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OVERLAPPING OPTIMIZED SCHWARZ METHODS FOR PARABOLIC EQUATIONS IN N-DIMENSIONS

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Abstract. We introduce in this paper a new tool to prove the convergence of the overlapping optimized Schwarz methods with multisubdomains. The technique is based on some estimates of the errors on the boundaries of the overlapping strips. Our guiding example is an n-dimensional linear parabolic equation.

1. Introduction

In the pioneer work [11], [12], [13], P. L. Lions laid the foundations for the continuous approach of Schwarz algorithms. With the development of parallel computers, the interest in Schwarz methods have grown rapidly, as these methods lead to inherently parallel algorithms. However, with classical Schwarz methods, high frequency components converge very fast, while low frequency components converge slowly and that slows down the performance of the methods. By replacing Dirichlet transmission condition in classical Schwarz methods by Robin or higher order transmission conditions, we can correct this weakness of Schwarz method. The new methods are called optimized Schwarz methods and have been introduced in [5], [6], [10]. Since then, the convergence properties of the optimized Schwarz methods have been studied thoroughly, based on the following two main tools: energy estimates and Laplace and Fourier transforms. Energy estimates allow us to study the convergence of the methods in the nonoverlapping case. With energy estimates, both linear and nonlinear problems have been studied and optimized Schwarz methods have been proven to converge, while applying to these equations (see for example, the papers [1], [8], [9]). On the other hand, Laplace and Fourier transforms allow us to study the convergence of the overlapping optimized Schwarz methods, but for only a few simple equations, where all coefficients are constants and the second order operators are usually Laplace ones (see, for example [2], [4], [5], [6]), and the convergence problem of the overlapping domain decomposition methods with Robin transmission conditions still remains an open problem up to now.

In this paper, we introduce a new tool to prove the convergence of the optimized Schwarz methods for multisubdomains and apply it to an n-dimensional linear parabolic equation. The idea of the technique is to estimate carefully the difference between the values of the errors at the boundaries of the overlapping strips. The

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technique has the potential to be applied to many other kinds of partial differential equations including nonlinear ones, with classical solutions of the equation. The variational setting will be consider in a forth coming paper (see [14]). Our long term goal is to construct some new tools to study the convergence problem of Schwarz methods and this technique takes us a step closer to it.

2. Problem Description and Main Results

We consider the following parabolic equation

(2.1)
$$\begin{cases} \frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{i,j}(t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t)\frac{\partial u}{\partial x_i} + c(t)u = f(t,x), \text{ in } (0,T)\times\Omega, \\ u(x,t) = g(x,t), \text{ on } \partial\Omega\times(0,T), \\ u(x,0) = g(x,0), \text{ on } \Omega, \end{cases}$$

where $\Omega = D\times(\alpha,\beta)$, D is a bounded and smooth domain in $\mathbb{R}^{n-1}$. We impose the following conditions on the coefficients of (2.1).
(A1) For all i, j in $\{1,\dots,n\}$, $a_{i,j}(t) = a_{j,i}(t)$. There exists $\nu_0 > 0$ such that $A(t) = (a_{i,j}(t)) \geq \nu_0 I$ for all t in $(0,T)$ in the sense of symmetric positive definite matrices.
(A2) The functions $a_{i,j}$, b_i, c are bounded in $C^\infty(\mathbb{R})$; f and g are bounded functions in $C^\infty(\overline{\Omega\times(0,T)})$.
With the conditions $(A1)$ and $(A2)$, Equation (2.1) has a unique bounded solution u in $C^\infty((0,T)\times\Omega)$. The proof of this result can be infered from Theorems 9 and 10, page 71 [3].

We now divide the domain Ω into M subdomains, with $\Omega_i = D\times(a_i,b_i)$ and $\alpha = a_1 < a_2 < b_1 < \dots < a_M < b_{M-1} < b_M = \beta$. The optimized Schwarz algorithm solves M equations in M subdomains instead of solving directly the main problem (2.1). The iterate #k in the l-th domain, denoted by u_l^k, is defined by

(2.2)
$$\begin{cases} \frac{\partial u_l^k}{\partial t} - \sum_{i,j=1}^n a_{i,j}(t)\frac{\partial^2 u_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t)\frac{\partial u_l^k}{\partial x_i} + c(t)u_l^k = f(t,x) & \text{in } \Omega_l\times(0,T), \\ \frac{\partial u_l^k(\cdot,a_l,\cdot)}{\partial x_n} + pu_l^k(\cdot,a_l,\cdot) = \frac{\partial u_{l-1}^{k-1}(\cdot,a_l,\cdot)}{\partial x_n} + pu_{l-1}^{k-1}(\cdot,a_l,\cdot) & \text{in } D\times(0,T), \\ \frac{\partial u_l^k(\cdot,b_l,\cdot)}{\partial x_n} + pu_l^k(\cdot,b_l,\cdot) = \frac{\partial u_{l+1}^{k-1}(\cdot,b_l,\cdot)}{\partial x_n} + pu_{l+1}^{k-1}(\cdot,b_l,\cdot) & \text{in } D\times(0,T), \end{cases}$$

here, p is a constant and for each vector x in $\mathbb{R}^n$, we denote $x = (X, x_n)$, with $X\in\mathbb{R}^{n-1}$ and $x_n\in\mathbb{R}$. Each iterate inherits the boundary conditions and the initial values of u:

$$u_l^k(x,t) = g(x,t) \text{ on } (\partial\Omega_j\cap\partial\Omega)\times(0,T), \quad u_l^k(x,0) = g(x,0) \text{ in } \Omega_j,$$

and a special treatment for the extreme subdomains,

$$u_1^k(\cdot,\alpha,\cdot) = g(\cdot,\alpha,\cdot), \quad u_M^k(\cdot,\beta,\cdot) = g(\cdot,\beta,\cdot).$$

A bounded initial guess h^0 in $C^\infty(\overline{\Omega \times (0,T)})$ is provided, *i.e.* we solve at the first iteration Equations (2.2), with boundary data on left and right

$$\frac{\partial u_l^1(\cdot, a_l, \cdot)}{\partial x_n} + p u_l^1(\cdot, a_l, \cdot) = h^0(\cdot, a_l, \cdot) \text{ in } D \times (0,T),$$
$$\frac{\partial u_l^1(\cdot, b_l, \cdot)}{\partial x_n} + p u_l^1(\cdot, b_l, \cdot) = h^0(\cdot, b_l, \cdot) \text{ in } D \times (0,T).$$

By using an induction argument and the same arguments as in Theorem 2, page 144 [3], we can see that each subproblem (2.2) in each iteration has a unique solution. Theorem 10, page 71 [3] shows that these solutions belong to $C^\infty(\Omega \times (0,T))$. This means that the algorithm is well-posed.

Denote by e_l^k the difference between u_l^k and u, and substract Equation (2.2) from the main equation (2.1), we get the following equation on e_l^k

(2.3)
$$\begin{cases} \frac{\partial e_l^k}{\partial t} - \sum_{i,j=1}^n a_{i,j}(t) \frac{\partial^2 e_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(t) \frac{\partial e_l^k}{\partial x_i} + c(t) e_l^k = 0 & \text{in } \Omega_l \times (0,T), \\ \frac{\partial e_l^k(\cdot, a_l, \cdot)}{\partial x_n} + p e_l^k(\cdot, a_l, \cdot) = \frac{\partial e_{l-1}^{k-1}(\cdot, a_l, \cdot)}{\partial x_n} + p e_{l-1}^{k-1}(\cdot, a_l, \cdot) & \text{in } D \times (0,T), \\ \frac{\partial e_l^k(\cdot, b_l, \cdot)}{\partial x_n} + p e_l^k(\cdot, b_l, \cdot) = \frac{\partial e_{l+1}^{k-1}(\cdot, b_l, \cdot)}{\partial x_n} + p e_{l+1}^{k-1}(\cdot, b_l, \cdot) & \text{in } D \times (0,T). \end{cases}$$

Similarly, each iterate inherits the boundary conditions and the initial values of u

$$e_l^k = 0 \text{ on } (\partial\Omega_l \cap \partial\Omega) \times (0,T), \quad e_l^k(\cdot, \cdot, 0) = 0 \text{ in } \Omega_l,$$

and the special treatment for the extreme subdomains,

$$e_1^k(\cdot, \alpha, \cdot) = 0, \quad e_M^k(\cdot, \beta, \cdot) = 0.$$

The following theorem states that the algorithm converges.

Theorem 2.1. *Let φ be a strictly positive function in $C^1(\mathbb{R})$ such that $-\max_{x_n \in \mathbb{R}} \left(\frac{\varphi'}{\varphi}(x_n)\right)$ is large enough, the optimized Schwarz method converges in the following sense*

$$\lim_{k \to \infty} \max_{l \in \{1,\ldots,M\}} \left\| \left(\frac{\partial(u_l^k - u)}{\partial x_n} \exp(p x_n) \right)^2 \varphi(t) \right\|_{C(\overline{\Omega_l \times (0,T)})} = 0.$$

Moreover, for l in $\{1, \ldots, M\}$, the sequence $\{u_l^k\}$ converges point-wisely to u as k tends to infinity.

Remark 2.2. We can see that if we choose $\varphi(x_n) = \exp(-\gamma x_n)$, then if γ is large enough, $-\max_{x_n \in \mathbb{R}} \left(\frac{\varphi'}{\varphi}(x_n)\right)$ is large enough. The condition of our theorem is then satisfied.

Remark 2.3. Since $a_{i,j}$, b_i are functions of t, and the domain is divided into n-subdomains, we cannot use Fourier and Laplace transforms. Moreover, since the subdomains are overlapping, the energy estimate method cannot be used in our case. In the next section, we introduce a new technique to prove the convergence of the algorithm, the technique is based on the observation that we can estimate the difference between the values of e_l^k on the boundary and in the interior.

Remark 2.4. The result in the theorem remains true if we let $a_{i,j}$, b_i be bounded and continuous functions of t and x, but not depend on the n-th space variable x_n, as we can see in the proof in the following section.

Remark 2.5. The idea of the proof is to transform the equations in subdomains with Robin boundary conditions into equations with Dirichlet boundary conditions and then to apply the maximum principle to get some boundary estimates for the errors. However, the algorithm with Robin transmission conditions is not equivalent to the classical algorithm with Dirichlet transmission conditions, since the transformed equation is an equation on $\frac{\partial \epsilon_l^k}{\partial x_n}$ and the form of the operator of the equation is completely changed.

3. The Convergence of the Algorithm

This section is devoted to the proof of Theorem 2.1. We divide the proof into two steps.

Step 1: The error estimates.

For k in $\mathbb{N}$ and i in $\{1,\dots,M\}$, setting ϵ_l^k to be $e_l^k\exp(px_n)$, we get $e_l^k = \epsilon_l^k\exp(-px_n)$. Equation (2.3) then leads to

$$(3.1)\quad \begin{cases} \frac{\partial \epsilon_l^k}{\partial t} - \sum_{i,j=1}^n a_{i,j}\frac{\partial^2 \epsilon_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1}(pa_{i,n}+b_i)\frac{\partial \epsilon_l^k}{\partial x_i} + (2pa_{n,n}+b_n)\frac{\partial \epsilon_l^k}{\partial x_n} \\ +(c-pb_n-p^2a_{n,n})\epsilon_l^k = 0, \text{ in } \Omega_l\times(0,T), \\ \frac{\partial \epsilon_l^k(\cdot,a_l,\cdot)}{\partial x_n} = \frac{\partial \epsilon_{l-1}^{k-1}(\cdot,a_l,\cdot)}{\partial x_n} \text{ in } D\times(0,T), \\ \frac{\partial \epsilon_l^k(\cdot,b_l,\cdot)}{\partial x_n} = \frac{\partial \epsilon_{l+1}^{k-1}(\cdot,b_l,\cdot)}{\partial x_n} \text{ in } D\times(0,T), \\ \epsilon_l^k(\cdot,\cdot,\cdot) = 0 \text{ on } (\partial\Omega_j\cap\partial\Omega)\times(0,T), \\ \epsilon_l^k(\cdot,\cdot,0) = 0 \text{ in } \Omega_j, \end{cases}$$

and for the extreme subdomains,

$$\epsilon_1^k(\cdot,a_1,\cdot) = 0, \quad \epsilon_M^k(\cdot,b_M,\cdot) = 0.$$

Setting $\nu_l^k = \frac{\partial \epsilon_l^k}{\partial x_n}$, we infer from Equation (3.1) that for l in $\{2,\dots,M-1\}$

$$(3.2)\quad \begin{cases} \frac{\partial \nu_l^k}{\partial t} - \sum_{i,j=1}^n a_{i,j}\frac{\partial^2 \nu_l^k}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1}(pa_{i,n}+b_i)\frac{\partial \nu_l^k}{\partial x_i} + (2pa_{n,n}+b_n)\frac{\partial \nu_l^k}{\partial x_n} \\ +(c-pb_n-p^2a_{n,n})\nu_l^k = 0, \text{ in } \Omega_l\times(0,T), \\ \nu_l^k(\cdot,a_l,\cdot) = \nu_{l-1}^{k-1}(\cdot,a_l,\cdot) \text{ in } D\times(0,T), \\ \nu_l^k(\cdot,b_l,\cdot) = \nu_{l+1}^{k-1}(\cdot,b_l,\cdot) \text{ in } D\times(0,T), \\ \nu_l^k(\cdot,\cdot,\cdot) = 0 \text{ on } (\partial\Omega_j\cap\partial\Omega)\times(0,T), \\ \nu_l^k(\cdot,\cdot,0) = 0 \text{ in } \Omega_j. \end{cases}$$

On $\overline{\Omega_l\times(0,T)}$, we define $\Phi = \left(\nu_l^k\right)^2\phi(x_n)\varphi(t)$, where ϕ is a strictly positive function in $C^2(\mathbb{R})$ to be chosen later, with the notice that $-\max_{x_n\in\mathbb{R}}\left(\frac{\varphi'}{\varphi}(x_n)\right)$ is large enough. Our purpose is to construct an operator $\mathfrak{L}$ of Φ, such that $\mathfrak{L}(\Phi)$ is negative and then on $\mathfrak{L}$, we can apply the maximum principle to get some estimates on the boundaries for Φ. We now consider the following operator

$$\mathfrak{L}_0(\Phi) := \frac{\partial \Phi}{\partial t} - \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}. \tag{3.3}$$

A simple calculation gives

$$\begin{aligned} \mathfrak{L}_0(\Phi) &= 2\nu_l^k \phi\varphi \left(\frac{\partial \nu_l^k}{\partial t} - \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 \nu_l^k}{\partial x_i x_j} \right) - \sum_{i,j=1}^{n} 2a_{i,j}\phi\varphi \frac{\partial \nu_l^k}{\partial x_i}\frac{\partial \nu_l^k}{\partial x_j} \\ & \quad - \sum_{i=1}^{n} 2a_{i,n}\phi'\varphi\nu_l^k \frac{\partial \nu_l^k}{\partial x_i} + \left(\frac{\varphi'}{\varphi} - a_{n,n}\frac{\phi''}{\phi} \right) \phi\varphi(\nu_l^k)^2. \end{aligned} \tag{3.4}$$

We observe that the second term on the right hand side of the previous inequality is negative, it directly leads to

$$\begin{aligned} \mathfrak{L}_0(\Phi) &\leq 2\nu_l^k \phi\varphi \left(\frac{\partial \nu_l^k}{\partial t} - \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2 \nu_l^k}{\partial x_i x_j} \right) \\ & \quad - \sum_{i=1}^{n} 2a_{i,n}\phi'\varphi\nu_l^k \frac{\partial \nu_l^k}{\partial x_i} + \left(\frac{\varphi'}{\varphi} - a_{n,n}\frac{\phi''}{\phi} \right) \phi\varphi(\nu_l^k)^2. \end{aligned} \tag{3.5}$$

We now replace (3.2) into (3.5) and get the following bound for $\mathfrak{L}_0(\Phi)$

$$\begin{aligned} & 2\nu_l^k\phi\varphi \left(- \sum_{i=1}^{n-1} (pa_{i,n} + b_i)\frac{\partial \nu_l^k}{\partial x_i} - (2pa_{n,n} + b_n)\frac{\partial \nu_l^k}{\partial x_n} - (c - pb_n - p^2 a_{n,n})\nu_l^k \right) \\ & - \sum_{i=1}^{n} 2a_{i,n}\phi'\varphi\nu_l^k \frac{\partial \nu_l^k}{\partial x_i} + \left(\frac{\varphi'}{\varphi} - a_{n,n}\frac{\phi''}{\phi} \right) \phi\varphi(\nu_l^k)^2. \end{aligned} \tag{3.6}$$

Substituting

$$\frac{\partial \Phi}{\partial x_i} = 2\phi\varphi\nu_l^k \frac{\partial \nu_l^k}{\partial x_i}, \text{ for } i \in \{1, \dots, n-1\},$$

and

$$\frac{\partial \Phi}{\partial x_n} = 2\phi\varphi\nu_l^k \frac{\partial \nu_l^k}{\partial x_n} + \phi'\varphi(\nu_l^k)^2,$$

into (3.6), we get

$$\begin{aligned} & \frac{\partial \Phi}{\partial t} - \sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 \Phi}{\partial x_i \partial x_j} + \left(2pa_{n,n} + b_n + a_{n,n}\frac{\phi'}{\phi} \right) \frac{\partial \Phi}{\partial x_n} \\ & + \sum_{i=1}^{n-1} \left(pa_{i,n} + b_i + a_{in}\frac{\phi'}{\phi} \right) \frac{\partial \Phi}{\partial x_i} \\ \leq \; & \phi\varphi(\nu_l^k)^2 \left(\frac{\varphi'}{\varphi} - a_{nn}\frac{\phi''}{\phi} + a_{n,n}\frac{\phi'^2}{\phi^2} - 2(c - pb_n - p^2 a_{n,n}) + (2pa_{n,n} + b_n)\frac{\phi'}{\phi} \right). \end{aligned} \tag{3.7}$$

We now get the formula for $\mathfrak{L}$

$$\begin{aligned}(3.8)\qquad \mathfrak{L}(\Phi) &:= \frac{\partial\Phi}{\partial t}-\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2\Phi}{\partial x_i\partial x_j}+(2pa_{n,n}+b_n+a_{nn}\frac{\phi'}{\phi})\frac{\partial\Phi}{\partial x_n}\\ &+\sum_{i=1}^{n-1}\left(pa_{i,n}+b_i+a_{in}\frac{\phi'}{\phi}\right)\frac{\partial\Phi}{\partial x_i},\end{aligned}$$

then if we choose φ such that $-\max_{x_n\in\mathbb{R}}\left(\frac{\varphi'}{\varphi}(x_n)\right)$ is large enough, since $a_{n,n}$, b_n, c, $\frac{\phi'}{\phi}$, $\frac{\phi''}{\phi}$ are all bounded in $C(\mathbb{R})$, we can obtain a negative sign on the right hand side of (3.7), which means $\mathfrak{L}(\Phi)$ is negative.

Since $\mathfrak{L}(\Phi)\leq 0$, the maximum of Φ can only be attained on the boundary of $\Omega_l\times(0,T)$. Using the fact that $\Phi=0$ on $\partial\Omega_l\cap\partial\Omega$ and on $\Omega\times\{0\}$, we have the following three estimates.

Estimate 1: $1\leq l\leq M$.

The maximum value of Φ can be achieved on both $\overline{D}\times\{a_l\}\times[0,T]$ and $\overline{D}\times\{b_l\}\times[0,T]$

$$\begin{aligned}(3.9)\qquad &(\nu_l^k(X,x_n,t))^2\phi(x_n)\varphi(t)\leq\\ \leq\quad &\max\{\max_{\bar{D}\times[0,T]}\{(\nu_l^k(X,a_l,t))^2\phi(a_l)\varphi(t)\},\max_{\bar{D}\times[0,T]}\{(\nu_l^k(X,b_l,t))^2\phi(b_l)\varphi(t)\}\}.\end{aligned}$$

Estimate 2: $l=1$.

The maximum value of Φ can be achieved on both $\overline{D}\times\{a_1\}\times[0,T]$ and $\overline{D}\times\{b_1\}\times[0,T]$. If the maximum of Φ is achieved on $\overline{D}\times\{a_1\}\times[0,T]$, then at the maximum point, we need that $\frac{\partial\Phi}{\partial n}>0$ due to Hopf's Lemma. We compute

$$\begin{aligned}\frac{\partial\Phi}{\partial n}(.,a_1,t) &= -\frac{\partial\nu_1^k}{\partial x_n}\nu_1^k\phi(a_1)\varphi(t)-(\nu_1^k)^2\phi'(a_1)\varphi(t)\\ &= -\phi(a_1)\varphi(t)\left[\frac{\partial^2\epsilon_1^k}{\partial x_n^2}\frac{\partial\epsilon_1^k}{\partial x_n}+\left(\frac{\partial\epsilon_1^k}{\partial x_n}\right)^2\frac{\phi'(a_1)}{\phi(a_1)}\right].\end{aligned}$$

Since

$$\frac{\partial\epsilon_1^k}{\partial t}(.,a_1,.)-\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2\epsilon_1^k}{\partial x_i\partial x_j}(.,a_1,.)+\sum_{i=1}^{n-1}(pa_{i,n}+b_i)\frac{\partial\epsilon_1^k}{\partial x_i}(.,a_1,.)$$

$$+(2pa_{n,n}+b_n)\frac{\partial\epsilon_1^k}{\partial x_n}(.,a_1,.)+(c-pb_n-p^2a_{n,n})\epsilon_1^k(.,a_1,.)=0,\ \text{in } D\times(0,T).$$

and $\epsilon_l^k(.,a_1,.)=0$ on $D\times(0,T)$, we deduce

$$-\frac{\partial^2\epsilon_1^k}{\partial x_n^2}(.,a_1,.)+\left(2p+\frac{b_n}{a_{n,n}}\right)\frac{\partial\epsilon_1^k}{\partial x_n}(.,a_1,.)=0,$$

and as a consequence, we can write $\frac{\partial\Phi}{\partial n}$ in a different way

$$\frac{\partial\Phi}{\partial n}(.,a_1,.)=-\phi(a_1)\varphi(t)\left(\frac{\partial\epsilon_1^k}{\partial x_n}\right)^2\left[\left(2p+\frac{b_n}{a_{n,n}}\right)+\frac{\phi'(a_1)}{\phi(a_1)}\right].$$

Choosing ϕ such that

$$\left(2p+\frac{b_n(t)}{a_{n,n}(t)}\right)+\frac{\phi'(a_1)}{\phi(a_1)}>0,$$

we can see that

$$\frac{\partial \Phi}{\partial n}(.,a_1,.) < 0;$$

which means that the maximum of Φ can be achieved only on $\overline{D} \times \{b_1\} \times [0,T]$, then

$$(\nu_1^k(X,x_n,t))^2\phi(x_n)\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_1^k(X,b_1,t))^2\phi(b_1)\varphi(t)\}. \tag{3.10}$$

Estimate 3: $l = M$.

The maximum value(s) of Φ can be achieved on both $\overline{D} \times \{a_M\} \times [0,T]$ and $\overline{D} \times \{b_M\} \times [0,T]$. If the maximum of Φ is achieved on $\overline{D} \times \{b_M\} \times [0,T]$, then at the maximum point, we need that $\frac{\partial \Phi}{\partial n} > 0$ due to Hopf's Lemma. Similar as in Estimate 2, we can get

$$\begin{aligned}\frac{\partial \Phi}{\partial n}(.,b_M,t) &= \frac{\partial \nu_1^k}{\partial x_n}\nu_1^k\phi(b_M)\varphi(t) + (\nu_1^k)^2\phi'(b_M)\varphi(t)\\ &= \phi(b_M)\varphi(t)\left(\frac{\partial \epsilon_1^k}{\partial x_n}\right)^2\left[\left(2p + \frac{b_n}{a_{n,n}}\right) + \frac{\phi'(b_M)}{\phi(b_M)}\right].\end{aligned}$$

With the function ϕ satisfying

$$\left(2p + \frac{b_n(t)}{a_{n,n}(t)}\right) + \frac{\phi'(b_M)}{\phi(b_M)} < 0,$$

we can see that

$$\frac{\partial \Phi}{\partial n}(.,b_M,.) < 0;$$

which means the maximum of Φ can be achieved only on $\overline{D} \times \{a_M\} \times [0,T]$, then

$$(\nu_M^k(X,x_n,t))^2\phi(x_n)\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_M^k(X,a_M,t))^2\phi(a_M)\varphi(t)\}. \tag{3.11}$$

Step 2: Proof of convergence,

$$\lim_{k\to\infty}\max_{l\in\{1,\dots,M\}}\left\|\left(\nu_l^k\right)^2\varphi(t)\right\|_{C(\overline{\Omega_l\times(0,T)})} = 0.$$

In the proof of convergence, we will use the three estimates (3.9), (3.10) and (3.11) by fixing φ and replacing ϕ by appropriate functions $\overline{\phi}_i$, $\tilde{\phi}_i$, $\overline{\phi}_*$, $\tilde{\phi}_*$ ($i \in \{1,\dots,M\}$) in each subdomain. We define

$$E_k = \max_{l\in\{1,\dots,M\}}||(\nu_l^k)^2\phi\varphi||_{C(\overline{\Omega_l\times(0,T)})}.$$

Step 2.1: Estimate of the right boundaries of the sub-domains.

Consider the M-th domain, at the k-th step, (3.11) implies

$$(\nu_M^k(X,x_n,t))^2\overline{\phi}_M(x_n)\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_M^k(X,a_M,t))^2\overline{\phi}_M(a_M)\varphi(t)\},$$

where $\overline{\phi}_M$ is a strictly positive function and will be chosen later.
Replacing x_n by b_{M-1}, we get

$$(\nu_M^k(X,b_{M-1},t))^2\overline{\phi}_M(b_{M-1})\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_M^k(X,a_M,t))^2\overline{\phi}_M(a_M)\varphi(t)\}.$$

Since $\nu_M^k(X,b_{M-1},t)=\nu_{M-1}^{k+1}(X,b_{M-1},t)$,

$$(\nu_{M-1}^{k+1}(X,b_{M-1},t))^2\overline{\phi}_M(b_{M-1})\varphi(t)\leq \max_{\bar{D}\times[0,T]}\{(\nu_M^k(X,a_M,t))^2\overline{\phi}_M(a_M)\varphi(t)\}.$$

The inequality becomes

$$(\nu_{M-1}^{k+1}(X,b_{M-1},t))^2\varphi(t)\leq \frac{\overline{\phi}_M(a_M)}{\overline{\phi}_M(b_{M-1})}\max_{\bar{D}\times[0,T]}\{(\nu_M^k(X,a_M,t))^2\varphi(t)\}.$$

We can choose $\overline{\phi}_M$ such that $\frac{\overline{\phi}_M(a_M)}{\overline{\phi}_M(b_{M-1})}<1$, and deduce

$$(\nu_{M-1}^{k+1}(X,b_{M-1},t))^2\varphi(t)\leq \frac{\overline{\phi}_M(a_M)}{\overline{\phi}_M(b_{M-1})}E_k. \tag{3.12}$$

Moreover, on the $(M-1)$-th domain, at the $(k+1)$-th step, (3.9) leads to

$$\begin{aligned}&(\nu_{M-1}^{k+1}(X,x_n,t))^2\overline{\phi}_{M-1}(x_n)\varphi(t)\leq\\ &\max\{\max_{\bar{D}\times[0,T]}\{(\nu_{M-1}^{k+1}(X,b_{M-1},t))^2\overline{\phi}_{M-1}(b_{M-1})\varphi(t)\},\\ &\max_{\bar{D}\times[0,T]}\{(\nu_{M-1}^{k+1}(X,a_{M-1},t))^2\overline{\phi}_{M-1}(a_{M-1})\varphi(t)\}\},\end{aligned}$$

where $\overline{\phi}_{M-1}$ is a strictly positive function that will be chosen later.
Since $\nu_{M-1}^{k+1}(X,b_{M-2},t)=\nu_{M-2}^{k+2}(X,b_{M-2},t)$,

$$\begin{aligned}&(\nu_{M-2}^{k+2}(X,b_{M-2},t))^2\overline{\phi}_{M-1}(b_{M-2})\varphi(t)\leq\\ &\max\{\max_{\bar{D}\times[0,T]}\{(\nu_{M-1}^{k+1}(X,b_{M-1},t))^2\overline{\phi}_{M-1}(b_{M-1})\varphi(t)\},\\ &\max_{\bar{D}\times[0,T]}\{(\nu_{M-1}^{k+1}(X,a_{M-1},t))^2\overline{\phi}_{M-1}(a_{M-1})\varphi(t)\}\}.\end{aligned}$$

Hence

$$\begin{aligned}&(\nu_{M-2}^{k+2}(X,b_{M-2},t))^2\varphi(t)\leq\\ &\max\left\{\frac{\overline{\phi}_{M-1}(b_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}\max_{\bar{D}\times[0,T]}\{(\nu_{M-1}^{k+1}(X,b_{M-1},t))^2\varphi(t)\},\right.\\ &\left.\max_{\bar{D}\times[0,T]}\left(\frac{\overline{\phi}_{M-1}(a_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}(\nu_{M-1}^{k+1}(X,a_{M-1},t))^2\varphi(t)\right)\right\}.\end{aligned}$$

Combining this inequality with (3.12), we get

$$(\nu_{M-2}^{k+2}(X,b_{M-2},t))^2\varphi(t)\leq\max\left\{\frac{\overline{\phi}_{M-1}(b_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}\frac{\overline{\phi}_M(a_M)}{\overline{\phi}_M(b_{M-1})}E_k,\frac{\overline{\phi}_{M-1}(a_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}E_{k+1}\right\}.$$

Choosing $\overline{\phi}_{M-1}$ such that

$$\frac{\overline{\phi}_{M-1}(b_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}\frac{\overline{\phi}_M(a_M)}{\overline{\phi}_M(b_{M-1})}=\frac{\overline{\phi}_{M-1}(a_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}<1,$$

we get

$$(\nu_{M-2}^{k+2}(X,b_{M-2},t))^2\varphi(t)\leq\frac{\overline{\phi}_{M-1}(a_{M-1})}{\overline{\phi}_{M-1}(b_{M-2})}\max\{E_k,E_{k+1}\}. \tag{3.13}$$

Using the same techniques as the ones we use to achieve (3.12) and (3.13), we can prove that

$$(\nu_{M-j}^{k+j}(X, b_{M-j}, t))^2\varphi(t) \le \frac{\overline{\phi}_{M-j+1}(a_{M-j+1})}{\overline{\phi}_{M-j+1}(b_{M-j})} \max\{E_k, \dots, E_{k+j-1}\}, \tag{3.14}$$

where $\overline{\phi}_{M-j+1}$ is a strictly positive function satisfying

$$\frac{\overline{\phi}_{M-j+1}(a_{M-j+1})}{\overline{\phi}_{M-j+1}(b_{M-j})} < 1,$$

with $j \in \{1, \dots, M-1\}$.

Now, with (3.10), we can choose a strictly positive function $\overline{\phi}_*$ such that $\overline{\phi}_*(b_M) > \overline{\phi}_*(a_M)$, then

$$(\nu_M^k(X, b_M, t))^2\overline{\phi}_*(b_M)\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_M^k(X, a_M, t))^2\overline{\phi}_*(a_M)\varphi(t)\},$$

and as a result

$$(\nu_M^k(X, b_M, t))^2\overline{\phi}_*(b_M)\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_{M-1}^{k-1}(X, a_M, t))^2\overline{\phi}_*(a_M)\varphi(t)\},$$

which implies

$$(\nu_M^k(X, b_M, t))^2\varphi(t) \le \frac{\overline{\phi}_*(a_M)}{\overline{\phi}_*(b_M)} E_{k-1}. \tag{3.15}$$

Step 2.2: Estimate of the left boundaries of the sub-domains.

Consider the first domain, at the k-th step, (3.11) implies

$$(\nu_1^k(X, x_n, t))^2\tilde{\phi}_1(x_n)\varphi(t) \le \max_{\bar{D}\times[0,T]}\{(\nu_1^k(X, b_1, t))^2\tilde{\phi}_1(b_1)\varphi(t)\},$$

where $\tilde{\phi}_1$ is a strictly positive function that will be chosen later.
Replacing x_n by a_2, we get

$$(\nu_1^k(X, a_2, t))^2\varphi(t) \le \frac{\tilde{\phi}_1(b_1)}{\tilde{\phi}_1(a_2)} \max_{\bar{D}\times[0,T]}\{(\nu_1^k(X, b_1, t))^2\varphi(t)\}.$$

Since $\nu_1^k(X, a_2, t) = \nu_2^{k+1}(X, a_2, t)$, then

$$(\nu_2^{k+1}(X, a_2, t))^2\varphi(t) \le \frac{\tilde{\phi}_1(b_1)}{\tilde{\phi}_1(a_2)} \max_{\bar{D}\times[0,T]}\{(\nu_1^k(X, b_1, t))^2\varphi(t)\}.$$

We choose $\tilde{\phi}_1$ such that

$$\frac{\tilde{\phi}_1(b_1)}{\tilde{\phi}_1(a_2)} < 1,$$

and deduce

$$(\nu_2^{k+1}(X, a_2, t))^2\varphi(t) \le \frac{\tilde{\phi}_1(b_1)}{\tilde{\phi}_1(a_2)} E_k. \tag{3.16}$$

Second, on the second domain, at the $(k+1)$-th step, (3.11) leads to

$$\begin{aligned} &(\nu_2^{k+1}(X, x_n, t))^2\tilde{\phi}_2(x_n)\varphi(t) \le \\ &\max\{\max_{\bar{D}\times[0,T]}\{(\nu_2^{k+1}(X, b_2, t))^2\tilde{\phi}_2(b_2)\varphi(t)\}, \\ &\max_{\bar{D}\times[0,T]}\{(\nu_2^{k+1}(X, a_2, t))^2\tilde{\phi}_2(a_2)\varphi(t)\}\}, \end{aligned}$$

where $\tilde{\phi}_2$ is a strictly positive function and will be chosen later.
Since $\nu_2^{k+1}(X, a_3, t) = \nu_3^{k+2}(X, a_3, t)$, then

$$\begin{aligned}&(\nu_3^{k+2}(X, a_3, t))^2\tilde{\phi}_2(a_3)\varphi(t) \leq \\ &\max\{\max_{\bar{D}\times[0,T]}\{(\nu_2^{k+1}(X, b_2, t))^2\tilde{\phi}_2(b_2)\varphi(t)\}, \\ &\max_{\bar{D}\times[0,T]}[(\nu_2^{k+1}(X, a_2, t))^2\tilde{\phi}_2(a_2)\varphi(t)]\}.\end{aligned}$$

Hence

$$\begin{aligned}&(\nu_3^{k+2}(X, a_3, t))^2\varphi(t) \leq \\ &\max\left\{\frac{\tilde{\phi}_2(b_2)}{\tilde{\phi}_2(a_3)}\max_{\bar{D}\times[0,T]}\{(\nu_2^{k+1}(X, b_2, t))^2\varphi(t)\}\right., \\ &\left.\frac{\tilde{\phi}_2(a_2)}{\tilde{\phi}_2(a_3)}\max_{\bar{D}\times[0,T]}[(\nu_2^{k+1}(X, a_2, t))^2\varphi(t)]\right\}.\end{aligned}$$

Combining this with (3.16), we get

$$(\nu_3^{k+2}(X, a_3, t))^2\varphi(t) \leq \max\left\{\frac{\tilde{\phi}_2(b_2)}{\tilde{\phi}_2(a_3)}E_{k+1}, \frac{\tilde{\phi}_2(a_2)}{\tilde{\phi}_2(a_3)}\frac{\tilde{\phi}_1(b_1)}{\tilde{\phi}_1(a_2)}E_k\right\}.$$

Choosing $\tilde{\phi}_2$ such that

$$\frac{\tilde{\phi}_2(b_2)}{\tilde{\phi}_2(a_3)} = \frac{\tilde{\phi}_2(a_2)}{\tilde{\phi}_2(a_3)}\frac{\tilde{\phi}_1(b_1)}{\tilde{\phi}_1(a_2)} < 1,$$

we then obtain

$$(\nu_3^{k+2}(X, a_3, t))^2\varphi(t) \leq \frac{\tilde{\phi}_2(b_2)}{\tilde{\phi}_2(a_3)}\max\{E_k, E_{k+1}\}. \tag{3.17}$$

Using the same techniques as the ones that we use to derive (3.16) and (3.17), we can prove that

$$(\nu_j^{k+j-1}(X, a_j, t))^2\varphi(t) \leq \frac{\tilde{\phi}_{j-1}(b_{j-1})}{\tilde{\phi}_{j-1}(a_j)}\max\{E_k, \dots, E_{k+j-2}\}, \tag{3.18}$$

where $\tilde{\phi}_{j-1}$ is a strictly positive function satisfying

$$\frac{\tilde{\phi}_{j-1}(b_{j-1})}{\tilde{\phi}_{j-1}(a_j)} < 1,$$

with $j \in \{1, \dots, M-1\}$.

Now, with (3.11), we can choose a strictly positive function $\tilde{\phi}_*$ such that $\tilde{\phi}_*(b_1) < \tilde{\phi}_*(a_1)$, and get

$$(\nu_1^k(X, a_1, t))^2\tilde{\phi}_*(a_1)\varphi(t) \leq \max_{\bar{D}\times[0,T]}\{(\nu_1^k(X, b_1, t))^2\tilde{\phi}_*(b_1)\varphi(t)\},$$

which is equivalent to

$$(\nu_1^k(X, a_1, t))^2\tilde{\phi}_*(a_1)\varphi(t) \leq \max_{\bar{D}\times[0,T]}\{(\nu_2^{k-1}(X, b_1, t))^2\tilde{\phi}_*(b_1)\varphi(t)\}.$$

That implies

$$(\nu_M^k(X,a_1,t))^2\varphi(t) \le \frac{\tilde{\phi}_*(b_1)}{\tilde{\phi}_*(a_1)}E_{k-1}. \tag{3.19}$$

Step 2.3: Convergence result.

From (3.14), (3.15), (3.18) and (3.19), there exists γ in $(0,1)$ such that

$$(\nu_l^{k+M}(X,a_l,t))^2\varphi(t) \le \gamma \max\{E_k,\dots,E_{k+M-1}\}, \text{ for } l \in \{1,\dots,M\}, \tag{3.20}$$

and

$$(\nu_l^{k+M}(X,b_l,t))^2\varphi(t) \le \gamma \max\{E_k,\dots,E_{k+M-1}\}, \text{ for } l \in \{1,\dots,M\}. \tag{3.21}$$

Using (3.9) for $\phi \equiv 1$, we have that

$$\begin{aligned}(\nu_l^k(X,x_n,t))^2\varphi(t) &\le \\ &\le \max\{\max_{\bar{D}\times[0,T]}\{(\nu_l^k(X,a_l,t))^2\varphi(t)\}, \max_{\bar{D}\times[0,T]}\{(\nu_l^k(X,b_l,t))^2\varphi(t)\}\}.\end{aligned} \tag{3.22}$$

Combining (3.20), (3.21) and (3.22), we get

$$E_{k+M} \le \gamma \max\{E_k,\dots,E_{k+M-1}\}. \tag{3.23}$$

Hence, E_k tends to 0 as k tends to infinity.

Step 3: Proof of convergence: for l in $\{1,\dots,M\}$, the sequence $\{e_l^k\}$ converges point-wisely to 0 as k tends to infinity.

Since for l in $\{1,\dots,M\}$,

$$\lim_{k\to\infty}\left\|\left(\nu_l^k\right)^2\varphi(t)\right\|_{C(\overline{\Omega_l\times(0,T)})} = 0,$$

the sequence $\left\{\left(\frac{\partial\epsilon_l^k}{\partial x_n}\right)^2\varphi(t)\right\}$ converges to 0 point-wisely and the sequence is bounded by a constant M_0. Since φ is strictly positive on $[0,T]$, there exist positive constants M_1, M_2 such that $M_1<\varphi<M_2$. That means the sequence $\left\{\left|\frac{\partial\epsilon_l^k}{\partial x_n}\right|\right\}$ converges to 0 point-wisely and is bounded by a constant M_3.

For $l=1$, with a fixed value of (X,t), we get from the Lebesgue Dominated Convergence Theorem that for x_n in $[a_1,b_1]$, $\int_{a_1}^{x_n}\frac{\partial\epsilon_1^k}{\partial x_n}(X,\zeta,t)d\zeta$ converges to 0 as k tends to infinity. Hence the sequence $\{\epsilon_1^k(X,x_n,t)-\epsilon_1^k(X,a_1,t)\}$ converges to 0 as k tends to infinity. Since $\epsilon_1^k(X,a_1,t)=0$, the sequence $\{\epsilon_1^k\}$ converges to 0 point-wisely.

For $l=2$, with a fixed value of (X,t), again by the Lebesgue Dominated Convergence Theorem, for x_n in $[a_2,b_2]$, the sequence

$$\left\{\int_{a_2}^{x_n t}\frac{\partial\epsilon_2^k}{\partial x_n}(X,\zeta,t)d\zeta\right\}$$

converges to 0 as k tends to infinity. Hence the sequence

$$\{\epsilon_2^k(X,x_n,t)-\epsilon_2^k(X,a_2,t)\}$$

converges to 0 as k tends to infinity. Since

$$\frac{\partial e_2^k}{\partial x_n}(X,a_2,t)+pe_2^k(X,a_2,t) = \frac{\partial e_1^{k-1}}{\partial x_n}(X,a_2,t)+pe_1^{k-1}(X,a_2,t),$$

and the sequences $\{e_1^k\}$, $\left\{\left|\frac{\partial\epsilon_l^k}{\partial x_n}\right|\right\}$ converge to 0 point-wisely for l in $\{1,\dots,M\}$, we can deduce that $\epsilon_2^k(X,x_n,t)$ converges to 0 as k tends to infinity.

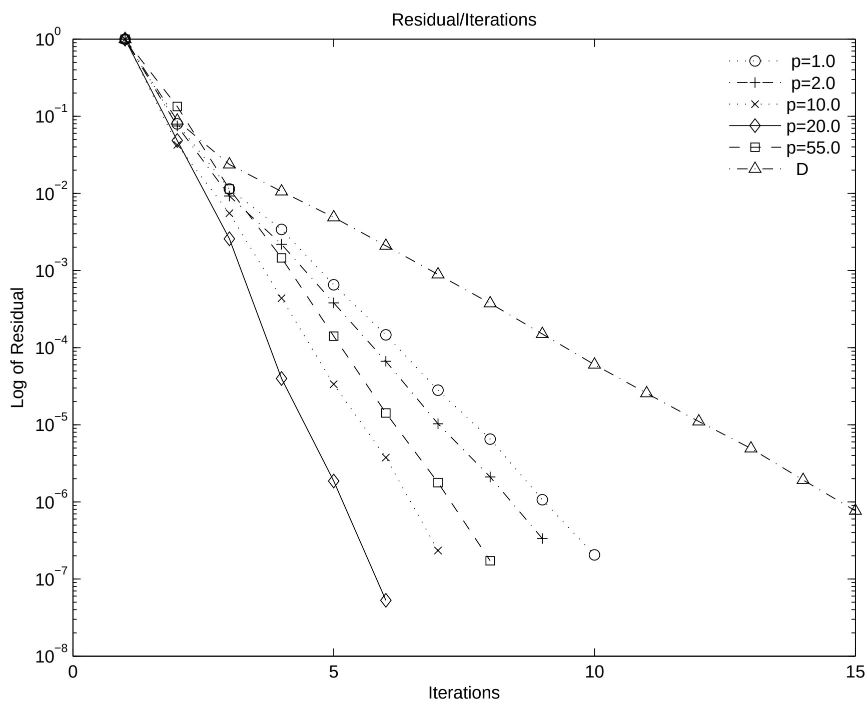
By similar processes, we can prove that for l in $\{1,\dots,M\}$, the sequence $\{e_l^k\}$ converges point-wisely to 0 as k tends to infinity. This concludes the proof.

4. Numerical Results

We give here some numerical results to illustrate the convergence of the method. Here we use the Python module *Optimism* developed by Loic Gouarin [7]. This code uses the MPI library to solve domain decomposition problems and can handle any number of subdomains. The problem in our example is the following

$$\frac{\partial u}{\partial t} - \triangle u + \nabla\cdot u + u = 0, \text{ in } (0,T)\times\Omega,$$

where $\Omega = (0,1)\times(0,1)$. The initial and boundary data are 0.
The code uses the finite element method to solve the problem and a triangular mesh is used. The discretization steps in space and time are $dx = dy = dt = 0.01$. We look only at the first iteration in time such that $T = dt$.
In our example, there are four subdomains ($M = 4$) and the decomposition in subdomains follows the x -direction. The overlapping length is $2\,dx$. It means that the first subdomain is $[0, 0.26]\times[0,1]$, the second one is $[0.24, 0.51]\times[0,1]$, the third one is $[0.49, 0.76]\times[0,1]$, and the fourth one is $[0.74, 1]$. We use random initial data h^0 on the boundaries a_l and b_l.



We consider the performance of the algorithm for several values of p including small and large ones: 1, 2, 10, 20, 55. On the same figure, we also plot the performance of the algorithm with Dirichlet transmission condition. According to this test, the algorithm with Robin transmission conditions reach the errors of 10^{-6}

after at most 9 iterations while the one with Dirichlet transmission conditions needs 15 iterations to reach this error.

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