho_{n^*}}{2}.$$

In order to see this, we consider two cases:
If

$$\int_{\mathbb{R}^3} f \mathcal{E}_p^{n^*} dp < \frac{3\rho_{n^*}}{2},$$

we deduce from the fact

$$\lim_{h \to 0} \int_{\mathbb{R}^3} |w_R - f| \mathcal{E}_p^{n^*} dp = 0,$$

that we can choose h_* small enough such that

$$\int_{\mathbb{R}^3} w_R \mathcal{E}_p^{n^*} dp < \frac{3\rho_{n^*}}{2}.$$

If, on the other hand, we have

$$\int_{\mathbb{R}^3} f \mathcal{E}_p^{n^*} dp = \frac{3\rho_{n^*}}{2},$$

we can choose R large enough such that

$$\int_{\mathbb{R}^3} f_R \mathcal{E}_p^{n^*} dp > \rho_{n^*},$$

which implies, by (2.99), that

$$\int\limits_{\mathbb{R}^3} Q[f_R] < 0.$$

As a consequence,

$$\int\limits_{\mathbb{R}^3} w_R \mathcal{E}_p^{n^*} dp < \int\limits_{\mathbb{R}^3} f \mathcal{E}_p^{n^*} dp = \frac{3\rho_{n^*}}{2}.$$

Finally, we have $w_R \in \mathcal{S}_T$ for all $0 < h < h_*$.

Now since

$$\lim_{R \to \infty} \frac{1}{h} \| w_R - f - hQ[f_R] \|_{\mathbb{L}^1_{2n}(\mathbb{R}^3)} = \lim_{R \to \infty} \| Q[f] - Q[f_R] \|_{\mathbb{L}^1_{2n}(\mathbb{R}^3)} = 0,$$

then for R large enough, $w_R \in B(f + hQ[f], h\epsilon)$, which implies $B(f + hQ[f], h\epsilon) \cap \mathcal{S}_T \backslash \{f + hQ[f]\}$. Condition ($\mathfrak{B}$) is verified.

2.4.3. Checking condition ($\mathfrak{C}$)

Condition ($\mathfrak{C}$) follows from Propositions 2.5, 2.6, and 2.7.

2.4.4. Checking condition ($\mathfrak{D}$)

By the Lebesgue dominated convergence theorem, we have that

$$\left[\varphi, \phi \right] \leq \int\limits_{\mathbb{R}^3} \varphi(p) \text{sign}(\phi(p))(1 + \mathcal{E}_p^n) dp, \tag{2.100}$$

which means that Condition ($\mathfrak{D}$) is satisfied if we have the following inequality

$$\mathcal{M}_0 := \int\limits_{\mathbb{R}^3} [Q[f](p) - Q[g](p)] \text{sign}((f - g)(p))(1 + \mathcal{E}_p^n) dp \leq C \| f - g \|_{\mathbb{L}^1_{2n}}. \tag{2.101}$$

Since $Q = C_{12} + C_{22}$, let us split

$$\mathcal{M}_0 = \mathcal{M}_1 + \mathcal{M}_2,$$

where

$$\mathcal{M}_1 := \int\limits_{\mathbb{R}^3} [C_{12}[f](p) - C_{12}[g](p)] \text{sign}((f - g)(p))(1 + \mathcal{E}_p^n) dp,$$

and

$$\mathcal{M}_2 := \int_{\mathbb{R}^3} [C_{22}[f](p) - C_{22}[g](p)]\text{sign}((f - g)(p))(1 + \mathcal{E}_p^n)dp.$$

Step 1: Estimating $\mathcal{M}_1$.

Define $\varphi_k(p) = \text{sign}((f - g)(p))\mathcal{E}_p^k$, $k \in \mathbb{Z}, k \geq 0$, $k \neq 1$. Let us consider the following generalized term of $\mathcal{M}_1$

$$\mathcal{N}_0 := \int_{\mathbb{R}^3} [C_{12}[f](p) - C_{12}[g](p)]\varphi_k(p)dp, \tag{2.102}$$

which by [Lemma 2.1](#) can be rewritten as

$$\begin{aligned}
\mathcal{N}_0 &:= \int_{\mathbb{R}^{3\times 3}} [R_{12}[f](p_1) - R_{12}[g](p_1)][\varphi_k(p_1) - \varphi_k(p_2) - \varphi_k(p_3)]dp_1 dp_2 dp_3 \\
&= \int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})[(f(p_2)f(p_3) - g(p_2)g(p_3)) \\
&\quad - 2(f(p_2)f(p_2 + p_3) - g(p_2)g(p_2 + p_3)) - (f(p_2 + p_3) - g(p_2 + p_3))] \times \\
&\quad \times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)]dp_2 dp_3. \tag{2.103}
\end{aligned}$$

Split $\mathcal{N}_0$ into the sum of three terms:

$$\begin{aligned}
\mathcal{N}_1 &:= \int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})[f(p_2)f(p_3) - g(p_2)g(p_3)] \\
&\quad \times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)]dp_2 dp_3, \tag{2.104}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_2 &:= -2\int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times [f(p_2)f(p_2 + p_3) - g(p_2)g(p_2 + p_3)][\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)]dp_2 dp_3, \tag{2.105}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}_3 &:= -\int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2 + p_3, p_2, p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})[f(p_2 + p_3) - g(p_2 + p_3)] \times \\
&\quad \times [\varphi_k(p_2 + p_3) - \varphi_k(p_2) - \varphi_k(p_3)]dp_2 dp_3.
\end{aligned}$$
$$\tag{2.106}$$

The same arguments as for [(2.80)](#) and [(2.81)](#) give

$$\mathcal{N}_1 \leq C\|f - g\|_{\mathbb{L}^1_{2k}(\mathbb{R}^3)}, \tag{2.107}$$

and

$$\mathcal{N}_2 \leq C\|f - g\|_{\mathbb{L}^1_{2k}(\mathbb{R}^3)}, \tag{2.108}$$

where C is a positive constant varying from line to line.

The third term $\mathcal{N}_3$ can be estimated as

$$
\begin{aligned}
\mathcal{N}_3 = \quad & -\int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2+p_3,p_2,p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})[f(p_2+p_3) - g(p_2+p_3)]\times \\
& \times [\mathcal{E}^k_{p_2+p_3}\mathrm{sign}((f(p_2+p_3) - g(p_2+p_3))) - \mathcal{E}^k_{p_2}\mathrm{sign}((f(p_2) - g(p_2)) - \\
& - \mathcal{E}^k_{p_3}\mathrm{sign}((f(p_3) - g(p_3))]dp_2dp_3 \\
\leq \quad & \int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2+p_3,p_2,p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})|f(p_2+p_3) - g(p_2+p_3)|\times \\
& \times [\mathcal{E}^k_{p_2} + \mathcal{E}^k_{p_3} - \mathcal{E}^k_{p_2+p_3}]dp_2dp_3. \tag{2.109}
\end{aligned}
$$

Now, let us consider the two cases $k = 0$ and $k > 1$ separately.

- If $k = 0$,

$$\mathcal{N}_3 \leq \int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2+p_3,p_2,p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3})|f(p_2+p_3) - g(p_2+p_3)|dp_2dp_3, \tag{2.110}$$

which, by the same arguments that lead to (2.82), can be bounded as

$$\mathcal{N}_3 \leq C\|f - g\|_{\mathbb{L}^1_3(\mathbb{R}^3)}. \tag{2.111}$$

- If $k > 1$, since $\mathcal{E}_{p_2+p_3} = \mathcal{E}_{p_2} + \mathcal{E}_{p_3}$, it is straightforward that

$$\mathcal{E}^k_{p_2} + \mathcal{E}^k_{p_3} - \mathcal{E}^k_{p_2+p_3} = \mathcal{E}^k_{p_2} + \mathcal{E}^k_{p_3} - (\mathcal{E}_{p_2} + \mathcal{E}_{p_3})^k \leq -k\mathcal{E}_{p_2}\mathcal{E}^{k-1}_{p_3} \leq 0.$$

As a consequence, we can estimate $\mathcal{N}_3$ as

$$
\begin{aligned}
\mathcal{N}_3 \leq \quad & \\
\leq \quad & -\int_{\mathbb{R}^{3\times 2}} \bar{K}^{12}(p_2+p_3,p_2,p_3)\delta(\mathcal{E}_{p_2+p_3} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
& \times |f(p_2+p_3) - g(p_2+p_3)|k\mathcal{E}_{p_2}\mathcal{E}^{k-1}_{p_3}dp_2dp_3 \\
\leq \quad & -\int_{\mathbb{R}^3}\int_{S^0_{p_1}} \bar{K}^{12}_0(p_1,p_2,p_1-p_2)|f(p_1) - g(p_1)|k\mathcal{E}_{p_2}\mathcal{E}^{k-1}_{p_1-p_2}d\sigma(p_2)dp_1. \tag{2.112}
\end{aligned}
$$

As a view of Lemma 2.2, we find the following bound on $\mathcal{N}_3$

$$\mathcal{N}_3 \leq -C \int_{\mathbb{R}^3} |f(p) - g(p)| \Big(|p|^{2k+1} \min\{1, |p|\}^{2k+7} \Big) dp. \tag{2.113}$$

Combining (2.107), (2.108), (2.111) and (2.113) for the two cases $k = 0$ and $k = n$, yields

$$\mathcal{M}_1 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| \Big(1 + |p| + |p|^3 + |p|^{2n} + |p|^{2n+1} \tag{2.114}$$
$$- |p|^{2n+1} \min\{1, |p|\}^{2n+7} \Big) dp.$$

Step 2: Estimating $\mathcal{M}_2$.

We can estimate $\mathcal{M}_2$ in a straightforward manner by employing Propositions 2.6 and 2.7, as follows

$$\mathcal{M}_2 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| \Big(1 + |p| + |p|^{2n} + |p|^{2n+1} \Big) dp. \tag{2.115}$$

Step 3: Estimating $\mathcal{M}_0$.

Combining (2.114) and (2.115) yields

$$\mathcal{M}_0 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| \Big(1 + |p| + |p|^3 + |p|^{2n} \tag{2.116}$$
$$- |p|^{2n+1} \min\{1, |p|\}^{2n+7} \Big) dp.$$

Since for $|p| \leq 1$,

$$1 + |p| + |p|^3 + |p|^{2n} - |p|^{4n+8} \leq 5,$$

and for $|p| > 1$, there exists $C > 0$ independent of p such that

$$1 + |p| + |p|^3 + |p|^{2n} - |p|^{2n+1} \leq C,$$

we find that the weight

$$1 + |p| + |p|^3 + |p|^{2n} - |p|^{2n+1} \min\{1, |p|\}^{2n+7}$$

of (2.116) is bounded uniformly in p by a universal positive constant C. As a consequence, Inequality (2.116) implies

$$\mathcal{M}_0 \leq C \int_{\mathbb{R}^3} |f(p) - g(p)| dp, \tag{2.117}$$

which concludes the proof of (2.101).

3. Proof of Theorem 1.2

Our proof is a extension and generalization of the framework proposed in [16]. The proof is divided into four parts:

Part 1: Fix a element v of $\mathcal{S}_T$, due to the Holder continuity property of Q, we have

$$\|Q(u)\| \leq \|Q(v)\| + C\|u - v\|^\beta, \qquad \forall u \in \mathcal{S}_T.$$

According to our assumption, $\mathcal{S}_T$ is bounded by a constant C_S. We deduce from the above inequality that

$$\|Q(u)\| \leq \|Q(v)\| + C \left(\|u\| + \|v\|\right)^\beta \leq \|Q(v)\| + C \left(C_S + \|v\|\right)^\beta =: C_Q, \ \forall u \in \mathcal{S}_T.$$

For an element u be in $\mathcal{S}_T$, there exists $\xi_u > 0$ such that for $0 < \xi < \xi_u$, $u + \xi Q(u) \in \mathcal{S}_T$, which implies

$$B(u + \xi Q(u), \delta) \cap \mathcal{S}_T \backslash \{u + \xi Q(u)\} \neq \text{‰},$$

for δ small enough. Choose $\epsilon = 2C((C_Q + 1)\xi)^\beta$, then $\|Q(u) - Q(v)\| \leq \frac{\epsilon}{2}$ if $\|u - v\| \leq (C_Q + 1)\xi$, by the Holder continuity of Q. Let $z \in B\left(u + \xi Q(u), \frac{\epsilon \xi}{2}\right) \cap \mathcal{S}_T \backslash \{u + \xi Q(u)\}$ and define

$$t \mapsto \vartheta(t) = u + \frac{t(z - u)}{\xi}, \qquad t \in [0, \xi].$$

Since $\mathcal{S}_T$ is convex, ϑ maps $[0, \xi]$ into $\mathcal{S}_T$. It is straightforward that

$$\|\vartheta(t) - u\| \leq \xi \|Q(u)\| + \frac{\epsilon \xi}{2} < (C_Q + 1)\xi,$$

which implies

$$\|Q(\vartheta(t)) - Q(u)\| \leq \frac{\epsilon}{2}, \quad \forall t \in [0, \xi].$$

The above inequality and the fact that

$$\|\dot{\vartheta}(t) - Q(u)\| \leq \frac{\epsilon}{2},$$

leads to

$$\|\dot{\vartheta}(t) - Q(\vartheta(t))\| \le \epsilon, \quad \forall t \in [0, \xi]. \tag{3.1}$$

Part 2: Let ϑ be a solution to (3.1) on $[0, \tau]$ constructed in Part 1. From Part 1, we have that

$$\|\vartheta(t)\|_* = |\vartheta(t)|_* = \left| u + \frac{t(z - u)}{\tau} \right|_* \le |u|_* + \left| \frac{t(z - u)}{\tau} \right|_* \le |u|_* + |u|_* \frac{tC_*}{2}$$
$$= \|\vartheta(0)\|_* \left(1 + \frac{tC_*}{2} \right).$$

We then obtain

$$\|\vartheta(t)\|_* \le (\|\vartheta(0)\|_* + 1)e^{C_* t} - 1 < (2R_* + 1)e^{C_* t}. \tag{3.2}$$

Using the procedure of Part 1, we assume that ϑ can be extended to the interval $[\tau, \tau + \tau']$.

The same arguments that lead to (3.2) imply

$$\|\vartheta(\tau + t)\|_* \le \left((\|\vartheta(\tau)\|_* + 1)e^{C_* t} - 1 \right), \quad t \in [0, \tau'].$$

Combining the above inequality with (3.2) yields

$$\begin{aligned}
\|\vartheta(\tau + t)\|_* &\le \left((\|\vartheta(0)\|_* + 1) e^{C_* \tau} - 1 + 1 \right) e^{C_* t} - 1 \\
&\le (\|\vartheta(0)\|_* + 1) e^{C_*(\tau + t)} - 1 \\
&< (2R_* + 1)e^{C_*(\tau + t)},
\end{aligned} \tag{3.3}$$

where the last inequality follows from the fact that $R_* \ge 1$.

Part 3: From Part 1, there exists a solution ϑ to the equation (3.1) on an interval $[0, h]$. Now, we have the following procedure.

- *Step 1:* Suppose that we can construct the solution ϑ of (3.1) on $[0, \tau]$ ($\tau < T$). Since $\vartheta(\tau) \in \mathcal{S}_T$, by the same process as in Part 1 and by (3.2) and (3.3), the solution ϑ could be extended to $[\tau, \tau + h_\tau]$ where $\tau + h_\tau \le T, h_\tau \le \tau$.
- *Step 2:* Suppose that we can construct the solution ϑ of (3.1) on a series of intervals $[0, \tau_1], [\tau_1, \tau_2], \cdots, [\tau_n, \tau_{n+1}], \cdots$. Observe that the increasing sequence $\{\tau_n\}$ is bounded by T, the sequence has a limit, defined by τ. Recall that $Q(\vartheta)$ is bounded by C_Q on $[\tau_n, \tau_{n+1}]$ for all $n \in \mathbb{N}$, then $\dot{\vartheta}$ is bounded by $\epsilon + C_Q$ on $[0, \tau)$. As a consequence $\vartheta(\tau)$ can be defined as

$$\vartheta(\tau) = \lim_{n \to \infty} \vartheta(\tau_n), \dot{\vartheta}(\tau) = \lim_{n \to \infty} \dot{\vartheta}(\tau_n),$$

which, together with the fact that $\mathcal{S}_T$ is closed, implies that ϑ is a solution of (3.1) on $[0, \tau]$.

By Step 2, if the solution ϑ can be defined on $[0, T_0)$, $T_0 < T$, it could be extended to $[0, T_0]$. Now, we suppose that $[0, T_0]$ is the maximal closed interval that ϑ could be defined, by Step 1, ϑ could be extended to a larger interval $[T_0, T_0 + T_h]$, which means that $T = T_0$ and ϑ is defined on the whole interval $[0, T]$.

Part 4: Finally, let us consider a sequence of solution $\{u^\epsilon\}$ to (3.1) on $[0, T]$. We will prove that this is a Cauchy sequence. Let $\{u^\epsilon\}$ and $\{v^\epsilon\}$ be two sequences of solutions to (3.1) on $[0, T]$. We note that u^ϵ and v^ϵ are affine functions on $[0, T]$. Moreover by the one-side Lipschitz condition

$$
\begin{aligned}
\frac{d}{dt}\|u^\epsilon(t) - v^\epsilon(t)\| &= \left[u^\epsilon(t) - v^\epsilon(t), \dot{u}^\epsilon(t) - \dot{v}^\epsilon(t)\right] \\
&\leq \left[u^\epsilon(t) - v^\epsilon(t), Q[u^\epsilon(t)] - Q[v^\epsilon(t)]\right] + 2\epsilon \\
&\leq C\|u^\epsilon(t) - v^\epsilon(t)\| + 2\epsilon,
\end{aligned}
$$

for a.e. $t \in [0, T]$, which leads to

$$
\|u^\epsilon(t) - v^\epsilon(t)\| \leq 2\epsilon \frac{e^{LT}}{L}.
$$

Let ϵ tend to 0, $u^\epsilon \to u$ uniformly on $[0, T]$. It is straightforward that u is a solution to (1.26).

Acknowledgments

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