# Nonlinear approximation theory for the homogeneous Boltzmann equation II

Minh-Binh Tran
Department of Mathematics
University of Wisconsin Madison
Email: minhbinh@math.wisc.edu

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**Abstract**

The current paper is the second part of our work on the nonlinear approximation theory for the homogeneous Boltzmann equation. In the first part, we introduced an adaptive wavelet spectral method for the numerical resolution of the Boltzmann equation. A complete convergence theory was provided and we also proved the approximate solution is bounded from below by a Maxwellian. The third part is devoted to the numerical study of the equation. This is part of the work, we associate the adaptive spectral method associated with a new wavelet filtering technique to preserve the some important properties of the solution: the propagation of polynomial and exponential momentums.

**Keyword** Boltzmann equation, wavelet, adaptive spectral method, Maxwell lower bound, propagation of polynomial moments, propagation of exponential moments, convergence to equilibrium, nonlinear approximation theory, numerical stability, convergence theory, wavelet filter.
**MSC:** 82C40, 65M70, 76P05, 41A46, 42C40.

## Contents

# 1 Introduction

The Boltzmann equation describes the behaviour of a dilute gas of particles when the binary elastic collisions are the only interactions taken into account. In this work, we are interested in the space homogeneous Boltzmann equation, which reads

$$\frac{\partial f}{\partial t} = Q(f,f), \quad v \in \mathbb{R}^3, \tag{1.1}$$

where $f := f(t,v)$ is the time-dependent particle distribution function for the phase space. The Boltzmann collision operator $Q$ is a quadratic operator defined as

$$Q(f,f)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|, \cos\theta)(f'_*f' - f_*f)d\sigma dv_*, \tag{1.2}$$

where $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$, $f'_* = f(v'_*)$ and

$$\begin{cases} v' = v - \frac{1}{2}(v - v_* - |v-v_*|\sigma), \\ v'_* = v - \frac{1}{2}(v - v_* + |v-v_*|\sigma), \end{cases}$$

with $\sigma \in \mathbb{S}^2$ and

$$\cos\theta = \left\langle \frac{v-v_*}{|v-v_*|, \sigma} \right\rangle.$$

We assume that

$$B(|u|, \cos\theta) = |u|^\gamma b(\cos\theta), \tag{1.3}$$

where $\gamma \in [0, 1]$ and $b$ is a smooth function satisfying

$$\int_0^\pi b(\cos\theta) \sin\theta d\theta < +\infty, \tag{1.4}$$

and assumptions (2.1)-(2.2) in [21]

$$\exists\theta_b > 0 \text{ such that } supp\{b(\cos\theta)\} \subset \{\theta \quad | \quad \theta_b \leq \theta \leq \pi - \theta_b\}. \tag{1.5}$$

Under these assumptions, the collision operator could be split as

$$Q(f, f) = Q^+(f, f) - L(f)f,$$

with

$$Q^+(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta) f'_* f' d\sigma dv_*$$

and

$$L(f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta) f_* d\sigma dv_*.$$

The main difficulty in the numerical resolution of the Boltzmann equation is due to the multidimensional structure of the collision operator. One of the main deterministic methods to resolve the Boltzmann equation numerically is Discrete Velocity Models - DVMs, which was first initiated in the early work of Carleman ([4], [3]). The DVMs were proved to be consistent ([22], [8]), i.e. the discrete collision term could be seen as an approximation of the real collision operator: and the approximate solutions are proved to converge weakly to the solution of the main equation ([18], [23], [7]). However, it is not easy to obtain an error estimate as well as the strong convergence of the approximate solutions to the global solution. DVMs are also expensive. Fourier Spectral Methods - FSMs is another well-known technique to approximate the solution of the Botlzmann equation numerically. The idea of the methods is to truncate the Boltzmann equation on the velocity space and periodize the solution on the truncated domain. The methods were first introduced in [25] developed in several works ([26], [20], [11], [20], [24], [9]). The analysis of the methods was provided in [10].
The main problem with deterministic methods like DVMs and FSMs that use a fixed discretization in the velocity space is that the velocity space is approximated by a finite region. Physically, the velocity space is $\mathbb{R}^3$ and

even if the initial condition is compactly supported, the collision operator indeed spreads out the supports by a factor $\sqrt{2}$ (see [28]). Therefore in order to use both DVMs and FSMs, we have to impose nonphysical conditions to keep the supports of the solutions in the velocity space uniformly compact. In our work, we introduce a new way to deal with the truncation problem: in stead of truncating the computational domain from $\mathbb{R}^d$ into a bounded domain $(-R,R)^d$, we choose a change of variable mapping $\varphi : \mathbb{R}^d \to (-1,1)^d$

$$v \to \bar{v} = \frac{v}{1+|v|}.$$

and construct a nonlinear wavelet basis for $(-R,R)^d$: Let $\{e_n\}$ be a wavelet on $L^2(-1,1)^d$, then $e_n(\varphi)$ will be our new wavelet basis on the entire space $\mathbb{R}^d$. The price that we need to pay after using this change of variable is the Jacobian $\frac{1}{(1+|v|)^4}$, a momentum of order $-4$, which goes naturally into the physics of the equation. Using this new wavelet basis, we can construct an adaptive wavelet spectral method to solve Boltzmann equation, the mesh around the origin is very fine, while the mesh away from the origin is very coarse. Theoretically, we know that the solution of the Boltzmann equation is bounded from below and above by Gaussians ([28],[19],[12]), this particular mesh is adapted to the Boltzmann equation.
Our work is divided into three parts: In the first part [29], we proved that the algorithm converges in the energy norm and the approximate solution is also bounded from below by a Gaussian. The second part [30] is devoted to the practical and numerical aspects of the theory. The current paper is the second part of the work, in which we introduce the filtering technique and how to preserve some important properties of the solutions: propagation of exponential and polynomial moments. We first recall these quantitative properties of the Boltzmann equation

- **Production of polynomial moments** (Povzner [27], Desvillettes [6], Wennberg [31], Mischler and Wennberg [19]): if the initial condition $f_0$ satisfies

$$\int_{\mathbb{R}^3} f_0(v)(1+|v|^2)dv < +\infty,$$

then

$$\forall s \geq 2, \forall t_0 > 0, \sup_{t\geq t_0} \int_{\mathbb{R}^3} f(t,v)(1+|v|^s) < +\infty,$$

or

$$\forall s \geq 2, \forall t_0 > 0, \sup_{t\geq t_0} \int_{(-1,1)^3} f(t,\bar{v}) \left(1 + \left|\frac{\bar{v}}{1-|\bar{v}|}\right|^s\right) < +\infty. \quad (1.6)$$

- **Propagation of exponential moments** (Bobylev, Gamba and Panferov [2], Gamba, Panferov and Villani [12], Alonso, Cañizo, Gamba and Mouhot[1]): Assume that the initial data satisfies for some $s \in [\gamma, 2]$
$$\int_{\mathbb{R}^3} f_0(v) \exp(a_0|v|^s) dv \leq C_0,$$
then there are some constants $C, a > 0$ such that
$$\int_{\mathbb{R}^3} f(t, v) \exp(a|v|^s) dv < C,$$
or
$$\int_{(-1,1)^3} f(t, \bar{v}) \exp\left( a \left| \frac{\bar{v}}{1 - |\bar{v}|} \right|^s \right) d\bar{v} < C. \tag{1.7}$$

Suppose that we approximate $f$ by its truncated Fourier series

$$f_N = \sum_{k_1,k_2,k_3=(-N,-N,-N)}^{(N,N,N)} \hat{f}_k \exp(i\pi k.\bar{v}),$$

with
$$\hat{f}_k = \frac{1}{8} \int_{(-1,1)^3} f(\bar{v}) \exp(-i\pi k.\bar{v}) d\bar{v}.$$

We can see that the approximate solution $f_N$ will never satisfy the properties that we mention above no matter how good $f$ is. The reason is that all components of the Fourier basis, i.e. the sin and cos functions are globally and smoothly defined on the whole interval $[-1, 1]$ and they encounter singular problems at the extremes $-1$ and $1$. This raises the need for a compactly supported wavelet basis and a new filtering technique. The idea of the technique is simple: we remove compactly supported wavelets which contain the singular points $-1$ and $1$. After having a good orthogonal basis based on this filtering technique, we can apply the normal spectral method to solve the equation. This filtering technique looks like a truncation technique, however it is more natural since we only need to remove some spectral components and different from classical approximations, the support of our approximate solutions spread to the whole space $\mathbb{R}^3$ gradually after each approximate level $N$. Our filtering technique is inspired by Zuazua's Fourier filtering technique ([32] and [33]) used to preserve the propagation, observation and control of waves.

## 2 Reformulating the Boltzmann equation

Let us define the change of variables mapping

$$\varphi : \mathbb{R}^3 \to (-1,1)^3,$$

$$\varphi(v) = (\varphi_1(v_1), \varphi_2(v_2), \varphi_3(v_3)) = \left( \frac{v_1}{1+|v|}, \frac{v_2}{1+|v|}, \frac{v_3}{1+|v|} \right), \tag{2.1}$$

where $|v| = \max\{|v_1|, |v_2|, |v_3|\}$ with $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The inverse mapping $\varphi^{-1}$ of $\varphi$ is then

$$\varphi^{-1} : (-1,1)^3 \to \mathbb{R}^3,$$

$$\varphi^{-1}(\bar{v}) = (\varphi_1(\bar{v}_1), \varphi_2(\bar{v}_2), \varphi_3(\bar{v}_3)) = \left( \frac{\bar{v}_1}{1-|\bar{v}|}, \frac{\bar{v}_2}{1-|\bar{v}|}, \frac{\bar{v}_3}{1-|\bar{v}|} \right).$$

We define

$$g(t, \bar{v}) = f(t, \varphi^{-1}(\bar{v})),$$

where $\bar{v}$ is the new variable in $(-1,1)^3$.
Since the Jacobian of the change of variable $\bar{v} \to v$ is $\frac{1}{(1+|v|)^4}$

$$\int_{(-1,1)^3} |g(\bar{v})|^p (1-|\bar{v}|)^{-s-4} d\bar{v} = \int_{\mathbb{R}^3} |f(v)|^p (1+|v|)^s dv.$$

We now set

$$L^p_s = \{ f \quad | \quad \int_{\mathbb{R}^3} |f(v)|^p (1+|v|)^{sp} dv < +\infty \},$$

$$\mathcal{L}^p_s = \{ f \quad | \quad \int_{(-1,1)^3} |f(\bar{v})|^p (1-|\bar{v}|)^{-sp} d\bar{v} < +\infty \},$$

$$L^p(W) = \{ f \quad | \quad \int_{\mathbb{R}^3} |f(v)|^p W^p(v) dv < +\infty \},$$

$$\mathcal{L}^p(W') = \{ f \quad | \quad \int_{(-1,1)^3} |f(\bar{v})|^p (W'(\bar{v}))^p d\bar{v} < +\infty \},$$

where $p$, $s$ are real numbers and $W$, $W'$ are some positive weights.
Moreover, we also need the usual notation

$$< v > = \sqrt{1+|v|^2}, \quad \forall v \in \mathbb{R}^3.$$

The Boltzmann equation for $g$ is now

$$\partial_t g(t,x,\bar v)+\frac{\bar v}{1-|\bar v|}\nabla_x g(t,x,\bar v)=\int_{(-1,1)^3}\int_{\mathbb{S}^2}\frac{B(|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|,\sigma)}{(1-|\bar v_*|)^4} \tag{2.2}$$
$$\times\left[g\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.$$
$$\left.\times g\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)-g(\bar v)g(\bar v_*)\right]d\sigma d\bar v_*.$$

Now define

$$h(t,\bar v)=g(t,\bar v)(1-|\bar v|)^{-4},$$

then the Boltzmann equation for $h$ then reads

$$\partial_t h(t,x,\bar v)+\frac{\bar v}{1-|\bar v|}\nabla_x h(t,x,\bar v)=\int_{(-1,1)^3}\int_{\mathbb{S}^2}B(|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|,\sigma)$$
$$\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right. \tag{2.3}$$
$$\left.\times h\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)-h(\bar v)h(\bar v_*)\right]d\sigma d\bar v_*,$$

where

$$\begin{aligned}\mathcal{C}(\bar v,\bar v_*,\sigma) &= \left[1-\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right]^4\\ &\quad\times\left[1-\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right]^4\\ &\quad\times(1-|\bar v|)^{-4}(1-|\bar v_*|)^{-4}. \end{aligned}\tag{2.4}$$

Define

$$\mathcal{B}(\bar v,\bar v_*,\sigma)=B(|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|,\sigma), \tag{2.5}$$

we get our second new formulation of the Boltzmann equation

$$\partial_t h(t,x,\bar v)+\frac{\bar v}{1-|\bar v|}\nabla_x h(t,x,\bar v)=\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)$$
$$\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right. \tag{2.6}$$
$$\left.\times h\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)-h(\bar v)h(\bar v_*)\right]d\sigma d\bar v_*.$$

The corresponding initial datum is

$$h_0(\bar{v}) = (1 - |\bar{v}|)^{-4} f_0(\varphi^{-1}(\bar{v})),$$

then

$$\int_{(-1,1)^3} h_0(\bar{v}) \left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right) d\bar{v} < +\infty.$$

# 3 The adaptive spectral method

## 3.1 Wavelets for $L^2((-1,1)^3)$

We first define a multiresolution analysis for $L^2((-1,1))$.

$$V_0^{per} \subset V_{-1}^{per} \subset V_{-2}^{per} \subset \cdots \to L^2(-1,1)$$

with

$$W_0^{per} \oplus V_0^{per} = V_{-1}^{per} \ldots$$

and $\{\phi_{0,0}^{per}\} \cup \{\psi_{j,k}^{per}; j \in -\mathbb{N}, k = 0, \ldots, 2^{|j|} - 1\}$ is an orthonormal basis of $L^2(-1,1)$.
Multiresolution analysis is a frame work developed by Mallat [13] and Meyer [16], we refer to these two pioneering works or the books [5], [17] for more details, examples and proofs.
Define by $S_j\varkappa$ the orthogonal project of a function $\varkappa$ in $L^1(-1,1)$ onto $V_j$, due to [5, Section 9.3] we then have the following remarkable property, which is not true with a Fourier basis

$$\|S_j\varkappa\|_{L^1(-1,1)} \leq C_S\|\varkappa\|_{L^1(-1,1)},$$

and for wavelets like Haar, we also have

$$\|S_j\varkappa\|_{L^\infty(-1,1)} \leq C_S\|\varkappa\|_{L^\infty(-1,1)},$$

where $C_S$ is a constant not depending on $j$ and $\varkappa$.
*We now construct a multiresolution analysis for* $L^2((-1,1)^3)$. Define

$$\Psi_{\bar{j},k}^{per}(\bar{y}) = \psi_{j_1,k_1}^{per}(\bar{y}_1)\psi_{j_2,k_2}^{per}(\bar{y}_2)\psi_{j_3,k_3}^{per}(\bar{y}_3),$$

and

$$\Phi_{\bar{j},k}^{per}(\bar{y}) = \phi_{j_1,k_1}^{per}(\bar{y}_1)\phi_{j_2,k_2}^{per}(\bar{y}_2)\phi_{j_3,k_3}^{per}(\bar{y}_3),$$

where $\bar{j} = (j_1, j_2, j_3) \in (-\mathbb{N})^3, k = (k_1, k_2, k_3) \in \{0, \ldots, 2^{|j|} - 1\}^3$, $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in (-1,1)^3$. Then $\{\Phi_{0,0}^{per}\} \cup \{\Psi_{\bar{j},k}^{per}\}$ is an orthonormal basis of

$L^2((-1,1)^3)$.
Set $j \in -\mathbb{N}$ and put

$$\mathcal{V}_{|j|} = \{\Phi_{|j|,k}(\bar{y}) = \Phi^{per}_{(j,j,j),k}(\bar{y}), k = (k_1, k_2, k_3) \in \{0, \dots, 2^{|j|} - 1\}^3\}.$$

then

$$\overline{\cup_{|j| \in \mathbb{N}} \mathcal{V}_{|j|}} = L^2((-1,1)^3),$$

which is the ladder of multiresolution spaces for $L^2((-1,1)^3)$ we need.
Define by $P_{|j|}\varrho$ the orthogonal project of a function $\varrho$ in $L^1((-1,1)^3)$ onto $\mathcal{V}_{|j|}$, we also have the following properties

$$\|P_{|j|}\varrho\|_{L^1((-1,1)^3)} \leq C_P \|\varrho\|_{L^1((-1,1)^3)}, \tag{3.1}$$

and for wavelets like Haar, we also have

$$\|P_{|j|}\varrho\|_{L^\infty((-1,1)^3)} \leq C_P \|\varrho\|_{L^\infty((-1,1)^3)}, \tag{3.2}$$

where $C_P$ is a constant not depending on $j$ or $\varrho$.
Notice that since $\phi$ is a positive function, the following property is true

$$\varrho \geq 0 \Rightarrow P_{|j|}\varrho \geq 0. \tag{3.3}$$

### 3.2 The nonlinear approximation for the homogeneous Boltzmann equation

First of all, we define the concept of a filter.

**Definition 3.1** *Let $\varsigma$ be a function in $\mathcal{V}_N$, $N \in \mathbb{N}$ and*

$$\varsigma = \sum_{k=(0,0,0)}^{(2^N-1, 2^N-1, 2^N-1)} \varsigma_{N,k} \Phi_{N,k},$$

*where*

$$\varsigma_{N,k} = \int_{(-1,1)^3} \varsigma \Phi_{N,k} d\bar{v}.$$

*Set $\mathfrak{A}_N$ to be the set of indices $\{k = (k_1, k_2, k_3) \quad | \quad 0 \leq k_1, k_2, k_3 \leq 2^N - 1\}$, and suppose that $\mathfrak{B}_N$ is a subset of $\mathfrak{A}_N$. Define*

$$F_N \varsigma = \sum_{k \in \mathfrak{A}_N \setminus \mathfrak{B}_N} \varsigma_{N,k} \Phi_{N,k},$$

*then $F_N$ is called a filter for $\varsigma$. In other words, a filter eliminates some components when we write $\varsigma$ as a linear combination of the basis $\{\Phi_{N,k}\}_{k\in\mathfrak{A}_N}$ of $\mathcal{V}_N$.*
*Since our idea is to remove wavelets containing the extreme points of $(-1,1)^3$, we suppose that after the filtering process, $F_N\varsigma$ is supported in $(-\zeta_N,\zeta_N)^3$ with $0<\zeta_N<1$ and $F_N 1$ is the characteristic function of $(-\zeta_N,\zeta_N)^3$. Notice that if $\bar v$ belongs to $(-\zeta_N,\zeta_N)^3$, then $v=\varphi^{-1}(\bar v)$ belongs to $(-\frac{\zeta_N}{1-\zeta_N},\frac{\zeta_N}{1-\zeta_N})^3$. For the sake of simplicity, we still denote*

$$\sum_{k\in\mathfrak{A}_N\setminus\mathfrak{B}_N} = \sum_{k=0}^{2^N-1}. \tag{3.4}$$

In this work, we only consider spectral methods for the homogeneous Boltzmann equation, which is written

$$\frac{\partial f}{\partial t} = Q(f,f), \quad v\in\mathbb{R}^3. \tag{3.5}$$

After performing the change of variables, we get

$$\begin{aligned}
&\partial_t h(t,\bar v) = \int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)\\
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\left.\times h\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)-h(\bar v)h(\bar v_*)\right]d\sigma d\bar v_*,
\end{aligned} \tag{3.6}$$

where $\mathcal{B}$, $\mathcal{C}$ are defined in (2.4).
Let $N$ be a positive integer and set

$$h_N = \left(1+\frac{|\bar v|^2}{(1-|\bar v|)^2}\right)^{-1} F_N P_N\left(\left(1+\frac{|\bar v|^2}{(1-|\bar v|)^2}\right)h\right),$$

where $F_N$ is a filter and $P_N$ is the orthogonal project onto the space $\mathcal{V}_N$.
Define

$$\tilde h_N = F_N P_N\left(\left(1+\frac{|\bar v|^2}{(1-|\bar v|)^2}\right)h\right),\quad \mathcal{P}_N = F_N P_N,\quad \eta(\bar v) = \left(1+\frac{|\bar v|^2}{(1-|\bar v|)^2}\right)^{-1}.$$

then

$$\partial_t\tilde h_N(t,\bar v)$$

$$
\begin{aligned}
&= \quad Q_N(\tilde{h}_N,\tilde{h}_N) = Q_N^+(\tilde{h}_N,\tilde{h}_N) - Q_N^-(\tilde{h}_N,\tilde{h}_N)\\
&:= \quad \mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right.\\
&\quad \times\left[\eta(\bar{v})^{-1}\mathcal{C}(\bar{v},\bar{v}_*,\sigma)\tilde{h}_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
&\quad \times\tilde{h}_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right) \qquad (3.7)\\
&\quad \times\eta\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\\
&\quad \left.\left.\times\eta\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-\tilde{h}_N(\bar{v})\tilde{h}_N(\bar{v}_*)\eta(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\},
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&\quad \partial_t h_N(t,\bar{v})\\
&= \quad Q_N(\tilde{h}_N,\tilde{h}_N) = Q_N^+(\tilde{h}_N,\tilde{h}_N) - Q_N^-(\tilde{h}_N,\tilde{h}_N)\\
&:= \quad \mathbb{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right.\\
&\quad \times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right. \qquad (3.8)\\
&\quad \left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\},
\end{aligned}
$$

where

$$\mathbb{P}_N(\varrho) = \eta\mathcal{P}_N(\eta^{-1}\varrho),$$

for some function $\varrho$, and the initial condition is

$$h_{0_N} = \mathbb{P}_N(h_0).$$

### 3.3 Assumptions on the multiresolution analysis and the filter

**Assumption 3.1** *(Energy preserving property) Define $\kappa = \eta(\bar{v})^{-1}\mathcal{P}_N\chi_{(-1,1)^3}$, where $\chi_{(-1,1)^3}$ is the characteristic function of $(-1,1)^3$. Set $\varkappa(v) = \kappa(\varphi(v))$, where $\varphi$ is the change of variables mapping defined in (2.1). In order to preserve the energy of the approximate solution, we impose the following assumption on $\mathcal{P}_N$*

$$\varkappa(v_*') + \varkappa(v') - \varkappa(v) - \varkappa(v_*) \leq 0, \quad \forall (v,v_*) \in \left(-\frac{\zeta_N}{1-\zeta_N},\frac{\zeta_N}{1-\zeta_N}\right)^6. \quad (3.9)$$

**Assumption 3.2** *(Coercivity preserving property) Let $N$ be a positive integer and $\vartheta$, $\vartheta'$ be two positive functions in $L^2((-1,1)^3)$. Define*

$$\vartheta_N = \mathcal{P}_N\vartheta, \text{ and } \vartheta_N = \mathcal{P}_N\vartheta'.$$

*Let $s$ be a constant. We impose the following assumption on the multiresolution analysis and the filter $F_N$: There exist constants $N_0$, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ not depending on $\vartheta$, $\vartheta'$ such that*

$$\begin{aligned} \forall N > N_0, \quad & \mathcal{K}_1(1-|\bar{v}|)^s \geq \mathcal{P}_N((1-|\bar{v}|)^s) \geq \mathcal{K}_2(1-|\bar{v}|)^s \text{ on } [-\zeta_N,\zeta_N]^3, \\ \text{and} \quad & \mathcal{K}_3\vartheta_N\vartheta'_N \geq \mathcal{P}_N(\vartheta_N\vartheta') \geq \mathcal{K}_4\vartheta_N\vartheta'_N. \end{aligned} \tag{3.10}$$

We refer the readers to the first part of our work [29], in which more explications about the above two assumption are given.

# 4 Propagation of polynomial moments

In [6], [31], it is proved that the solution $f$ of (3.5) satisfies the following property

$$\forall s > 0, \forall t_0 > 0, \sup_{t\geq t_0}\int_{\mathbb{R}^3} f(t,v)(1+|v|^s)dv < +\infty,$$

or equivalently

$$\forall s > 0, \forall t_0 > 0, \sup_{t\geq t_0}\int_{(-1,1)^3} h(t,\bar{v})(1-|\bar{v}|)^{-s}d\bar{v} < +\infty.$$

We will establish some conditions on the filter $F_N$ such that the above property is satisfied with the solution $h_N$ of the approximate problem (3.8). The idea of constructing $F_N$ is, again, to remove some components of the wavelet representation which are close to the extreme points of $(-1,1)^3$, or in other words, to restrict $h_N$ onto $[-\zeta_N,\zeta_N]^3$ with $0 < \zeta_N < 1$.

## 4.1 Assumption

First, we establish some properties on the filter $F_N$.

**Assumption 4.1** *Let $n$ be a positive integer, we suppose the following assumption on the multiresolution analysis and the filter $F_N$: There exists a constant $\epsilon(N)$ such that*

$$\lim_{N\to\infty}\epsilon(N) = 0,$$

*and*

$$\|\mathcal{P}_N^C(\eta^{-n}(\bar{v}))\|_{L^\infty([-\zeta_N,\zeta_N]^3)} < \epsilon(N),$$

*where*

$$\mathcal{P}_N^C(\eta^{-n}(\bar{v})) := \eta^{-n}(\bar{v}) - \mathcal{P}_N(\eta^{-n}(\bar{v})).$$

We now point out an example which satisfies this assumption. Consider again the Haar function in (**??**), (**??**), (**??**) and (**??**). According to the definition of the filter $F_N$, the approximate function $\mathcal{P}_N(\eta^{-n}(\bar{v}))$ is supported in $[-2^{-N}(2\hat{k}_N+1), 2^{-N}(2\hat{k}_N+1)]^3$.

**Proposition 4.1** *Let $\Delta$ be some constant in $(0,1)$ and suppose that*

$$\hat{k}_N = \left[\frac{\Delta 2^N - 1}{2}\right],$$

*where $[\frac{\Delta 2^N-1}{2}]$ denotes the largest integer smaller than $\frac{\Delta 2^N-1}{2}$. There exists a constant $\epsilon(N)$ such that*

$$\lim_{N\to\infty} \epsilon(N) = 0,$$

*and*

$$\|\mathcal{P}_N^C(\eta^{-n}(\bar{v}))\|_{L^\infty([-2^{-N}(2\hat{k}_N+1),2^{-N}(2\hat{k}_N+1)]^3)} < \epsilon(N).$$

**Remark 4.1** *This technique of wavelets filtering is inspired by the Fourier filtering technique introduced in [32], [33], [14], [15]. In order to preserve the propagation, observation and control of waves, Zuazua introduced a new Fourier filter: Suppose that the solution $u$ defined on $(0,1)$ could be written under the form of Fourier series*

$$u(x) = \sum_{-\infty}^{\infty} a_m \exp(-2\pi m i),$$

*and its approximation is*

$$u_N(x) = \sum_{-N}^{N} a_m \exp(-2\pi m i).$$

*Zuazua's Fourier filter is defined by removing all of the indices $m$ such that $|m| > [\Delta(N+1)]$ where $\Delta$ is a constant in $(0,1)$*

$$F_N u_N(x) = \sum_{-[\Delta(N+1)]}^{[\Delta(N+1)]} a_m \exp(-2\pi m i).$$

**Proof** Set

$$\mathcal{P}_N\left[\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n\right] = \sum_{k=-\hat{k}_N}^{\hat{k}_N} d_k \Phi_{N,k},$$

where

$$d_k = \int_{(-1,1)^3}\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n \Phi_{N,k} d\bar{v}.$$

Suppose that

$$\Phi_{N,k}(\bar{v}) = \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3),$$

with $|k_1| \geq |k_2| \geq |k_3|$. Hence, $|\bar{v}| = \max\{|\bar{v}_1|, |\bar{v}_2|, |\bar{v}_3|\} \in [2^{-N}(2|k_1|-1), 2^{-N}(2|k_1|+1)]$ if $k_1 \neq 2^{N-1}$ and $|\bar{v}| \in [0, 2^{-N}]$ if $k_1 = 2^{N-1}$.
If $k_1 \neq 2^{N-1}$ and $|\bar{v}| = \max\{|\bar{v}_1|, |\bar{v}_2|, |\bar{v}_3|\} \in [2^{-N}(2|k_1|-1), 2^{-N}(2|k_1|+1)]$.

$$\begin{aligned}
&\left|\mathcal{P}^C_N\left[\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n\right]\right| \qquad (4.1)\\
= &\left|\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n - d_k\Phi_{N,k}\right|\\
\leq &\left|\left|\frac{2^{-N}(2|k_1|-1)}{1-2^{-N}(2|k_1|-1)}\right|^{2n} - \left|\frac{2^{-N}(2|k_1|+1)}{1-2^{-N}(2|k_1|+1)}\right|^{2n}\right|\\
\leq &\left|\frac{2^{-N}(2|k_1|-1)}{1-2^{-N}(2|k_1|-1)} - \frac{2^{-N}(2|k_1|+1)}{1-2^{-N}(2|k_1|+1)}\right|\\
&\times \sum_{i=0}^{2n-1}\left|\frac{2^{-N}(2|k_1|-1)}{1-2^{-N}(2|k_1|-1)}\right|^i \left|\frac{2^{-N}(2|k_1|+1)}{1-2^{-N}(2|k_1|+1)}\right|^{2n-1-i}\\
\leq &\frac{2^{1-N}}{(1-2^{-N}(2|k_1|-1))(1-2^{-N}(2|k_1|+1))}\\
&\times \sum_{i=0}^{2n-1}\left|\frac{2^{-N}(2|k_1|-1)}{1-2^{-N}(2|k_1|-1)}\right|^i \left|\frac{2^{-N}(2|k_1|+1)}{1-2^{-N}(2|k_1|+1)}\right|^{2n-1-i}\\
\leq &\ 2^{1-N}\sum_{i=0}^{2n-1}\frac{|2^{-N}(2|k_1|-1)|^i}{|1-2^{-N}(2|k_1|-1)|^{i+1}}\frac{|2^{-N}(2|k_1|+1)|^{2n-1-i}}{|1-2^{-N}(2|k_1|+1)|^{2n-i}}.
\end{aligned}$$

Now consider the function

$$\varrho(y) = \frac{y^j}{(1-y)^{j+1}}, \quad y \in (0,1),$$

for any positive integer $j$. Since

$$\varrho'(y) = \frac{jy^{j-1}(1-y)^{j+1} + (j+1)y^j(1-y)^j}{(1-y)^{2j+2}} > 0,$$

the function $\varrho$ is increasing. Apply this into (4.1), we get

$$\begin{aligned}
& \left|\mathcal{P}_N^C\left[\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n\right]\right| \\
\leq \quad & 2^{1-N}\sum_{i=0}^{2n-1}\frac{|2^{-N}(2\hat{k}_N+1)|^i}{|1-2^{-N}(2\hat{k}_N+1)|^{i+1}}\frac{|2^{-N}(2\hat{k}_N+1)|^{2n-1-i}}{|1-2^{-N}(2\hat{k}_N+1)|^{2n-i}} \\
\leq \quad & 2^{1-N}2n\frac{|2^{-N}(2\hat{k}_N+1)|^{2n-1}}{|1-2^{-N}(2\hat{k}_N+1)|^{2n+1}} \qquad (4.2)\\
\leq \quad & 2^{1-N}2n\frac{\Delta^{2n-1}}{|1-\Delta|^{2n+1}}.
\end{aligned}$$

If $k_1 = 2^{N-1}$ and $|\bar{v}| \in [0, 2^{-N}]$.

$$\begin{aligned}
& \left|\mathcal{P}_N^C\left[\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n\right]\right| \\
= \quad & \left|\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n - \mathcal{P}_N\left[\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^n\right]\right| \qquad (4.3)\\
\leq \quad & \left|\frac{2^{-N}}{1-2^{-N}}\right|^{2n}.
\end{aligned}$$

The conclusion of the proposition follows from (4.2) and (4.3). ■

## 4.2 Propagation of polynomial moments

**Theorem 4.1** *Assuming assumptions* 3.1, 3.2, 4.1 *on the multiresolution analysis and the filter, then we get the following propagation of polynomial moments property*

$$\forall s > 0, \forall t_0 > 0, \exists N_0, \text{ such that } \sup_{t\geq t_0, N>N_0}\int_{(-1,1)^3} h_N(t,\bar{v})(1-|\bar{v}|)^{-s}d\bar{v} < +\infty. \qquad (4.4)$$

*If*

$$\int_{(-1,1)^3} h_0(t,\bar{v})(1-|\bar{v}|)^{-s}d\bar{v} < +\infty,$$

*then* $t_0$ *could be chosen to be* 0.

**Remark 4.2** *Using theorem* 4.1*, by the same argument as theorem 4.2, [29], we can have the following property*

$$\forall s>0, \forall t_0>0, \exists N_0, \ s.t. \ \sup_{t\geq t_0, N>N_0} \int_{(-1,1)^3} |h_N(t,\bar{v})|^2(1-|\bar{v}|)^{-s+4} d\bar{v} < +\infty. \tag{4.5}$$

*Moreover, by theorem* 4.1*, we can expect that*

$$\sup_{t\in[0,T]} \lim_{N\to\infty} \|h_N(t)-h(t)\|_{\mathcal{L}^1_s} = 0, \forall T\in\mathbb{R},$$

*if* $h_0 \in \mathcal{L}^1_s$.

**Proof** We only prove the theorem for integer values of $s$, $s>1$, the non-integer cases could be deduced directly from the integer cases by classical interpolation arguments. First, we observe that $f_{0_N}$ are uniformly bounded with respect to $N$ in $L^1_s$ if $f_0$ belongs to $L^1_s$. We prove the theorem in two steps.

**Step 1:** Transforming (3.7).

We recall (3.7)

$$\begin{aligned}
&\partial_t h_N(t,\bar{v})\\
= \ & \eta\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\eta^{-1}\right.\\
&\times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
&\left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\},
\end{aligned} \tag{4.6}$$

and use $\eta^{-s}$ ($s\in\mathbb{N}$) as a test function for (4.6) to obtain

$$\begin{aligned}
&\int_{(-1,1)^3}\partial_t h_N(t,\bar{v})\eta^{-s}d\bar{v}\\
= \ & \int_{(-1,1)^3}\eta^{1-s}\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\eta^{-1}\right.\\
&\times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
&\left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\}d\bar{v}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{(-1,1)^3} \mathcal{P}_N[\eta^{1-s}] \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar v, \bar v_*, \sigma)\eta^{-1} \right.\\
&\quad \times \left[ \mathcal{C}(\bar v, \bar v_*, \sigma) h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right)\right.\\
&\quad \left.\left. \times h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right) - h_N(\bar v) h_N(\bar v_*) \right] d\sigma d\bar v_* \right\} d\bar v \\
&= \int_{(-1,1)^6\times\mathbb{S}^2} \mathcal{B}(\bar v, \bar v_*, \sigma)\eta^{-s} \\
&\quad \times \left[ \mathcal{C}(\bar v, \bar v_*, \sigma) h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right)\right.\\
&\quad \left. \times h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right) - h_N(\bar v) h_N(\bar v_*) \right] d\sigma d\bar v_* d\bar v \\
&\quad - \int_{(-1,1)^3} \mathcal{P}_N^C[\eta^{1-s}] \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar v, \bar v_*, \sigma)\eta^{-1} \right. \\
&\quad \times \left[ \mathcal{C}(\bar v, \bar v_*, \sigma) h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right)\right.\\
&\quad \left.\left. \times h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right) \right] d\sigma d\bar v_* \right\} d\bar v \\
&\quad + \int_{(-1,1)^3} \mathcal{P}_N^C[\eta^{1-s}] \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar v, \bar v_*, \sigma)\eta^{-1} h_N(\bar v) h_N(\bar v_*) d\sigma d\bar v_* \right\} d\bar v.
\end{aligned} \tag{4.7}
$$

Now, consider the second term on the right hand side of (4.7), we have

$$
\begin{aligned}
&- \int_{(-1,1)^3} \mathcal{P}_N^C[\eta^{1-s}] \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar v, \bar v_*, \sigma)\eta^{-1} \right. \\
&\quad \times \left[ \mathcal{C}(\bar v, \bar v_*, \sigma) h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right)\right.\\
&\quad \left.\left. \times h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right) \right] d\sigma d\bar v_* \right\} d\bar v \\
= &- \int_{(-1,1)^3} \mathcal{P}_N^C[\eta^{1-s}] \chi_{(-\zeta_N,\zeta_N)^3} \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar v, \bar v_*, \sigma)\eta^{-1} \right. \\
&\quad \times \left[ \mathcal{C}(\bar v, \bar v_*, \sigma) h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right)\right.\\
&\quad \left.\left. \times h_N \left( \varphi \left( \frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2} \right)\right) \right] d\sigma d\bar v_* \right\} d\bar v
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{(-1,1)^3}\eta^{1-s}\chi_{\mathbb{R}^3\setminus(-\zeta_N,\zeta_N)^3}\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)\eta^{-1}\right. \\
& \quad\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right. \\
& \quad\left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right]d\sigma d\bar v_*\right\}d\bar v \\
\leq\ & -\int_{(-1,1)^3}\mathcal{P}_N^C[\eta^{1-s}]\chi_{(-\zeta_N,\zeta_N)^3}\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)\eta^{-1}\right. \\
& \quad\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right. \\
& \quad\left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right]d\sigma d\bar v_*\right\}d\bar v \\
\leq\ & \epsilon(N)\int_{(-1,1)^6\times\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)\eta^{-1} \\
& \quad\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right. \\
& \quad\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right]d\sigma d\bar v_* d\bar v,
\end{aligned}
\tag{4.8}
$$

where we use that fact

$$\mathcal{P}_N^C[\eta^{1-s}]\chi_{\mathbb{R}^3\setminus(-\zeta_N,\zeta_N)^3}=(Id-\mathcal{P}_N)[\eta^{1-s}]\chi_{\mathbb{R}^3\setminus(-\zeta_N,\zeta_N)^3}=\eta^{1-s}\chi_{\mathbb{R}^3\setminus(-\zeta_N,\zeta_N)^3},$$

since $\mathcal{P}_N[\eta^{1-s}]$ is supported in $(-\zeta_N,\zeta_N)^3$, after that assumption 4.1 is applied to get the final inequality.
According to assumption 4.1, the third term on the right hand side of (4.7) could be bounded in the following way

$$
\begin{aligned}
& \int_{(-1,1)^3}\mathcal{P}_N^C[\eta^{1-s}]\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)\eta^{-1}h_N(\bar v)h_N(\bar v_*)d\sigma d\bar v_*\right\}d\bar v \\
\leq\ & \int_{(-1,1)^6\times\mathbb{S}^2}\epsilon(N)\eta^{-1}\mathcal{B}(\bar v,\bar v_*,\sigma)h_N(\bar v)h_N(\bar v_*)d\sigma d\bar v_* d\bar v,
\end{aligned}
\tag{4.9}
$$

with the notice that since $h_N(\bar v)$ is supported in $(-\zeta_N,\zeta_N)^3$, we can suppose that $\eta^{1-s}(\bar v)$ is supported in $(-\zeta_N,\zeta_N)^3$ as well and hence assumption 4.1 is applicable.
We use again the change of variables mapping $\varphi$ to define

$$f_N(v)=h_N(\varphi(v))(1+|v|)^{-4}.$$

Inequalities (4.7), (4.8) and (4.9) lead to

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \partial_t f_N |v|^{2s} dv \qquad\qquad (4.10)\\
\leq\ & \int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f'_{N*}f'_N - f_{N*}f_N]|v|^{2s} d\sigma dv_* dv\\
&+\epsilon(N)\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f'_{N*}f'_N + f_{N*}f_N]|v|^{2} d\sigma dv_* dv\\
\leq\ & \int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f'_{N*}f'_N - f_{N*}f_N](|v|^{2s}+\epsilon(N)|v|^2) d\sigma dv_* dv\\
&+2\epsilon(N)\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)f_{N*}f_N|v|^{2} d\sigma dv_* dv\\
\leq\ & \frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)f_{N*}f_N(|v'_*|^{2s}+|v'|^{2s}-|v_*|^{2s}-|v|^{2s}) d\sigma dv_* dv\\
&+2\epsilon(N)\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)f_{N*}f_N|v|^{2} d\sigma dv_* dv,
\end{aligned}
$$

where the last inequality follows from the usual change of variables $(v, v_*) \to (v', v'_*)$.

**Step 2:** Using Povzner's inequality.

By Povzner's inequality (Theorem 4.1 [31]), we get from (4.10) that

$$
\begin{aligned}
\int_{\mathbb{R}^3} \partial_t f_N |v|^{2s} dv \ \leq\ & C\int_{\mathbb{R}^6} f_{N*}f_N|v_*|^{2s-1}|v||v-v_*|^{\gamma} dv_* dv \qquad (4.11)\\
&-C\int_{\mathbb{R}^6} f_{N*}f_N(|v_*|^{2s}+|v|^{2s})|v-v_*|^{\gamma} dv_* dv\\
&+C\epsilon(N)\int_{\mathbb{R}^6} f_{N*}f_N|v|^{2}|v-v_*|^{\gamma} dv_* dv.
\end{aligned}
$$

Since

$$
|v_*|^{2s-1}|v||v-v_*|^{\gamma} \leq |v_*|^{2s-1}|v|(1+|v|+|v_*|) \leq (1+|v_*|^{2s})(1+|v|^2),
$$

the first term on the right hand side of (4.11) could be bounded by

$$
C\left(1+\int_{\mathbb{R}^3} f_N|v|^{2s}dv\right). \qquad (4.12)
$$

We estimate the second term on the right hand side of (4.11)

$$
\int_{\mathbb{R}^6\times\mathbb{S}^2} f_{N*}f_N|v_*|^{2s}|v-v_*|^{\gamma} d\sigma dv_* dv
$$

$$\geq \quad C\int_{\mathbb{R}^3} f_N|v|^{2s+\gamma} \geq C\left(f_N|v|^{2s}\right)^{\frac{2s+\gamma}{2s}}, \tag{4.13}$$

with the notice that the results of lemma **??** still hold with $\lambda = \infty$.
We now estimate the third term on the right hand side of (4.11)

$$\begin{aligned} C\epsilon(N)\int_{\mathbb{R}^6} f_{N*}f_N|v|^2|v-v_*|^\gamma dv_*dv &\leq C\epsilon(N)\int_{\mathbb{R}^3} f_N(C_\epsilon + \epsilon|v|^{2+\gamma})dv \quad (4.14)\\ &\leq C\epsilon(N)\int_{\mathbb{R}^3} f_N(C_\epsilon + \epsilon|v|^{2s+\gamma})dv. \end{aligned}$$

where the inequality follows from the fact that the energy of $f_N$ is uniformly bounded with respect to $N$ and Young's inequality with a small positive constant $\epsilon$, with the notice that $s > 1$. Combine (4.11), (4.12), (4.13) and (4.14) with the assumption that $N$ is sufficiently large or $\epsilon(N)$ is sufficiently small we get

$$\int_{\mathbb{R}^3} \partial_t f_N|v|^{2s}dv \leq C\left(\int_{\mathbb{R}^3} f_N|v|^{2s}dv + 1\right) - C\left(\int_{\mathbb{R}^3} f_N|v|^{2s}dv\right)^{1+\frac{\gamma}{2s}}. \tag{4.15}$$

Define

$$Y(t) = \int_{\mathbb{R}^3} f_N|v|^{2s}dv,$$

inequality (4.15) becomes

$$\frac{dY}{dt} \leq C(Y+1) - CY^{1+\frac{\gamma}{2s}}.$$

Proceed similarly as the classical case [31], we get the conclusion of the theorem. ■

# 5 Propagation of exponential moments

In [1] and [12], it is proved that the solution of the homogeneous Boltzmann equation is bounded from above by a Maxwellian. Let us recall theorem 2 [1].

**Theorem 2 [1]** (Propagation of exponential moments)
*Let $f$ be an energy-conserving solution to the homogeneous Boltzmann equation* (3.5) *on* $[0, +\infty)$ *with initial data* $f_0 \in L^1_2$. *Assume moreover that the initial data satisfies for some* $s \in [\gamma, 2]$

$$\int_{\mathbb{R}^3} f_0(v)(a_0|v|^s)dv \leq C_0. \tag{5.1}$$

*Then there are some constants $C, a > 0$ which depend only on $b, \gamma$ and the initial mass, energy and $a_0$, $C_0$ in* (5.1) *such that*

$$\int_{\mathbb{R}^3} f(t,v)\exp(a|v|^s)dv < C. \tag{5.2}$$

Our task in this section is to preserve this property at the numerical level

$$\int_{(-1,1)^3} h_N(t,\bar{v})\exp\left(a\left(\frac{|\bar{v}|}{1-|\bar{v}|}\right)^s\right)d\bar{v} < C, \tag{5.3}$$

or if we use the $f_N$ formulation

$$f_N(v) = h_N(\varphi(v))(1+|v|)^{-4},$$

we should have

$$\int_{\mathbb{R}^3} f_N(t,v)\exp(a|v|^s)dv < C. \tag{5.4}$$

## 5.1 Assumption

Since

$$\int_{\mathbb{R}^3} f_0\exp(a_0|v|^s)dv \le C_0,$$

we have

$$\int_{(-1,1)^3} h_0\exp\left(a\left(\frac{|\bar{v}|}{1-|\bar{v}|}\right)^s\right)d\bar{v} < C_0.$$

Therefore, we need the following property as well for each initial datum of the approximate equation (3.8)

$$\int_{(-1,1)^3} \eta\mathcal{P}_N[h_0\eta^{-1}]\exp\left(a\left(\frac{|\bar{v}|}{1-|\bar{v}|}\right)^s\right)d\bar{v} < C,$$

or equivalently

$$\int_{(-1,1)^3} h_0\eta^{-1}\mathcal{P}_N\left[\eta\exp\left(a\left(\frac{|\bar{v}|}{1-|\bar{v}|}\right)^s\right)\right]d\bar{v} < C. \tag{5.5}$$

In order to have (5.5) we establish some further properties on the multiresolution analysis and the filter $F_N$ (notice that we always assume assumptions 3.1, 3.2, 4.1 are satisfied).

**Assumption 5.1** *Let $q$, $a$ be positive constants. We impose the following assumption on the multiresolution analysis and the filter $F_N$: There exist constants $N_0$, $\bar{\mathcal{K}}$, such that*

$$\forall N > N_0, \mathcal{P}_N[\eta \exp(a\eta^q)] \le \bar{\mathcal{K}} \eta \exp(a\eta^q). \tag{5.6}$$

A consequence of this assumption is that (5.5) follows directly from (5.1). We will point out an example which satisfies this assumption. Consider again the Haar function in (**??**), (**??**), (**??**) and (**??**). According to the definition of the filter $F_N$, the approximate function $\mathcal{P}_N[\eta \exp(a\eta^q)]$ is supported in $[-2^{-N}(2\hat{k}_N+1), 2^{-N}(2\hat{k}_N+1)]^3$.

**Proposition 5.1** *Let $\Delta$ be some constant in $(1/2, 1)$ and suppose that*

$$\hat{k}_N = \left[\frac{\Delta 2^N - 1}{2}\right],$$

*where $[\frac{\Delta 2^N-1}{2}]$ denotes the largest integer smaller than $\frac{\Delta 2^N-1}{2}$.*
*There exist constants $N_0$, $\bar{\mathcal{K}}$, such that*

$$\forall N > N_0, \mathcal{P}_N[\eta \exp(a\eta^q)] \le \bar{\mathcal{K}} \eta \exp(a\eta^q).$$

**Proof** Set

$$\mathcal{P}_N\left[\eta \exp(a\eta^q)\right] = \sum_{k=-\hat{k}_N}^{\hat{k}_N} d_k \Phi_{N,k},$$

where

$$d_k = \int_{(-1,1)^3} \eta \exp(a\eta^q) \Phi_{N,k} d\bar{v}.$$

Suppose that

$$\Phi_{N,k}(\bar{v}) = \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3),$$

with $k = (k_1, k_2, k_3)$ and $|k_1| \ge |k_2| \ge |k_3|$. Hence, $|\bar{v}| = \max\{|\bar{v}_1|, |\bar{v}_2|, |\bar{v}_3|\} \in [2^{-N}(2|k_1|-1), 2^{-N}(2|k_1|+1)]$ if $k_1 \ne 2^{N-1}$ and $|\bar{v}| \in [0, 2^{-N}]$ if $k_1 = 2^{N-1}$. If $k_1 \ne 2^{N-1}$ and $|\bar{v}| = \max\{|\bar{v}_1|, |\bar{v}_2|, |\bar{v}_3|\} \in [2^{-N}(2|k_1|-1), 2^{-N}(2|k_1|+1)]$.

$$\begin{aligned} & \frac{d_k \Phi_{N,k}(\bar{v})}{\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^{-1} \exp\left(a\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^q\right)} \qquad (5.7)\\ \le \quad & \frac{1 + \frac{(2^{-N}(2|k_1|+1))^2}{(1-2^{-N}(2|k_1|+1))^2}}{1 + \frac{(2^{-N}(2|k_1|-1))^2}{(1-2^{-N}(2|k_1|-1))^2}} \left[\exp\left(a\left(\frac{(2^{-N}(2|k_1|+1))^2}{(1-2^{-N}(2|k_1|+1))^2}\right)^q\right)\right. \end{aligned}$$

$$
\begin{aligned}
& - \exp\left(a\left(\frac{(2^{-N}(2|k_1|-1))^2}{(1-2^{-N}(2|k_1|-1))^2}\right)^q\right)\Bigg] \\
\leq \quad & \frac{1+\frac{(2^{-N}(2|k_1|+1))^2}{(1-2^{-N}(2|k_1|+1))^2}}{1+\frac{(2^{-N}(2|k_1|-1))^2}{(1-2^{-N}(2|k_1|-1))^2}}\left[2\exp\left(a\left(\frac{\Delta}{1-\Delta}\right)^{2q}\right)\right] \leq C,
\end{aligned}
$$

where the second inequality follows from a similar argument as proposition 4.1.
If $k_1 = 2^{N-1}$ and $|\bar{v}| \in [0, 2^{-N}]$.

$$
\begin{aligned}
& \frac{d_k \Phi_{N,k}(\bar{v})}{\left(1+\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^{-1}\exp\left(a\left(\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^q\right)} \\
\leq \quad & \left(1+\frac{|2^{-N}|^2}{(1-2^{-N})^2}\right)\exp\left(a\left(\frac{|2^{-N}|^2}{(1-2^{-N})^2}\right)^q\right) \leq C.
\end{aligned}
\tag{5.8}
$$

The two inequalities (5.7) and (5.8) imply the conclusion of the proposition. ■

## 5.2 Propagation of exponential moments

**Theorem 5.1** *Assume that the assumptions* 3.1*,* 3.2*,* 4.1*,* 5.1 *are all satisfied. Assume moreover that the initial data satisfies for some* $s \in [\gamma, 1]$

$$
\int_{\mathbb{R}^3} f_0(v)(a_0|v|^s)dv \leq C_0. \tag{5.9}
$$

*Then there are some constants* $C, a, N_0 > 0$ *which depend only on the equation, the initial mass, momentum energy and* $a_0$*,* $C_0$ *in* (5.1) *such that*

$$
\int_{(-1,1)^3} h_N(t,\bar{v})\exp\left(a\left(\frac{|\bar{v}|}{1-|\bar{v}|}\right)^s\right)d\bar{v} < C, \quad \forall N > N_0. \tag{5.10}
$$

**Proof** We define

$$
m_p^N(t) = \int_{\mathbb{R}^3} f_N(t,v)|v|^p dv, \quad p \in \mathbb{R}_+. \tag{5.11}
$$

We now prove the theorem in two steps.
**Step 1:** Estimate $m_{sp}^N$, with $0 < s \leq 1$ and $p \geq 2/s$.
A similar argument as theorem 4.1 gives

$$
\int_{\mathbb{R}^3} \partial_t f_N |v|^{sp}
$$

$$\begin{aligned}
&\le \int_{\mathbb{R}^6\times\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) f_{N*} f_N [|v_*'|^{sp} + |v'|^{sp} - |v_*|^{sp} - |v|^{sp}] d\sigma dv_* dv \\
&\quad +2\epsilon(N) \int_{\mathbb{R}^6} |v-v_*|^\gamma b(\cos\theta) f_{N*} f_N |v|^2 d\sigma dv_* dv. \qquad (5.12)
\end{aligned}$$

We recall the sharp Povzner Lemma (Lemma 3 [1]) for $q \ge 1$

$$\int_{\mathbb{S}^2} (|v'|^{2q} + |v_*'|^{2q}) b(\cos\theta) d\sigma \le \gamma_q (|v|^2 + |v_*|^2)^q, \qquad (5.13)$$

where $\gamma_q$ are positive constants satisfying $q \to \gamma_q$ is strictly decreasing and tends to 0 as $q \to \infty$.
Apply (5.13) to (5.12), we get

$$\begin{aligned}
\frac{d}{dt} m_{sp}^N &\le \gamma_{\frac{sp}{2}} \int_{\mathbb{R}^6} f_N f_{N*} \left[ (|v|^2 + |v_*|^2)^{\frac{sp}{2}} - |v|^{sp} - |v_*|^{sp} \right] |v-v_*|^\gamma dv_* dv \\
&\quad -2\left(1-\gamma_{\frac{sp}{2}}\right) \int_{\mathbb{R}^6} f_N f_{N*} |v|^{sp} |v-v_*|^\gamma dv_* dv \qquad (5.14) \\
&\quad +2\epsilon(N) \int_{\mathbb{R}^6\times\mathbb{S}^2} |v-v_*|^\gamma f_{N*} f_N |v|^2 dv_* dv,
\end{aligned}$$

with the normalized assumption

$$\int_{\mathbb{S}^2} b(\cos\theta) d\sigma = 1.$$

By the inequalities

$$(|v|^2 + |v_*|^2)^{\frac{sp}{2}} \le (|v|^s + |v_*|^s)^p, \text{ for } 0 < s \le 1,$$

and

$$\sum_{k=1}^{\left[\frac{p+1}{2}\right]-1} C_p^k (a^k b^{p-k} + a^{p-k} b^k) \le (a+b)^p - a^p - b^p \le \sum_{k=1}^{\left[\frac{p+1}{2}\right]} C_p^k (a^k b^{p-k} + a^{p-k} b^k),$$

(see Lemma 2 in [2]), we can bound the first term on the right hand side of (5.14)

$$\begin{aligned}
&\gamma_{\frac{sp}{2}} \int_{\mathbb{R}^6} f_N f_{N*} \left[ (|v|^2 + |v_*|^2)^{\frac{sp}{2}} - |v|^{sp} - |v_*|^{sp} \right] |v-v_*|^\gamma dv_* dv \\
&\le 2\gamma_{\frac{sp}{2}} \sum_{k=1}^{\left[\frac{p+1}{2}\right]} C_p^k (m_{sk+\gamma}^N m_{s(p-k)}^N + m_{sk}^N m_{s(p-k)+\gamma}^N). \qquad (5.15)
\end{aligned}$$

We then define

$$S_{s,p} := \sum_{k=1}^{[\frac{p+1}{2}]} C_p^k (m_{sk+\gamma}^N m_{s(p-k)}^N + m_{sk}^N m_{s(p-k)+\gamma}^N).$$

The second term could be bounded from below by

$$2\left(1-\gamma_{\frac{sp}{2}}\right)\int_{\mathbb{R}^6} f_N f_{N*}|v|^{sp}|v-v_*|^\gamma \geq 2\left(1-\gamma_{\frac{sp}{2}}\right)\bar{C}_\gamma[m_{sp+\gamma}^N + C], \quad (5.16)$$

where $\bar{C}_\gamma$ depends on $\gamma$ and the initial data and we use lemma **??**, with the assumption that $N$ is sufficiently large.
We can also estimate the third term

$$2\epsilon(N)\int_{\mathbb{R}^6\times\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) f_{N*} f_N |v|^2 dv_* dv \leq 2C\epsilon(N)[m_{\gamma+2}^N + C]. (5.17)$$

Combine (5.14), (5.15), (5.16) and (5.17), with the assumption that $\epsilon(N)$ is small enough, we get

$$\frac{d}{dt} m_{sp}^N \quad \leq \quad 2\gamma_{\frac{sp}{2}} S_{s,p} - 2\left(1-\gamma_{\frac{sp}{2}}\right)\bar{C}_\gamma m_{sp+\gamma}^N + 2C\epsilon(N) m_{\gamma+2}^N. \quad (5.18)$$

**Step 2:** Reduce the problem to the classical case of [1].
We define

$$E_s^m(t,z) := \sum_{p=0}^m m_{sp}^N(t)\frac{z^p}{p!},$$

and

$$I_{s,\gamma}^m(t,z) := \sum_{p=0}^m m_{sp+\gamma}^N(t)\frac{z^p}{p!}.$$

Let $s$ be in $[\gamma, 1]$ and $p_0 > \frac{2}{s}$. We reuse (5.18) with $a < \min\{1, a_0\}$ to get

$$\begin{aligned}
\frac{d}{dt}\sum_{p=p_0}^m m_{sp}\frac{a^p}{p!} \quad &\leq \quad \sum_{p=p_0}^m \frac{a^p}{p!}(2\gamma_{\frac{\gamma p}{2}} S_{\gamma,p} - 2\bar{C}_\gamma\left(1-\gamma_{\frac{sp}{2}}\right) m_{sp+\gamma}^N + 2C\epsilon(N) m_{\gamma+2}^N) \\
&\leq \quad \sum_{p=p_0}^m \frac{a^p}{p!} 2\gamma_{\frac{sp}{2}} S_{s,p} - K_1 I_{s,\gamma}^m + K_1 \sum_{p=0}^{p_0-1} m_{sp+\gamma}^N \frac{a^p}{p!} \\
&\quad + \left(\sum_{p=p_0}^m \frac{a^p}{p!}\right) K_2 \epsilon(N) m_{\gamma+2}^N
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{p=p_0}^{m} \frac{a^p}{p!} 2\gamma_{\frac{sp}{2}} S_{s,p} - K_1 I^m_{s,\gamma} + K_1 \sum_{p=0}^{p_0-1} m^N_{sp+\gamma} \frac{a^p}{p!} \qquad (5.19)\\
&\quad + \left( \sum_{p=p_0}^{m} \frac{a^p}{p!} \right) K_2 \epsilon(N)(C(\epsilon) m^N_\gamma + \epsilon m^N_{sp_0+\gamma})\\
&\leq \sum_{p=p_0}^{m} \frac{a^p}{p!} 2\gamma_{\frac{sp}{2}} S_{\gamma,p} - K_3 I^n_{s,\gamma} + K_4 \sum_{p=0}^{p_0-1} m^N_{sp+\gamma} \frac{a^p}{p!},
\end{aligned}
$$

where in the third inequality, we use Young's inequality

$$
m^N_{\gamma+2} \leq C(\epsilon) m^N_\gamma + \epsilon m^N_{sp_0+\gamma},
$$

in the fourth inequality, we suppose that $N$ is sufficiently large, such that $K_1$ could absorb the constants of the last term. Once we have (5.19) the proof could be proceeded in exactly the same way as the proof of the classical case (Theorem 2 [1]). ■

# 6 Conclusion

In this paper, we complete the second part of our work on the nonlinear approximation theory for the homogeneous Boltzmann equation. We introduce and then prove that our filtering technique preserves some important properties of the solution of the Boltzmann equation: the propagation of polynomial and exponential momentums. Our results are the complement of the theory introduced in [29], in which we prove that the approximate converges strongly to the exact solution [30] and the approximate is also bounded from below by a Maxwellian. In the third part of the work [30], we will give a formulation of the algorithm in the concrete case of the Haar wavelet and provide numerical tests to confirm the theoretical results.
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## References


[1] Ricardo Alonso, José A. Cañizo, Irene Gamba, and Clément Mouhot. A new approach to the creation and propagation of exponential moments in the Boltzmann equation. *Comm. Partial Differential Equations*, 38(1):155–169, 2013.

[2] A. V. Bobylev, I. M. Gamba, and V. A. Panferov. Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions. *J. Statist. Phys.*, 116(5-6):1651–1682, 2004.

[3] T. Carleman. *Problèmes mathématiques dans la théorie cinétique des gaz.* Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.

[4] Torsten Carleman. Sur la théorie de l'équation intégrodifférentielle de Boltzmann. *Acta Math.*, 60(1):91–146, 1933.

[5] Ingrid Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[6] L. Desvillettes. Some applications of the method of moments for the homogeneous Boltzmann and Kac equations. *Arch. Rational Mech. Anal.*, 123(4):387–404, 1993.

[7] L. Desvillettes and S. Mischler. About the splitting algorithm for Boltzmann and B.G.K. equations. *Math. Models Methods Appl. Sci.*, 6(8):1079–1101, 1996.

[8] Laura Fainsilber, Pär Kurlberg, and Bernt Wennberg. Lattice points on circles and discrete velocity models for the Boltzmann equation. *SIAM J. Math. Anal.*, 37(6):1903–1922 (electronic), 2006.

[9] Francis Filbet. On Deterministic Approximation of the Boltzmann Equation in a Bounded Domain. *Multiscale Model. Simul.*, 10(3):792–817, 2012.

[10] Francis Filbet and Clément Mouhot. Analysis of spectral methods for the homogeneous Boltzmann equation. *Trans. Amer. Math. Soc.*, 363(4):1947–1980, 2011.

[11] Francis Filbet, Clément Mouhot, and Lorenzo Pareschi. Solving the Boltzmann equation in $N \log_2 N$. *SIAM J. Sci. Comput.*, 28(3):1029–1053 (electronic), 2006.

[12] I. M. Gamba, V. Panferov, and C. Villani. Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation. *Arch. Ration. Mech. Anal.*, 194(1):253–282, 2009.

[13] Stephane G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$. *Trans. Amer. Math. Soc.*, 315(1):69–87, 1989.

[14] Aurora Marica and Enrique Zuazua. Localized solutions and filtering mechanisms for the discontinuous Galerkin semi-discretizations of the $1-d$ wave equation. *C. R. Math. Acad. Sci. Paris*, 348(19-20):1087–1092, 2010.

[15] Aurora Marica and Enrique Zuazua. Localized solutions for the finite difference semi-discretization of the wave equation. *C. R. Math. Acad. Sci. Paris*, 348(11-12):647–652, 2010.

[16] Yves Meyer. Ondelettes, fonctions splines et analyses graduées. *Rend. Sem. Mat. Univ. Politec. Torino*, 45(1):1–42 (1988), 1987.

[17] Yves Meyer. *Wavelets*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1993. Algorithms & applications, Translated from the French and with a foreword by Robert D. Ryan.

[18] Stéphane Mischler. Convergence of discrete-velocity schemes for the Boltzmann equation. *Arch. Rational Mech. Anal.*, 140(1):53–77, 1997.

[19] Stéphane Mischler and Bernt Wennberg. On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(4):467–501, 1999.

[20] Clément Mouhot and Lorenzo Pareschi. Fast methods for the Boltzmann collision integral. *C. R. Math. Acad. Sci. Paris*, 339(1):71–76, 2004.

[21] Clément Mouhot and Cédric Villani. Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. *Arch. Ration. Mech. Anal.*, 173(2):169–212, 2004.

[22] Andrzej Palczewski, Jacques Schneider, and Alexandre V. Bobylev. A consistency result for a discrete-velocity model of the Boltzmann equation. *SIAM J. Numer. Anal.*, 34(5):1865–1883, 1997.

[23] Vladislav A. Panferov and Alexei G. Heintz. A new consistent discrete-velocity model for the Boltzmann equation. *Math. Methods Appl. Sci.*, 25(7):571–593, 2002.

[24] L. Pareschi, G. Toscani, and C. Villani. Spectral methods for the non cut-off Boltzmann equation and numerical grazing collision limit. *Numer. Math.*, 93(3):527–548, 2003.

[25] Lorenzo Pareschi and Benoit Perthame. A Fourier spectral method for homogeneous Boltzmann equations. In *Proceedings of the Second International Workshop on Nonlinear Kinetic Theories and Mathematical Aspects of Hyperbolic Systems (Sanremo, 1994)*, volume 25, pages 369–382, 1996.

[26] Lorenzo Pareschi and Giovanni Russo. Numerical solution of the Boltzmann equation. I. Spectrally accurate approximation of the collision operator. *SIAM J. Numer. Anal.*, 37(4):1217–1245, 2000.

[27] A. Ja. Povzner. On the Boltzmann equation in the kinetic theory of gases. *Mat. Sb. (N.S.)*, 58 (100):65–86, 1962.

[28] Ada Pulvirenti and Bernt Wennberg. A Maxwellian lower bound for solutions to the Boltzmann equation. *Comm. Math. Phys.*, 183(1):145–160, 1997.

[29] Minh-Binh Tran. Nonlinear approximation theory for the homogeneous Boltzmann equation I. *Submitted*.

[30] Minh-Binh Tran. Nonlinear approximation theory for the homogeneous Boltzmann equation III. *Submitted*.

[31] Bernt Wennberg. Entropy dissipation and moment production for the Boltzmann equation. *J. Statist. Phys.*, 86(5-6):1053–1066, 1997.

[32] Enrique Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Rev.*, 47(2):197–243 (electronic), 2005.

[33] Enrique Zuazua. Control and numerical approximation of the wave and heat equations. In *International Congress of Mathematicians. Vol. III*, pages 1389–1417. Eur. Math. Soc., Zürich, 2006.