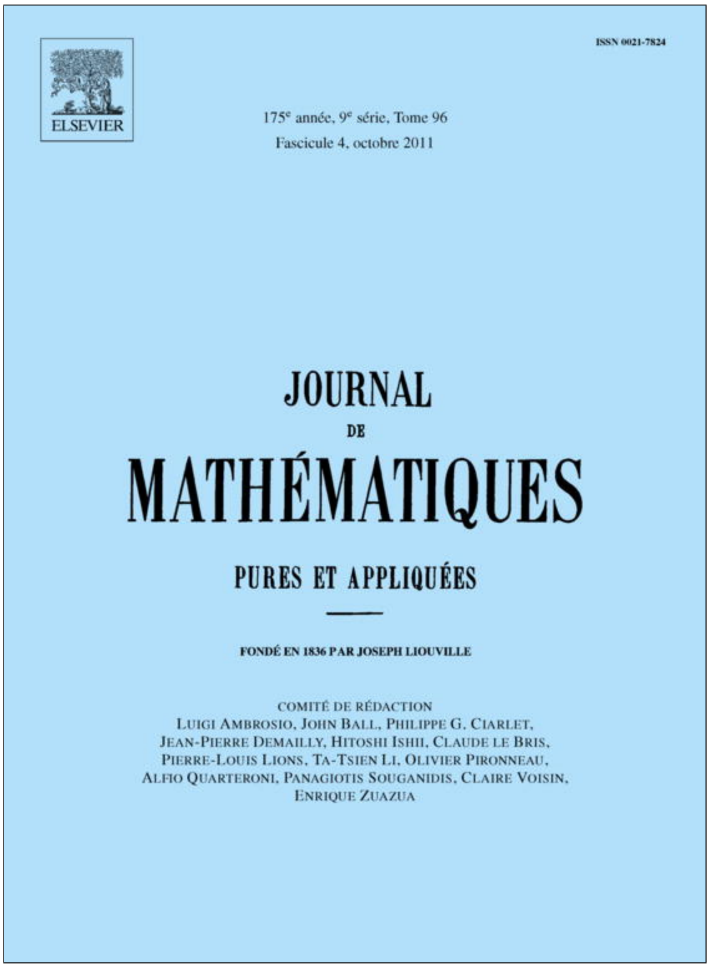

ISSN 0021-7824

175e année, 9e série, Tome 96
Fascicule 4, octobre 2011

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

FONDÉ EN 1836 PAR JOSEPH LIOUVILLE

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J. Math. Pures Appl. 96 (2011) 377–394

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

A parallel four step domain decomposition scheme for coupled forward–backward stochastic differential equations

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Received 27 September 2010

Available online 14 April 2011

Abstract

Motivated by the idea of imposing paralleling computing on solving stochastic differential equations (SDEs), we introduce a new domain decomposition scheme to solve forward–backward stochastic differential equations (FBSDEs) parallel. We reconstruct the four step scheme in Ma et al. (1994) [1] and then associate it with the idea of domain decomposition methods. We also introduce a new technique to prove the convergence of domain decomposition methods for systems of quasilinear parabolic equations and use it to prove the convergence of our scheme for the FBSDEs.

Résumé

Motivés par l'idée d'utiliser le calcul parallèle pour résoudre les équations différentielles stochastiques, on introduit un nouveau schéma de décomposition de domaines pour les FBSDEs. On reconstruit le schéma à quatre étapes de Ma et al. (1994) [1] et on le combine avec la méthode de décomposition de domaines. On introduit aussi une nouvelle technique pour étudier la convergence des méthodes de décomposition de domaines pour les systèmes d'équations paraboliques quasi-linéaires, et on l'utilise pour montrer la convergence de notre schéma.

MSC: 60H35; 65M12; 65C30; 35K55

Keywords: Domain decomposition; Waveform relaxation; Schwarz methods; Stochastic differential equations

1. Introduction

The theory of forward–backward stochastic differential equations (FBSDEs) is a very active field of research since the first work of Pardoux and Peng [2] and Antonelli [3] came out in the early 1990s. These equations appear in a large number of application fields in stochastic control and financial mathematics. We refer to the monograph [4] for details, further development and applications. Such systems strongly couple a forward stochastic differential equation with a backward one; and they can be written as a kind of stochastic two-point boundary value problems:

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0021-7824/$ – see front matter
doi:10.1016/j.matpur.2011.04.002

$$\begin{cases} dX_t = b(t, X_t, Y_t, Z_t)\,dt + \sigma(t, X_t, Y_t)\,dW_t, \\ dY_t = -\hat{b}(t, X_t, Y_t, Z_t)\,dt - Z_t\,dW_t, \\ X_0 = x, \quad Y_T = g(X_T). \end{cases} \tag{1.1}$$

Together with the theoretical studies on the systems (see [1–3,5–9]) finding an efficient numerical scheme for FBSDEs has also become an important part of the theory. In order to solve a system of FBSDEs, we need to use the "decoupling PDE" technique, based on the so-called four step scheme (see [1,8,9]). In which, the system of FBSDEs is associated with a quasilinear parabolic system of the following type:

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + \displaystyle\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2\theta}{\partial x_i\partial x_j} + \big\langle\nabla\theta, b(t,x,\theta,\nabla\theta)\big\rangle + \hat{b}(t,x,\theta,\nabla\theta) = 0, \quad \text{in } (0,T)\times\mathbb{R}^n, \\ \theta(T,x) = g(x), \quad \text{on } \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $\theta(t,x)$ is a vector of m components $\theta = (\theta^1, \ldots, \theta^m)$, $m \in \mathbb{N}$.

From here, there are two directions to solve FBSDEs. The first trend is to solve FBSDEs by using the decoupling technique combining with some probability methods to avoid treating the PDEs directly (see [6,10–12]). The second trend is to solve directly the PDEs. The first paper in this direction is the one of Douglas, Ma, Protter [13], in which the PDE is treated by a finite difference method. Later in 2008, Ma, Shen and Zhao proposed a new approach based on the Hermite-spectral method to treat the PDE (see [14]), that is then proved to be much more better than the previous one.

These systems of nonlinear PDEs are in high dimension and due to the large number of unknowns involved in the computation, it is an absolute necessity to split the computation between several processors. Therefore, transmission rules between the processors must be defined, which from a mathematical viewpoint means domain decomposition and transmission conditions between the sub-domains. In this paper, we present a new approach, still based on the second trend, to the coupled FBSDEs problem, by combining the classical four step scheme with domain decomposition methods or Schwarz methods, with waveform relaxation. The idea is to impose parallel computing on solving SDEs numerically. We reconstruct the four step scheme with some new conditions and then associate it with Schwarz waveform relaxation methods to parallelize the system of quasilinear parabolic equations (1.2): System (1.2) is divided into I subproblems, and each problem is solved separately. The scheme is then proved to be well-posed and stable. To our knowledge, this is the first attempt trying to apply domain decomposition algorithms to stochastic differential equations.

In the pioneer work [15–17], P.-L. Lions laid the foundations of the modern theory for the continuous approach of Schwarz algorithms. With the development of parallel computers, the interest in Schwarz methods has grown rapidly, as these methods lead to inherently parallel algorithms. During the last two decades, many domain decomposition algorithms have been introduced, but the problem of convergence of Schwarz methods still remains an open problem up to now. In his pioneer work [15–17], P.-L. Lions has proved that the classical Schwarz method for linear Laplace equation is in fact equivalent to a sequence of projections in a Hilbert space. Moreover, he also observed that the Schwarz sequences of linear elliptic equations is related to minimum methods over product spaces. This observation was used later by L. Badea in [18] to prove the convergence of the classical Schwarz method for a class of linear elliptic equations.

Later, in [19] and [20], M.J. Gander and A.M. Stuart and E. Giladi and H.B. Keller applied Schwarz methods to the 1-dimensional linear heat and advection–diffusion equations. Referring to the paper [21], they call Schwarz methods adapted to parabolic equations Schwarz waveform relaxation algorithms. In these papers, the convergence of the algorithms is proven by Laplace and Fourier transforms. An extension to the nonlinear reaction–diffusion equation in 1-dimension was considered in [22]. With the hypothesis $f'(c) \leqslant C$, proofs of linear convergence on unbounded time domains, and superlinear convergence on finite time intervals were then given in case of n sub-domains, based on some explicit computations on the linearized equations. Recently, an extension to systems of 1-dimensional semilinear reaction–diffusion equations was investigated in [23]. This is the first paper trying to apply Schwarz methods to a system of PDEs in 1-dimension and the proof of convergence is based strongly on the technique introduced in [22]. Another extension to nonlinear PDEs in higher dimension with monotone iterations was considered by Lui in [24–26]. The main idea of the papers is based on the sub–super solutions method in the theory of partial differential equations and the initial guesses are usually sub- or super-solutions of the equations.

In order to solve FBSDEs by Schwarz methods, we encounter the system of quasilinear parabolic equations (1.2) in n-dimension. To our knowledge, a good tool to study the convergence of Schwarz methods has not appeared, and the convergence problem remains to be a difficult problem up to now. We then introduce a new technique, that allows us to study the convergence of Schwarz algorithms for systems of nonlinear equations in n-dimension. Different from the systems and equations considered previously, our parabolic system contains nonlinearity in all advection and diffusion terms. The technique is based on an exponential decay estimate (Step 1 in the proof of Theorem 3.1) and an estimate of the errors on the boundaries (Step 2 in the same proof). Our long term goal is to construct some new tools to study the convergence problem of Schwarz methods and this technique takes us a step closer to it.

Our paper is organized as follows:

Section 2 introduces the FBSDEs we want to study. Definitions of the problem will be given in Sections 2.1 and 2.2 will give existence and uniqueness proofs.

In Section 3, we introduce our four step domain decomposition scheme and its proof of convergence. The scheme will be defined in Section 3.1 and the proofs will be given in Sections 3.2, 3.3 and 3.4.

2. Forward–backward stochastic differential equations

The structure of this section is as follows: In Section 2.1, we will give the definition of forward–backward stochastic differential equations, then state some results on the existence and uniqueness of the equations; these results will be proved in Section 2.2.

2.1. Existence and uniqueness results

Let $\{W_t\colon t \geqslant 0\}$ be a d-dimensional Brownian motion defined on the probability space $(\Omega, \mathfrak{F}, P)$. We define by $\{\mathfrak{F}_t\}$ the σ-field generated by W. We suppose that $\{\mathfrak{F}_t\}$ contains all the null sets of $\mathfrak{F}$ and consider the following forward–backward SDEs:

$$\begin{cases} X_t = x + \displaystyle\int_0^t b(s, X_s, Y_s)\,ds + \int_0^t \sigma(s, X_s, Y_s)\,dW_s, \\ Y_t = g(X_T) + \displaystyle\int_t^T \hat{b}(s, X_s, Y_s)\,ds + \int_t^T \hat{\sigma}(s, X_s, Y_s, Z_s)\,dW_s, \end{cases} \tag{2.1}$$

where t belongs to $[0, T]$; the processes X, Y, Z take values in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^{m\times d}$, respectively and b, $\hat{b}$, σ, $\hat{\sigma}$, g take values in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^{n\times d}$, $\mathbb{R}^{m\times d}$ and $\mathbb{R}^m$, respectively.

Since we are only looking for *ordinary adapted solutions* of the FBSDEs (2.1) (i.e. solutions which are $\{\mathfrak{F}_t\}$-adapted and square-integrable, and satisfy (2.1) P-almost surely), we can write (2.1) in the following form:

$$\begin{cases} dX_t = b(t, X_t, Y_t)\,dt + \sigma(t, X_t, Y_t)\,dW_t, \\ dY_t = -\hat{b}(t, X_t, Y_t)\,dt - \hat{\sigma}(t, X_t, Y_t, Z_t)\,dW_t, \\ X_0 = x, \quad Y_T = g(X_T). \end{cases} \tag{2.2}$$

Now we state the conditions that we impose on (2.1) and (2.2):

(A1) The functions b, $\hat{b}$, σ, $\hat{\sigma}$, g are bounded C^1-functions; and g is bounded in $C^{2+\delta}(\mathbb{R}^m)$ for some δ in (0, 1).

(A2) The matrix σ satisfies

$$\bigl|\sigma(t, x, y)\bigr| \leqslant C,$$

and

$$\sigma(t, x, y)\sigma^T(t, x, y) \geqslant \nu\bigl(|y|\bigr)I, \quad \forall(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

where ν is a positive continuous function, and C is a positive constant.

(A3) There exist a positive function η and a positive constant C such that for all $(t, x, y, 0)$ and $(t, x, 0, z)$ in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$,

$$\bigl|b(t, x, y, 0)\bigr| \leqslant \eta\bigl(|y|\bigr),$$

and

$$\left|\hat{b}(t,x,0,z)\right| \leqslant C.$$

(A4) We suppose also that for all $k \in \{1,\dots,m\}$ and for all $(t,x,y_1,\dots,y_{k-1},y_{k+1},\dots,y_m,\xi)$ in $[0,T]\times\mathbb{R}^n\times\mathbb{R}^{m-1}\times\mathbb{R}^{m\times n}$,

$$\hat{b}^k(t,x,y_1,\dots,y_{k-1},0,y_{k+1},\dots,y_m,\xi)=0,$$

and $\hat{b}^k(t,x,y_1,\dots,y_{k-1},y_k,y_{k+1},\dots,y_m,\xi)$ is decreasing in y_k.

Assuming that Y_t takes the form $\theta(t,X_t)$, P-almost surely, for all t in $[0,T]$, by Itô's formula, we can transform the backward SDE in (2.2) into the following system of PDEs:

$$\begin{cases} \dfrac{\partial\theta}{\partial t}+\displaystyle\sum_{i,j=1}^{n} a_{i,j}(t,x,\theta)\frac{\partial^2\theta}{\partial x_i\partial x_j}+\big\langle\nabla\theta, b(t,x,\theta,\nabla\theta)\big\rangle+\hat{b}(t,x,\theta,\nabla\theta)=0, & \text{in } (0,T)\times\mathbb{R}^n,\\ \theta(T,x)=g(x), \quad \text{on } \mathbb{R}^n, \end{cases} \tag{2.3}$$

where we define $\sigma^T(t,x,\theta)$ to be the transposed matrix of $\sigma(t,x,\theta)$ and $(a_{i,j})(t,x,\theta)$ to be $\frac{1}{2}\sigma(t,x,\theta)\sigma^T(t,x,\theta)$. The result on the existence and uniqueness of solutions of (2.2) then follows.

Theorem 2.1 *(Existence and uniqueness theorem). Suppose that conditions* (A1)–(A4) *above are satisfied, then Eq.* (2.2) *admits a unique solution* (X,Y,Z) *defined as follows*:

The process X is the solution of the following forward SDE,

$$X_t = x+\int_0^t b\big(s,X_s,\theta(s,X_s),\nabla\theta(s,X_s)\big)\,ds+\int_0^t \sigma\big(s,X_s,\theta(s,X_s)\big)\,ds, \tag{2.4}$$

where θ is the unique solution of (2.3).

The processes Y_t, Z_t are then $\theta(t,X_t)$ and $z(t,X_t,\theta(t,X_t),\nabla\theta(t,X_t))$, where z is a smooth mapping from $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$ to $\mathbb{R}^{m\times d}$ defined as follows:

$$z(t,x,y,\xi)=-\xi\sigma(t,x,y), \quad \forall(t,x,y,\xi)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}. \tag{2.5}$$

Remark 2.1. Assumptions (A1)–(A3) are similar to Assumptions (A1)–(A4) in [1]. (A4) will be used later in the proof of convergence for the parallel four steps domain decomposition scheme.

2.2. Proof of existence and uniqueness results

This subsection is devoted to the proof of the existence and uniqueness results stated in Section 2.1. The following proposition states a result on the existence and uniqueness of solutions to (2.3):

Proposition 2.1. *Suppose that* (A1)–(A4) *hold. Then the system* (2.3) *admits a unique classical solution $\theta(t,x)$, such that $\theta(t,x)$, $\frac{\partial}{\partial t}\theta(t,x)$, $\nabla\theta(t,x)$, $\Delta\theta(t,x)$ are bounded in $C((0,T)\times\mathbb{R}^n)$.*

Proof. We first recall a useful result. Letting ω be a bounded and smooth enough domain of $\mathbb{R}^n$, we consider the system:

$$\begin{cases} \dfrac{\partial\phi}{\partial t}+\displaystyle\sum_{i,j=1}^{n} a_{i,j}(t,x,\theta)\frac{\partial^2\phi}{\partial x_i\partial x_j}+\big\langle\nabla\phi, b(t,x,\phi,\nabla\phi)\big\rangle+\hat{b}(t,x,\phi,\nabla\phi)=0, & \text{in } (0,T)\times\omega,\\ \phi(t,x)=g(t,x), \quad \text{on } [0,T]\times\partial\omega,\\ \phi(T,x)=g(T,x), \quad \text{on } \omega. \end{cases} \tag{2.6}$$

Recalling Theorem 7.1, Chapter VII of [27] and Lemma 3.2 of [1], we have the following lemma:

Lemma 2.1. *Suppose that all the functions a_{ij}, b_i, $\hat{b}$ are smooth, g is bounded in $C^{1+\delta,2+\delta}([0,T]\times\omega)$ with δ belonging to $(0,1)$; and for all $(t,x,y)\in[0,t]\times\mathbb{R}^n\times\mathbb{R}^m$, we have:*

$$\nu_1(|y|)I \leqslant (a_{ij}(t,x,y)) \leqslant \nu_2(|y|)I, \tag{2.7}$$

$$\bigl|b\bigl(t,x,y,z(t,x,y,\xi)\bigr)\bigr| \leqslant \mu(|y|)(1+|\xi|), \tag{2.8}$$

$$\left|\frac{\partial}{\partial x_l}a_{ij}(t,x,y)\right| + \left|\frac{\partial}{\partial y_k}a_{ij}(t,x,y)\right| \leqslant \mu(|y|), \tag{2.9}$$

for some continuous positive functions $\nu_1(.)$, $\nu_2(.)$, $\mu(.)$, and

$$\bigl|\hat{b}\bigl(t,x,y,z(t,x,y,\xi)\bigr)\bigr| \leqslant C_1(1+|y|), \tag{2.10}$$

$$\bigl|\bigl\langle\hat{b}\bigl(t,x,y,z(t,x,y,\xi)\bigr),y\bigr\rangle\bigr| \leqslant C_2(1+|y|^2), \tag{2.11}$$

for some positive constants C_1, C_2. Then (2.6) admits a unique classical solution ϕ in $C^{1,2}((0,T)\times\omega)$. Moreover, ϕ, $\frac{\partial\phi}{\partial t}$, $\nabla\phi$, $\Delta\phi$ are bounded in $C((0,T)\times\mathbb{R}^n)$ by a constant which does not depend on ω, and there exists a positive number δ' in $(0,1)$ such that ϕ belongs to $C^{1+\delta',2+\delta'}((0,T)\times\omega)$.

Now, we will apply this lemma to our case. First of all, we verify that the conditions of Lemma 2.1 hold. We can see that (2.7) is a consequence of (A2), (2.8) is a consequence of (A4) and (2.9) is a consequence of (A1). We only need to prove (2.10) and (2.11).

Conditions (A1) and (A4) infer that

$$\bigl|\hat{b}(t,x,y,z)\bigr| \leqslant \bigl|\hat{b}(t,x,y,z)-\hat{b}(t,x,0,z)\bigr| + \bigl|\hat{b}(t,x,0,z)\bigr| \leqslant C_1(1+|y|),$$

and this implies,

$$\bigl|\bigl\langle\hat{b}(t,x,y,z),y\bigr\rangle\bigr| \leqslant C_1'(1+|y|)|y| \leqslant C_2(1+|y|^2),$$

where C_1' is a positive constant. Lemma 2.1 then implies that there exists a solution $\phi(t,x)$ for all ω in $\mathbb{R}^n$ bounded and smooth enough.

We now have a convergence argument similar as in [1]: we consider the ball B_R with center at the origin and radius to be R; using Lemma 2.1 with ω to be B_R, we see that there exists a unique bounded solution $\theta(t,x;R)$ for (2.6) and $\theta(t,x;R)$, $\frac{\partial}{\partial t}\theta(t,x;R)$, $\nabla\theta(t,x;R)$, $\Delta\theta(t,x;R)$ are bounded uniformly; then a diagonalization argument shows that there exists a subsequence $\theta(t,x;R)$ converging to $\theta(t,x)$ as R tends to infinity. We deduce that (2.3) admits a unique classical solution θ, such that θ, $\frac{\partial}{\partial t}\theta$, $\nabla\theta$, $\Delta\theta$ are bounded in $C((0,T)\times\mathbb{R}^n)$. □

We consider the forward SDE on X from (2.2), with the assumption that Y_t can be written under the form $\theta(t,X_t)$:

$$\begin{cases} dX_t = b\bigl(t,X_t,\theta(t,X_t),\nabla\theta(t,X_t)\bigr)\,dt + \sigma\bigl(t,X_t,\theta(t,X_t)\bigr)\,dW_t, \\ X_0 = x. \end{cases} \tag{2.12}$$

From the Lipschitz condition (A1), we can conclude that (2.12) has a unique solution X, which belongs to $\mathbb{L}^2(0,T)$. Basing on the previous proposition and following a similar argument as in [1], we get Theorem 2.1. □

3. The parallel four step domain decomposition scheme

This section is devoted to the construction of the parallel four step domain decomposition scheme and its proof of well-posedness and stability.

3.1. Definition of the scheme

We now define a new parallel four step domain decomposition scheme, based on the results obtained in Section 2.

Step 1. Let z be the smooth mapping satisfying (2.5).

Step 2. Choose l to be a constant large enough and consider the domain $\mathcal{O}_l = (-l, l)^n$. On $\mathcal{O}_l$, consider the following problem instead of (2.3):

$$\begin{cases} \dfrac{\partial \theta^l}{\partial t} + \displaystyle\sum_{i,j=1}^n a_{i,j}\big(t, x, \theta^l\big) \dfrac{\partial^2 \theta^l}{\partial x_i \partial x_j} + \big\langle \nabla\theta^l, b\big(t, x, \theta^l, \nabla\theta^l\big)\big\rangle + \hat{b}\big(t, x, \theta^l, \nabla\theta^l\big) = 0, \quad \text{in } (0, T) \times \mathcal{O}_l, \\ \theta^l(t, x) = g(x), \quad \text{on } (0, T) \times \partial\mathcal{O}_l, \\ \theta^l(T, x) = g(x), \quad \text{on } \mathcal{O}_l. \end{cases} \tag{3.1}$$

The same arguments as in the proof of Proposition 2.1 show that (3.1) has a unique classical solution $\theta^l(t, x)$, where $\theta^l(t, x)$, $\frac{\partial}{\partial t}\theta^l(t, x)$, $\nabla\theta^l(t, x)$, $\Delta\theta^l(t, x)$ are bounded. Suppose that $\theta^l(t, x) = g(x)$ on $(0, T) \times (\mathbb{R}^n \setminus \mathcal{O}_l)$, then these arguments also show that

$$\lim_{l\to\infty} \big\|\theta^l - \theta\big\|_{(L^\infty((0,T)\times\mathbb{R}^n))^m} = 0,$$
$$\lim_{l\to\infty} \big\|\nabla\theta^l - \nabla\theta\big\|_{(L^\infty((0,T)\times\mathbb{R}^n))^m} = 0.$$

Step 3. Solve Eq. (3.1) iteratively in the following manner:

- Divide $\mathcal{O}_l$ into I sub-domains,

$$\mathcal{O}_l = \bigcup_{p=1}^I \Omega_p = \bigcup_{p=1}^I (-l, l)^{n-1} \times (a_p, b_p),$$

  where $-l = a_1 < a_2 < b_1 < \cdots < a_I < b_{I-1} < b_I = l$. Denote S_i to be $b_i - a_{i+1}$ for i belonging to $\{1, \ldots, I - 1\}$ and L_i to be $b_i - a_i$ for i belonging to $\{1, \ldots, I\}$.
- Choose a bounded initial guess θ_0^l in $C^\infty(\mathbb{R}^n)$ at step 0. Associate each sub-domain Ω_p with a function $\theta_{p,0}^l$ such that $\theta_{p,0}^l = \theta_0^l$ on Ω_p.
- Solve the following p-th subproblem at iteration #q,

$$\begin{cases} \dfrac{\partial \theta_{p,q}^l}{\partial t} + \displaystyle\sum_{i,j=1}^n a_{i,j}\big(t, x, \theta_{p,q}^l\big) \dfrac{\partial^2 \theta_{p,q}^l}{\partial x_i \partial x_j} + \big\langle \nabla\theta_{p,q}^l, b\big(t, x, \theta_{p,q}^l, \nabla\theta_{p,q}^l\big)\big\rangle + \hat{b}\big(t, x, \theta_{p,q}^l, \nabla\theta_{p,q}^l\big) = 0, \\ \quad \text{in } (0, T) \times \Omega_p, \\ \theta_{p,q}^l(.,., a_p) = \theta_{p-1,q-1}^l(.,., a_p), \quad \text{on } (0, T) \times (-l, l)^{n-1}, \\ \theta_{p,q}^l(.,., b_p) = \theta_{p+1,q-1}^l(.,., b_p), \quad \text{on } (0, T) \times (-l, l)^{n-1}, \\ \theta_{p,q}^l(t, x) = g(x), \quad \text{on } (0, T) \times \big(\partial\mathcal{O}_l \setminus \big((0, T) \times (-l, l)^{n-1} \times \big(\{a_p\} \cup \{b_p\}\big)\big)\big), \\ \theta_{p,q}^l(T, x) = g(x), \quad \text{on } (-l, l)^{n-1} \times (a_p, b_p). \end{cases} \tag{3.2}$$

  For the extreme sub-domain Ω_1 (resp. Ω_I), we consider the boundary condition $\theta_{1,q}^l(t, x, a_1) = g(x)$ on the left (resp. $\theta_{I,q}^l(t, x, b_I) = g(x)$ on the right) in (3.2).
- Suppose that we stop at the iteration #q while solving (3.2).

The following two theorems insist that Step 2 of the algorithm is well-posed and show that the solutions of the subproblems (3.2) converge to the solution of the main problem (3.1) when q tends to infinity.

Theorem 3.1 *(Well-posedness theorem). Suppose that* (A1)–(A4) *hold, then at each iteration #q in each sub-domain #p, there exists a unique classical solution* $\theta_{p,q}^l(t, x)$ *for* (3.2)*, such that* $\theta_{p,q}^l(t, x)$, $\frac{\partial}{\partial t}\theta_{p,q}^l(t, x)$, $\nabla\theta_{p,q}^l(t, x)$, $\Delta\theta_{p,q}^l(t, x)$ *are bounded and the sequence* $\{\theta_{p,q}^l\}_{p\in\{1,\ldots,I\};\, q\in\mathbb{N}}$ *is uniformly bounded* (*with respect to* p *and* q) *in* $C((0, T) \times \mathcal{O}_l)$.

Theorem 3.2. *Under Assumptions* (A1)–(A4), *we have the convergence*:

$$\lim_{q\to\infty}\sup_{p\in\{1,\dots,I\}}\left\|\theta^l_{p,q}-\theta^l\right\|_{(L^\infty((0,T)\times(-l,l)^{n-1}\times(a_p,b_p)))^m}=0,$$
$$\lim_{q\to\infty}\sup_{p\in\{1,\dots,I\}}\left\|\nabla\theta^l_{p,q}-\nabla\theta^l\right\|_{(L^\infty((0,T)\times(-l,l)^{n-1}\times(a_p,b_p)))^m}=0. \tag{3.3}$$

Step 4. We will continue with the values $\theta^l_{p,q}$, $p\in\{1,\dots,I\}$, that we get at the end of Step 2.

- Let θ^l_q be a function defined on $[0,T]\times\mathcal{O}_l$ such that $\theta^l_q(t,x)=\theta^l_{p,q}(t,x)$ on $[0,T]\times(\Omega_p\setminus(\Omega_{p-1}\cup\Omega_{p+1}))$ for $p\in\{2,\dots,I-1\}$, on $[0,T]\times(\Omega_p\setminus\Omega_{p-1})$ for $p=I$, and on $[0,T]\times(\Omega_p\setminus\Omega_{p+1})$ for $p=1$. We can choose $\theta^l_q(t,x)$ such that it is Lipschitz, differentiable with respect to x and t in $\mathbb{R}^n$ and $\mathbb{R}$, and

$$\lim_{q\to\infty}\left\|\theta^l_q-\theta^l\right\|_{(L^\infty([0,T]\times\mathcal{O}_l))^m}=0,$$
$$\lim_{q\to\infty}\left\|\nabla\theta^l_q-\nabla\theta^l\right\|_{(L^\infty([0,T]\times\mathcal{O}_l))^m}=0. \tag{3.4}$$

- Use θ^l_q, solve the following forward SDE:

$$X^{q,l}_t=x+\int_0^t\bar{b}_q\left(s,X^{q,l}_s\right)ds+\int_0^t\bar{\sigma}_q\left(s,X^{q,l}_s\right)dW_s, \tag{3.5}$$

where $\bar{b}_q$ is $b(t,x,\theta^l_q(t,x),\nabla\theta^l_q(t,x))$ and $\bar{\sigma}_q(t,x)$ is $\sigma(t,x,\theta^l_q(t,x))$.
Using the same arguments as the ones used for (2.12), we can conclude that (3.5) has a unique solution in $\mathbb{L}^2(0,T)$.

Set $Y^{q,l}_t=\theta^l_q(t,X^{q,l}_t)$ and $Z^{q,l}_t=z(t,X^{q,l}_t,\theta^l_q(t,X^{q,l}_t),\nabla\theta^l_q(t,X^{q,l}_t))$. The following theorem says that the sequence $(X^{q,l}_t,Y^{q,l}_t,Z^{q,l}_t)$ converges to (X_t,Y_t,Z_t) as q and l tend to infinity.

Theorem 3.3 *(Convergence theorem). Suppose that all the assumptions in Section* 2.1 *hold, then as q and l tend to infinity,* $(X^{q,l}_t,Y^{q,l}_t,Z^{q,l}_t)$ *converges to the solution* (X_t,Y_t,Z_t) *of* (2.1) *in the following sense*:

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E\left(\left|X^{q,l}_t-X_t\right|^2\right)dt=0,$$
$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E\left(\left|Y^{q,l}_t-Y_t\right|^2\right)dt=0,$$
$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E\left(\left|Z^{q,l}_t-Z_t\right|^2\right)dt=0.$$

3.2. Proof of Theorem 3.1

First of all, we introduce some useful notations. We set:

$$M_0=\max\left\{\left\|\theta^l_0\right\|_{C([0,T]\times\mathbb{R}^n)},\|g\|_{C(\mathbb{R}^n)}\right\},$$

and define $\rho_{p,q}(t,x)$ to be $\theta^l_{p,q}(T-t,x)$ for $p\in\{1,\dots,I\}$ and $q\in\mathbb{N}$.

We can reformulate systems (3.2) into the following form:

$$\begin{cases} -\dfrac{\partial\rho_{p,q}}{\partial t}+\displaystyle\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2\rho_{p,q}}{\partial x_i\partial x_j}+\langle\nabla\rho_{p,q},b(t,x,\rho_{p,q},\nabla\rho_{p,q})\rangle+\hat{b}(t,x,\rho_{p,q},\nabla\rho_{p,q})=0, & \text{in } (0,T)\times\Omega_p,\\ \rho_{p,q}(.,.,a_p)=\rho_{p-1,q-1}(.,.,a_p), \quad \text{on } (0,T)\times(-l,l)^{n-1},\\ \rho_{p,q}(.,.,b_p)=\rho_{p+1,q-1}(.,.,b_p), \quad \text{on } (0,T)\times(-l,l)^{n-1},\\ \rho_{p,q}(t,x)=g(x), \quad \text{on } (0,T)\times\big(\partial\mathcal{O}_l\setminus\big((-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\})\big)\big),\\ \rho_{p,q}(0,x)=g(x), \quad \text{on } (-l,l)^{n-1}\times(a_p,b_p). \end{cases} \tag{3.6}$$

One can see that (3.6) are parabolic systems with the initial condition g.

Now, we will prove the theorem by induction.

At iteration #1, and in the p-th sub-domain, using the same argument as in Proposition 2.1, we can prove that (3.2) admits a unique classical solution $\theta^l_{p,1}(t,x)$, where $\theta^l_{p,1}(t,x)$, $\frac{\partial}{\partial t}\theta^l_{p,1}(t,x)$, $\nabla\theta^l_{p,1}(t,x)$, $\Delta\theta^l_{p,1}(t,x)$ are bounded. Consider the following k-th equation of (3.6), for k in $\{1,\dots,m\}$:

$$\begin{cases} -\dfrac{\partial\rho^k_{p,1}}{\partial t}+\displaystyle\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2\rho^k_{p,1}}{\partial x_i\partial x_j}+\langle\nabla\rho^k_{p,1},b(t,x,\rho_{p,1},\nabla\rho_{p,1})\rangle+\hat{b}(t,x,\rho_{p,1},\nabla\rho_{p,1})=0, & \text{in } (0,T)\times\Omega_p,\\ \rho^k_{p,1}(.,.,a_p)=\rho^k_{p-1,0}(.,.,a_p), \quad \text{on } (0,T)\times(-l,l)^{n-1},\\ \rho^k_{p,1}(.,.,b_p)=\rho^k_{p+1,0}(.,.,b_p), \quad \text{on } (0,T)\times(-l,l)^{n-1},\\ \rho^k_{p,1}(t,x)=g^k(x), \quad \text{on } (0,T)\times\big(\partial\mathcal{O}_l\setminus\big((-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\})\big)\big),\\ \rho^k_{p,1}(0,x)=g^k(x), \quad \text{on } (-l,l)^{n-1}\times(a_p,b_p). \end{cases} \tag{3.7}$$

Using (A4), we deduce from (3.7) that

$$-\frac{\partial\rho^k_{p,1}}{\partial t}+\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2\rho^k_{p,1}}{\partial x_i\partial x_j}+\langle\nabla\rho^k_{p,1},b(t,x,\rho_{p,1},\nabla\rho_{p,1})\rangle+c(t,x)\rho^k_{p,1}=0,$$

where

$$c(t,x)=\begin{cases} \dfrac{\hat{b}^k(t,x,\rho_{p,1},\nabla\rho_{p,1})-\hat{b}^k(t,x,\rho^1_{p,1},\dots,\rho^{k-1}_{p,1},0,\rho^{k+1}_{p,1},\dots,\rho^m_{p,1},\nabla\rho_{p,1})}{\rho^k_{p,1}} & \text{if } \rho^k_{p,1}(t,x)\neq 0,\\ 0 & \text{otherwise.} \end{cases}$$

Since $c(t,x)$ is negative, by applying the maximum principle (see [28]) to this equation, we can see that the maximum and minimum of $\rho^k_{p,1}$ can be obtained on the boundaries, for all k in $\{1,\dots,m\}$. This means that $\|\rho_{p,1}\|_{C([0,T]\times\Omega_p)}$ is bounded by M_0, and then M_0 is also an upper bound of $\|\theta^l_{p,1}\|_{C([0,T]\times\Omega_p)}$.

Suppose that up to iteration $\#q_0$, the unique classical solution $\theta^l_{p,q_0}(t,x)$ exists, $\theta^l_{p,q_0}(t,x)$, $\frac{\partial}{\partial t}\theta^l_{p,q_0}(t,x)$, $\nabla\theta^l_{p,q_0}(t,x)$, $\Delta\theta^l_{p,q_0}(t,x)$ are bounded, and for all p in $\{1,\dots,I\}$, $\|\theta^l_{p,q_0}\|_{C([0,T]\times\Omega_p)}$ is bounded by M_0. We will show that the conclusion is still correct for the step q_0+1. The existence and uniqueness of θ^l_{p,q_0+1} can be inferred by using the same argument as in iteration #1 and Proposition 2.1. Now, we consider the following equation, for k in $\{1,\dots,m\}$,

$$\begin{cases} -\dfrac{\partial\rho^k_{p,q_0+1}}{\partial t}+\displaystyle\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2\rho^k_{p,q_0+1}}{\partial x_i\partial x_j}+\langle\nabla\rho^k_{p,q_0+1},b(t,x,\rho_{p,q_0+1},\nabla\rho_{p,q_0+1})\rangle+\hat{b}(t,x,\rho_{p,q_0+1},\nabla\rho_{p,q_0+1})=0,\\ \quad\text{in } (0,T)\times\Omega_p,\\ \rho^k_{p,q_0+1}(.,.,a_p)=\rho^k_{p-1,q_0}(.,.,a_p), \quad \text{on } (0,T)\times(-l,l)^{n-1},\\ \rho^k_{p,q_0+1}(.,.,b_p)=\rho^k_{p+1,q_0}(.,.,b_p), \quad \text{on } (0,T)\times(-l,l)^{n-1},\\ \rho^k_{p,q_0+1}(t,x)=g^k(x), \quad \text{on } (0,T)\times\big(\partial\mathcal{O}_l\setminus\big((0,T)\times(-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\})\big)\big),\\ \rho^k_{p,q_0+1}(0,x)=g^k(x), \quad \text{on } (-l,l)^{n-1}\times(a_p,b_p). \end{cases} \tag{3.8}$$

Again, by a maximum principle argument applied to Eq. (3.8), we can see that the maximum and minimum of ρ^k_{p,q_0+1} can only be obtained on the boundaries, for all k in $\{1,\dots,m\}$. However, we know that $\|\theta^l_{p,q_0+1}\|_{C([0,T]\times\Omega_p)}$ is bounded by M_0 for all p in $\{1,\dots,I\}$, from the definition of M_0. We then deduce that $\|\theta^l_{p,q_0+1}\|_{C([0,T]\times\Omega_p)}$ is bounded by M_0.

We conclude that at each iteration #q in the p-th sub-domain, there exists a unique classical solution $\theta_{p,q}(t,x)$ for (3.2) and the sequence $\{\theta^l_{p,q}\}_{p\in\overline{1,I};\ q\in\mathbb{N}}$ is uniformly bounded with respect to p and q in $C((0,T)\times\mathcal{O}_l)$ by M_0. This finishes the proof.

3.3. *Proof of Theorem 3.2*

In this section, we introduce a completely new framework to study the convergence of domain decomposition methods. The framework contains 2 steps.

Step 1. *An exponential decay estimate.*

Setting $e_{p,q}$ to be $\theta^l_{p,q}-\theta^l$, we deduce the system:

$$\begin{cases} \dfrac{\partial e_{p,q}}{\partial t}+\displaystyle\sum_{i,j=1}^n a_{i,j}\big(t,x,\theta^l_{p,q}\big)\frac{\partial^2 e_{p,q}}{\partial x_i\partial x_j}+\big\langle\nabla e_{p,q}, b\big(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\big)\big\rangle+c\big(t,x,\theta^l_{p,q},\theta^l\big)=0, & \text{in } (0,T)\times\Omega_p,\\ e_{p,q}(.,.,a_p)=e_{p-1,q-1}(.,.,a_p), & \text{on } (0,T)\times(-l,l)^{n-1},\\ e_{p,q}(.,.,b_p)=e_{p+1,q-1}(.,.,b_p), & \text{on } (0,T)\times(-l,l)^{n-1},\\ e_{p,q}(t,x)=0, & \text{on } (0,T)\times\big(\partial\mathcal{O}_l\setminus\big((0,T)\times(-l,l)^{n-1}\times\big(\{a_p\}\cup\{b_p\}\big)\big)\big),\\ e_{p,q}(T,x)=0, & \text{on } (-l,l)^{n-1}\times(a_p,b_p), \end{cases} \tag{3.9}$$

where

$$\begin{aligned} c\big(t,x,\theta^l_{p,q},\theta^l\big)=&\left[\sum_{i,j=1}^n\big[a_{i,j}\big(t,x,\theta^l_{p,q}\big)-a_{i,j}\big(t,x,\theta^l\big)\big]\frac{\partial^2\theta^l}{\partial x_i\partial x_j}\right]\\ &+\big\langle\nabla\theta^l,\big[b\big(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\big)-b\big(t,x,\theta^l,\nabla\theta^l\big)\big]\big\rangle\\ &+\big[\hat b\big(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\big)-\hat b\big(t,x,\theta^l,\nabla\theta^l\big)\big]. \end{aligned}$$

Now, defining $\epsilon_{p,q}(t,x,y,z)$ to be $e_{p,q}(T-t,x,y,z)$, we change the system into,

$$\begin{cases} \dfrac{\partial\epsilon_{p,q}}{\partial t}-\displaystyle\sum_{i,j=1}^n a_{i,j}\big(t,x,\theta^l_{p,q}\big)\frac{\partial^2\epsilon_{p,q}}{\partial x_i\partial x_j}-\big\langle\nabla\epsilon_{p,q}, b\big(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\big)\big\rangle-c\big(t,x,\theta^l_{p,q},\theta^l\big)=0, & \text{in } (0,T)\times\Omega_p,\\ \epsilon_{p,q}(.,.,a_p)=\epsilon_{p-1,q-1}(.,.,a_p), & \text{on } (0,T)\times(-l,l)^{n-1},\\ \epsilon_{p,q}(.,.,b_p)=\epsilon_{p+1,q-1}(.,.,b_p), & \text{on } (0,T)\times(-l,l)^{n-1},\\ \epsilon_{p,q}(t,x)=0, & \text{on } (0,T)\times\big(\partial\mathcal{O}_l\setminus\big((-l,l)^{n-1}\times\big(\{a_p\}\cup\{b_p\}\big)\big)\big),\\ \epsilon_{p,q}(0,x)=0, & \text{on } (-l,l)^{n-1}\times(a_p,b_p). \end{cases} \tag{3.10}$$

We define:

$$\Phi_{p,q}(t,x)=\sum_{k=1}^m\big(\epsilon^k_{p,q}\big)^2\exp\big(\beta(x_n-\omega)-\gamma t\big),$$

where β, ω, γ will be fixed below, and consider the following parabolic operator,

$$\mathfrak{L}(\Phi)=\frac{\partial\Phi}{\partial t}-\big\langle\nabla\Phi, b\big(t,x,\theta^l,\nabla\theta^l\big)\big\rangle-\sum_{i,j=1}^n a_{ij}\frac{\partial^2\Phi}{\partial x_i x_j}+\sum_{i=1}^n\beta a_{i,n}\frac{\partial\Phi}{\partial x_i}. \tag{3.11}$$

A direct computation gives:

$$\begin{aligned}\mathfrak{L}(\Phi_{p,q}) &= \sum_{k=1}^{m}\bigl(-\gamma-\beta b^n\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)+\beta a_{n,n}-\beta^2 a_{n,n}\bigr)\bigl(\epsilon^k_{p,q}\bigr)^2\exp\bigl(\beta(x_n-\omega)-\gamma\bigr)\\ &\quad+2\epsilon^k_{p,q}\Biggl[-\bigl\langle\nabla\epsilon^k_{p,q},b\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)\bigr\rangle-\sum_{i,j=1}^{n}a_{i,j}\epsilon^k_{p,q}+\frac{\partial\epsilon^k_{p,q}}{\partial t}\Biggr]\\ &\quad-\sum_{i,j=1}^{n}2a_{i,j}\bigl(t,x,\theta^l_{p,q}\bigr)\frac{\partial\epsilon^k_{p,q}}{\partial x_i}\frac{\partial\epsilon^k_{p,q}}{\partial x_j}\\ &\leqslant\sum_{k=1}^{m}\bigl[\bigl(-\gamma-\beta b^n\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)-\beta^2 a_{n,n}\bigr)\bigl(\epsilon^k_{p,q}\bigr)^2\exp\bigl(\beta(x_n-\omega)-\gamma\bigr)+2\epsilon^k_{p,q}c^k\bigl(t,x,\theta^l_{p,q},\theta^l\bigr)\bigr],\end{aligned}\tag{3.12}$$

where c^k is the k-th component of the vector c.

We consider the following term of (3.12):

$$\begin{aligned}A &= \sum_{k=1}^{m}\bigl\{\bigl(-\gamma-\beta b^n\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)+\beta a_{n,n}-\beta^2 a_{n,n}\bigr)\bigl(\epsilon^k_{p,q}\bigr)^2+2\epsilon^k_{p,q}c^k\bigl(t,x,\theta^l_{p,q},\theta^l\bigr)\bigr\}\\ &= \sum_{k=1}^{m}\Biggl\{\bigl(-\gamma-\beta b^n\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)+\beta a_{n,n}-\beta^2 a_{n,n}\bigr)\bigl(\epsilon^k_{p,q}\bigr)^2\\ &\quad+2\epsilon^k_{p,q}\Biggl[\sum_{i,j=1}^{n}\bigl[a_{i,j}\bigl(t,x,\theta^l_{p,q}\bigr)-a_{i,j}\bigl(t,x,\theta^l\bigr)\bigr]\frac{\partial^2\theta^{l,k}}{\partial x_i\partial x_j}\\ &\quad+\bigl\langle\nabla\theta^{l,k},\bigl[b\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)-b\bigl(t,x,\theta^l,\nabla\theta^l\bigr)\bigr]\bigr\rangle\\ &\quad+\bigl[\hat{b}\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)-\hat{b}\bigl(t,x,\theta^l,,\nabla\theta^l\bigr)\bigr]\Biggr]\Biggr\}\\ &\leqslant\Biggl\{\sum_{k=1}^{m}\bigl(-\gamma-\beta b^n\bigl(t,x,\theta^l_{p,q},\nabla\theta^l_{p,q}\bigr)+\beta a_{n,n}-\beta^2 a_{n,n}\bigr)\bigl(\epsilon^k_{p,q}\bigr)^2\\ &\quad+N_1\bigl(\epsilon^k_{p,q}\bigr)^2\bigl\|\Delta\theta^l\bigr\|_{C(\mathbb{R}^n)}+N_2\bigl(\epsilon^k_{p,q}\bigr)^2\bigl\|\nabla\theta^l\bigr\|_{C(\mathbb{R}^n)}+N_3\bigl(\epsilon^k_{p,q}\bigr)^2\Biggr\}\\ &\leqslant\sum_{k=1}^{m}\bigl\{\bigl(-\gamma-\beta b^n\bigl(t,x,\theta^l_{p,q}\bigr)+\beta a_{n,n}-\beta^2 a_{n,n}\bigr)\bigl(\epsilon^k_{p,q}\bigr)^2+N_4\bigl(\epsilon^k_{p,q}\bigr)^2\bigr\},\end{aligned}\tag{3.13}$$

where N_1, N_2, N_3, N_4 are constants depending only on the coefficients of the system and the bound M_0 of $\theta^l_{p,q}$ and g in $C(\mathbb{R}^n)$. Since all solutions of the subproblems $\{\theta^l_{p,q}\}$ are uniformly bounded, according to classical results on a priori estimates (for example [29, Theorem 5, p. 64]), $\{\nabla\theta^l_{p,q}\}$ are uniformly bounded also. Hence A is negative when γ is large enough and β is suitable chosen. This implies that $\mathfrak{L}(\Phi_{p,q})$ is negative. According to the maximum principle, the maximum of $\Phi_{p,q}$ can only be attained on the boundary of the domain. Which means that the maximum of

$$\sum_{k=1}^{m}\bigl(e^k_{p,q}\bigr)^2\exp\bigl(\beta(x_n-\omega)-\gamma t\bigr)$$

can only be attained on $\{0\}\times\mathbb{R}^{n-1}\times[a_p,b_p]$, on $(0,T)\times(\partial\mathcal{O}_l\setminus((-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\})))$ or on $([0,T]\times\mathbb{R}^{n-1}\times\{a_p\})\cup([0,T]\times\mathbb{R}^{n-1}\times\{b_p\})$.

Since $\Phi_{p,q}(t,x)$ is equal to 0 on $\{0\}\times\mathbb{R}^{n-1}\times[a_p,b_p]$ and on $(0,T)\times(\partial\mathcal{O}_l\setminus((-l,l)^{n-1}\times(\{a_p\}\cup\{b_p\})))$, we have the following cases:

If $1 < p < I$,

$$\sum_{k=1}^{m}\big(e_{p,q}^k(t,x)\big)^2\exp\big(\beta(x_n-\omega)-\gamma t\big)$$
$$\leqslant \max\Big\{\max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{a_p\}}\sum_{k=1}^{m}\big(e_{p,q}^k(t,x)\big)^2\exp\big(\beta(a_p-\omega)-\gamma t\big),$$
$$\max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{b_p\}}\sum_{k=1}^{m}\big(e_{p,q}^k(t,x)\big)^2\exp\big(\beta(b_p-\omega)-\gamma t\big)\Big\}. \tag{3.14}$$

If $p = 1$,

$$\sum_{k=1}^{m}\big(e_{1,q}^k(t,x)\big)^2\exp\big(\beta(x_n-\omega)-\gamma t\big)$$
$$\leqslant \max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{b_1\}}\sum_{k=1}^{m}\big(e_{1,q}^k(t,x)\big)^2\exp\big(\beta(b_1-\omega)-\gamma t\big). \tag{3.15}$$

If $p = I$,

$$\sum_{k=1}^{m}\big(e_{I,q}^k(t,x)\big)^2\exp\big(\beta(x_n-\omega)-\gamma t\big)$$
$$\leqslant \max_{(t,x)\in[0,T]\times[-l,l]^{n-1}\times\{a_I\}}\sum_{k=1}^{m}\big(e_{I,q}^k(t,x)\big)^2\exp\big(\beta(a_I-\omega)-\gamma t\big). \tag{3.16}$$

Step 2. *Proof of the convergence*.

Step 2.1. Estimate of the right boundaries of the sub-domains.

For x in $[-l,l]^n$, we denote x by (X, x_n), where $X \in [-l,l]^{n-1}$ and $x_n \in [-l,l]$. Moreover, we define:

$$E_q = \max_{p\in\{1,\ldots,I\}}\Big\{\max_{(t,x)\in[0,T]\times[-l,l]^n}\sum_{k=1}^{m}\big(e_{p,q}^k(t,x)\big)^2\exp(-\gamma t)\Big\}.$$

Considering the I-th domain, at the q-th step, we can see that (3.16) infers:

$$\sum_{k=1}^{m}\big(e_{I,q}^k(t,X,x_n)\big)^2\exp\big(\beta(x_n-a_I)-\gamma t\big)\leqslant \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e_{I,q}^k(t,X,a_I)\big)^2\exp(-\gamma t),$$

where ω is replaced by a_I.

Replacing x_n by b_{I-1} in the previous inequality, we obtain:

$$\sum_{k=1}^{m}\big(e_{I,q}^k(t,X,b_{I-1})\big)^2\exp\big(\beta(b_{I-1}-a_I)-\gamma t\big)\leqslant \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e_{I,q}^k(t,X,a_I)\big)^2\exp(-\gamma t).$$

Since $e_{I,q}^k(t,X,b_{I-1})$ is equal to $e_{I-1,q+1}^k(t,X,b_{I-1})$, then

$$\sum_{k=1}^{m}\big(e_{I-1,q+1}^k(t,X,b_{I-1})\big)^2\exp\big(\beta(b_{I-1}-a_I)-\gamma t\big)\leqslant \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e_{I,q}^k(t,X,a_I)\big)^2\exp(-\gamma t).$$

We define β_1 to be $\sqrt{\frac{\gamma}{2}}$ and let β in this case be β_1; then if we choose γ large, $\gamma-\beta^2$ is large, the inequality becomes:

$$\sum_{k=1}^{m}\big(e_{I-1,q+1}^k(t,X,b_{I-1})\big)^2\exp(-\gamma t)\leqslant \exp(-\beta_1 S_{I-1})\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e_{I,q}^k(t,X,a_I)\big)^2\exp(-\gamma t).$$

We deduce that

$$\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,b_{I-1})\big)^2\exp(-\gamma t)\leqslant\exp(-\beta_1 S_{I-1})E_q. \tag{3.17}$$

Moreover, on the $(I-1)$-th domain, at the $(q+1)$-th step, (3.14) leads to,

$$\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,x_n)\big)^2\exp\big(\beta(x_n-a_{I-1})-\gamma t\big)$$
$$\leqslant\max\Bigg\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,a_{I-1})\big)^2\exp(-\gamma t),$$
$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,b_{I-1})\big)^2\exp\big(\beta(b_{I-1}-a_{I-1})-\gamma t\big)\Bigg\},$$

where ω is replaced by a_{I-1}.

Since $e^k_{I-1,q+1}(t,X,b_{I-2})$ is equal to $e^k_{I-2,q+2}(t,X,b_{I-2})$, then

$$\sum_{k=1}^{m}\big(e^k_{I-2,q+2}(t,X,b_{I-2})\big)^2\exp\big(\beta(b_{I-2}-a_{I-1})-\gamma t\big)$$
$$\leqslant\max\Bigg\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,a_{I-1})\big)^2\exp(-\gamma t),$$
$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,b_{I-1})\big)^2\exp(\beta L_{I-1}-\gamma t)\Bigg\};$$

thus

$$\sum_{k=1}^{m}\big(e^k_{I-2,q+2}(t,X,b_{I-2})\big)^2\exp(\beta S_{I-2}-\gamma t)$$
$$\leqslant\max\Bigg\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,a_{I-1})\big)^2\exp(-\gamma t),$$
$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,b_{I-1})\big)^2\exp(\beta L_{I-1}-\gamma t)\Bigg\}.$$

Combining this with (3.17) and the fact that

$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{I-1,q+1}(t,X,a_{I-1})\big)^2\exp(-\gamma t)\leqslant E_{q+1},$$

we obtain that

$$\sum_{k=1}^{m}\big(e^k_{I-2,q+2}(t,X,b_{I-2})\big)^2\exp(\beta S_{I-2}-\gamma t)\leqslant\max\big\{E_q\exp(\beta L_{I-1}-\beta_1 S_{I-1}),E_{q+1}\big\}.$$

Thus

$$\sum_{k=1}^{m}\big(e^k_{I-2,q+2}(t,X,b_{I-2})\big)^2\exp(-\gamma t)\leqslant\max\big\{E_q\exp\big(\beta(L_{I-1}-S_{I-2})-\beta_1 S_{I-1}\big),E_{q+1}\exp(-\beta S_{I-2})\big\}.$$

Defining β_2 to be $\beta_1\frac{S_{I-1}}{L_{I-1}}$ and choosing β to be β_2 such that

$$\beta_2(-L_{I-1}+S_{I-2})+\beta_1 S_{I-1}=\beta_2 S_{I-2},$$

we infer:

$$\sum_{k=1}^{m}\big(e^k_{I-2,q+2}(t,X,b_{I-2})\big)^2\exp(-\gamma t)\leqslant \max\{E_k,E_{k+1}\}\exp(-\beta_2 S_{I-2}). \tag{3.18}$$

Using the same techniques as the ones that we use to achieve (3.17) and (3.18), we can prove that

$$\sum_{k=1}^{m}\big(e^k_{I-j,q+j}(t,X,b_{I-j})\big)^2\exp(-\gamma t)\leqslant \max\{E_k,\dots,E_{k+j-1}\}\exp(-\beta_j S_{I-j}), \tag{3.19}$$

where

$$\beta_j=\beta_1\frac{S_{I-1}}{L_{I-1}}\cdots\frac{S_{I-j+1}}{L_{I-j+1}},\qquad j\in\{2,\dots,I-1\}.$$

Step 2.2. Estimate of the left boundaries of the sub-domains.

Consider the 1-th domain, at the k-th step. Then (3.15) infers that

$$\sum_{k=1}^{m}\big(e^k_{1,q}(t,X,x_n)\big)^2\exp\big(\beta(x_n-b_1)-\gamma t\big)\leqslant \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{1,q}(t,X,b_1)\big)^2\exp(-\gamma t),$$

we notice here that ω is replaced by b_1.

Replacing x_n by a_2, we obtain that

$$\sum_{k=1}^{m}\big(e^k_{1,q}(t,X,a_2)\big)^2\exp\big(\beta(a_2-b_1)-\gamma t\big)\leqslant \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{1,q}(t,X,b_1)\big)^2\exp(-\gamma t).$$

Since $e^k_{1,q}(t,X,a_2)$ is equal to $e^k_{2,q+1}(t,X,a_2)$,

$$\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,a_2)\big)^2\exp(-\gamma t)\leqslant \max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{1,q}(t,X,b_1)\big)^2\exp(-\gamma t)\exp(\beta S_1).$$

We define β_1' to be $-\sqrt{\frac{\gamma}{2}}$ and let β be β_1' in this case. If we choose γ large, $\gamma-\beta^2$ is large. The inequality becomes:

$$\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,a_2)\big)^2\exp(-\gamma t)\leqslant \exp\big(-\beta_1' S_1\big)\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{1,q}(t,X,b_1)\big)^2\exp(-\gamma t).$$

We deduce:

$$\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,a_2)\big)^2\exp(-\gamma t)\leqslant \exp\big(-\beta_1' S_1\big)E_q. \tag{3.20}$$

Moreover, on the 2-th domain, at the $(q+1)$-th step, (3.14) leads to,

$$\begin{aligned}&\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,x_n)\big)^2\exp\big(\beta(x_n-a_2)-\gamma t\big)\\&\quad\leqslant \max\Bigg\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,a_2)\big)^2\exp(-\gamma t),\\&\qquad\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,b_2)\big)^2\exp\big(\beta(b_2-a_2)-\gamma t\big)\Bigg\},\end{aligned}$$

notice that ω is replaced by a_2.

Since $e^k_{2,q+1}(t,X,a_3)$ is equal to $e^k_{3,q+2}(t,X,a_3)$, then

$$\sum_{k=1}^{m}\big(e^k_{3,q+2}(t,X,a_3)\big)^2\exp\big(\beta(a_3-a_2)-\gamma t\big)$$
$$\leqslant \max\Bigg\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,a_2)\big)^2\exp(-\gamma t),$$
$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,b_2)\big)^2\exp\big(\beta(b_2-a_2)-\gamma t\big)\Bigg\}.$$

Hence

$$\sum_{k=1}^{m}\big(e^k_{3,q+2}(t,X,a_3)\big)^2\exp(-\gamma t)$$
$$\leqslant \exp\big(-\beta(L_2-S_2)\big)\max\Bigg\{\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,a_2)\big)^2\exp(-\gamma t),$$
$$\max_{(t,X)\in[0,T]\times[-l,l]^{n-1}}\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,b_2)\big)^2\exp(\beta L_2-\gamma t)\Bigg\}.$$

Combining this inequality, (3.20) and the fact that

$$\sum_{k=1}^{m}\big(e^k_{2,q+1}(t,X,b_2)\big)^2\exp(-\gamma t)\leqslant E_{q+1},$$

we deduce:

$$\sum_{k=1}^{m}\big(e^k_{3,q+2}(t,X,a_3)\big)^2\exp(-\gamma t)\leqslant \exp\big(-\beta(L_2-S_2)\big)\max\big\{\exp\big(-\beta_1' S_1\big)E_q,\exp(\beta L_2)E_{q+1}\big\}.$$

Define β_2' to be $\beta_1'\frac{S_1}{L_2}$ and choose β to be $-\beta_2'$. We infer that

$$-\beta(L_2-S_2)-\beta_1' S_1=\beta S_2.$$

This implies:

$$\sum_{k=1}^{m}\big(e^k_{3,q+2}(t,X,a_3)\big)^2\exp(-\gamma t)\leqslant \max\{E_k,E_{k+1}\}\exp\big(-\beta_2' S_2\big). \tag{3.21}$$

Using the same techniques as the ones that we use to achieve (3.20) and (3.21), we can prove that

$$\sum_{k=1}^{m}\big(e^k_{j,q+j-1}(t,X,a_j)\big)^2\exp(-\gamma t)\leqslant \max\{E_k,\dots,E_{k+j-2}\}\exp\big(-\beta_{j-1}' S_{j-1}\big), \tag{3.22}$$

where

$$\beta_j'=\beta_1'\frac{S_1}{L_2}\cdots\frac{S_{j-1}}{L_j},\qquad j\in\{2,\dots,I-1\}.$$

Step 2.3. *Convergence result.*

Setting:

$$\bar\epsilon=\sqrt{\frac{\gamma}{2}}\frac{S_1\cdots S_{I-1}}{L_2\cdots L_{I-1}},$$

and

$$\bar{E}_k = \max_{j\in\{0,\dots,I-1\}} \{E_{k+j}\},$$

we infer from (3.19) and (3.22) that

$$\bar{E}_{k+1} \leqslant \bar{E}_k \exp(-\bar{\epsilon}), \quad \forall k \in \mathbb{N}.$$

Therefore

$$\bar{E}_n \leqslant \bar{E}_0 \exp(-n\bar{\epsilon}), \quad \forall n \in \mathbb{N}.$$

Hence E_k tends to 0 as k tends to infinity. Which gives that

$$\lim_{q\to\infty} \max_{p=\overline{1,I}} \sum_{k=1}^{m} \| e^k_{p,q} \|_{C([0,T]\times\mathbb{R}^{n-1}\times[a_p,b_p])} = 0.$$

Therefore, according to classical results on a priori estimates (for example [29, Theorem 5, p. 64]), $\{\nabla e^k_{p,q}\}$ are uniformly bounded, and

$$\lim_{q\to\infty} \max_{p=\overline{1,I}} \sum_{k=1}^{m} \| \nabla e^k_{p,q} \|_{C([0,T]\times\mathbb{R}^{n-1}\times[a_p,b_p])} = 0$$

that concludes the proof.

3.4. *Proof of Theorem 3.3*

Theorem 3.1 infers that $\{\theta^l_q\}$ and $\{\nabla\theta^l_q\}$ converge uniformly to $\{\theta^l\}$ and $\{\nabla\theta^l\}$. Moreover, $\{\theta^l\}$ and $\{\nabla\theta^l\}$ converge uniformly to $\{\theta\}$ and $\{\nabla\theta\}$. Hence, (2.5) implies that

$$\lim_{l\to\infty} \lim_{q\to\infty} \int_0^T E\left(\left|Z^{q,l}_t - Z_t\right|^2\right) dt = 0.$$

We start proving,

$$\lim_{l\to\infty} \lim_{q\to\infty} \int_0^T E\left(\left|X^{q,l}_t - X_t\right|^2\right) dt = 0.$$

Subtracting (2.4) and (3.5), we get:

$$\begin{aligned} X_t - X^{q,l}_t = {} & \int_0^t \left[b\left(s, X_s, \theta(s,X_s), \nabla\theta(s,X_s)\right) - b\left(s, X^{q,l}_s, \theta^{q,l}\left(s, X^{q,l}\right), \nabla\theta^{q,l}\left(s, X^{q,l}\right)\right)\right] ds \\ & + \int_0^t \left[\sigma\left(s, X_s, \theta(s,X_s)\right) - \sigma\left(s, X^{q,l}_s, \theta^{q,l}\left(s, X^{q,l}_s\right)\right)\right] dW_s, \end{aligned} \tag{3.23}$$

which leads to

$$\begin{aligned} \left|X_t - X^{q,l}_t\right|^2 \leqslant {} & 2\left(\int_0^t \left[b\left(s, X_s, \theta(s,X_s), \nabla\theta(s,X_s)\right) - b\left(s, X^{q,l}_s, \theta^l_q\left(s, X^{q,l}_s\right), \nabla\theta^{q,l}\left(s, X^{q,l}\right)\right)\right] ds \right)^2 \\ & + 2\left(\int_0^t \left[\sigma\left(s, X_s, \theta(s,X_s)\right) - \sigma\left(s, X^{q,l}_s, \theta^l_q\left(s, X^{q,l}_s\right)\right)\right] dW_s \right)^2. \end{aligned} \tag{3.24}$$

A simple calculation gives:

$$\begin{aligned}
E\big(\big|X_t - X_t^{q,l}\big|^2\big) &\leqslant N_5 E\int_0^t \big|b\big(s, X_s, \theta(s, X_s), \nabla\theta(s, X_s)\big) - b\big(s, X_s^{q,l}, \theta_q^l\big(s, X_s^{q,l}\big), \nabla\theta_q^l\big(s, X_s^{q,l}\big)\big)\big|^2\, ds \\
&\quad + N_5 E\int_0^t \big|\sigma\big(s, X_s, \theta(s, X_s)\big) - \sigma\big(s, X_s^{q,l}, \theta_q^l\big(s, X_s^{q,l}\big)\big)\big|^2\, ds \\
&\leqslant N_6 E\int_0^t \big[\big|X_s - X_s^{q,l}\big|^2 + \big|\theta(s, X_s) - \theta_q^l\big(s, X_s^{q,l}\big)\big|^2 + \big|\nabla\theta(s, X_s) - \nabla\theta_q^l\big(s, X_s^{q,l}\big)\big|^2\big]\, ds \\
&\leqslant N_6 E\int_0^t \big|X_s - X_s^{q,l}\big|^2\, ds \\
&\quad + N_7 E\int_0^t \big[\big|\theta\big(s, X_s^{q,l}\big) - \theta_q^l\big(s, X_s^{q,l}\big)\big|^2 + \big|\theta(s, X_s) - \theta\big(s, X_s^{q,l}\big)\big|^2\big]\, ds \\
&\quad + N_7 E\int_0^t \big[\big|\nabla\theta\big(s, X_s^{q,l}\big) - \nabla\theta_q^l\big(s, X_s^{q,l}\big)\big|^2 + \big|\nabla\theta(s, X_s) - \nabla\theta\big(s, X_s^{q,l}\big)\big|^2\big]\, ds \\
&\leqslant N_8 E\int_0^t \big[\big|X_s - X_s^{q,l}\big|^2 + \big|\theta\big(s, X_s^{q,l}\big) - \theta_q^l\big(s, X_s^{q,l}\big)\big|^2 + \big|\nabla\theta\big(s, X_s^{q,l}\big) - \nabla\theta_q^l\big(s, X_s^{q,l}\big)\big|^2\big]\, ds,
\end{aligned} \tag{3.25}$$

where N_5, N_6, N_7, N_8 are positive constants.

Since the sequence $\{\theta_q^l\}$ converges uniformly to θ^l, and the sequence $\{\theta^l\}$ converges uniformly to θ, then for all positive numbers ϵ, there exists $Q(\epsilon)$ such that

$$\big|\theta\big(s, X_s^{q,l}\big) - \theta_q^l\big(s, X_s^{q,l}\big)\big| + \big|\nabla\theta\big(s, X_s^{q,l}\big) - \nabla\theta_q^l\big(s, X_s^{q,l}\big)\big| < \sqrt{\epsilon}, \quad \forall q, l > Q(\epsilon).$$

Inequality (3.25) leads to,

$$E\big(\big|X_t - X_t^{q,l}\big|^2\big) \leqslant N_8 E\int_0^t \big[\big|X_s - X_s^{q,l}\big|^2 + \epsilon\big]\, ds, \quad \forall q, l > Q(\epsilon). \tag{3.26}$$

Now, fixing q, l greater than $Q(\epsilon)$ and setting,

$$H(t) = \int_0^t E\big(\big|X_s - X_s^{q,l}\big|^2\big)\, ds,$$

we obtain:

$$H'(t) \leqslant N_9 H(t) + N_9\epsilon, \tag{3.27}$$

where N_9 is a positive constant. Therefore

$$H'(t)\exp(-N_9 t) - N_9 H(t)\exp(-N_9 t) - N_9\epsilon\exp(-N_9 t) \leqslant 0.$$

This implies,

$$\big(H(t)\exp(-N_9 t) + \epsilon\exp(-N_9 t)\big)' \leqslant 0,$$

and this inequality then leads to,

$$H(t)\exp(-N_9 t) + \epsilon \exp(-N_9 t) \leqslant \epsilon.$$

Consequently,

$$H(t) \leqslant \epsilon\big(\exp(N_9 T) - 1\big),$$

for q, l greater than $Q(\epsilon)$, which leads to,

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E\big(\big|X_t - X_t^{q,l}\big|^2\big)\,dt = 0.$$

We next prove that

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E\big(\big|Y_t^{q,l} - Y_t\big|^2\big)\,dt = 0.$$

We can see that

$$\int_0^T E\big(\big|Y_t^{q,l} - Y_t\big|^2\big)\,dt = \int_0^T E\big(\big|\theta_q^l\big(t, X_t^{q,l}\big) - \theta(t, X_t)\big|^2\big)\,dt$$
$$\leqslant \int_0^T E\big(\big|\theta_q^l\big(t, X_t^{q,l}\big) - \theta\big(t, X_t^{q,l}\big)\big|^2\big)\,dt + \int_0^T E\big(\big|\theta\big(t, X_t^{q,l}\big) - \theta(t, X_t)\big|^2\big)\,dt.$$

Therefore

$$\lim_{l\to\infty}\lim_{q\to\infty}\int_0^T E\big(\big|Y_t^{q,l} - Y_t\big|^2\big)\,dt = 0.$$

This concludes the proof.

4. Conclusion

We have introduced a new domain decomposition method for a system of SDEs. The method has been studied theoretically and proved to be well-posed and stable. We have also proposed a new technique to prove the convergence of domain decomposition methods for systems of nonlinear parabolic equations in n-dimension. The method has the potential to be used to prove the convergence of domain decomposition methods for many kinds of nonlinear problems.

Acknowledgements

The author would like to express his gratitude to Professor Laurence Halpern for her help and encouragement. He is grateful to Dr Eulalia Nualart for her kindness and careful reading of the manuscript and Professor Huyên Pham for his advice. He would also like to thank the editor and the referee for their help and useful comments.

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