Nonlinear approximation theory for the homogeneous Boltzmann equation I

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Abstract

A challenging problem in solving the Boltzmann equation numerically is that the velocity space is approximated by a finite region. Therefore, most methods are based on a truncation technique and the computational cost is then very high if the velocity domain is large. Moreover, sometimes, non-physical conditions have to be imposed on the equation in order to keep the velocity domain bounded. The current paper is the first part of our work on the nonlinear approximation theory for the homogeneous Boltzmann equation. In this part, we introduced an adaptive, non-truncated wavelet spectral method for the numerical resolution of the equation. An complete convergence theory is provided.

Keyword Boltzmann equation, wavelet, adaptive spectral method, Maxwell lower bound, nonlinear approximation theory, numerical stability, wavelet filter.
MSC: 82C40, 65M70, 76P05, 41A46, 42C40.

1 Introduction

The Boltzmann equation describes the behaviour of a dilute gas of particles when the binary elastic collisions are the only interactions taken into account. In this work, we are interested in the space homogeneous Boltzmann equation, which reads

$$\frac{\partial f}{\partial t} = Q(f,f), \quad v \in \mathbb{R}^3, \tag{1.1}$$

where $f := f(t,v)$ is the time-dependent particle distribution function for the phase space. The Boltzmann collision operator Q is a quadratic operator defined as

$$Q(f,f)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|,\cos\theta)(f'_*f' - f_*f)d\sigma dv_*, \tag{1.2}$$

where $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$, $f'_* = f(v'_*)$ and

$$\begin{cases} v' = v - \frac{1}{2}(v - v_* - |v-v_*|\sigma), \\ v'_* = v - \frac{1}{2}(v - v_* + |v-v_*|\sigma), \end{cases}$$

with $\sigma \in \mathbb{S}^2$ and

$$\cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

We assume that

$$B(|u|, \cos\theta) = |u|^\gamma b(\cos\theta), \tag{1.3}$$

where $\gamma \in [0, 1]$ and b is a smooth function satisfying

$$\int_0^\pi b(\cos\theta)\sin\theta d\theta < +\infty, \tag{1.4}$$

and assumptions (2.1)-(2.2) in [48]

$$\exists\theta_b > 0 \text{ such that } supp\{b(\cos\theta)\} \subset \{\theta \quad | \quad \theta_b \leq \theta \leq \pi - \theta_b\}. \tag{1.5}$$

Under these assumptions, the collision operator could be split as

$$Q(f, f) = Q^+(f, f) - L(f)f,$$

with

$$Q^+(f, f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta) f'_* f' d\sigma dv_*$$

and

$$L(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta) f_* d\sigma dv_*.$$

Numerical resolution methods for the Boltzmann equation plays a very important role in the practical and theoretical study of the theory of rarefied gas. The main difficulty in the approximation of the Boltzmann equation is due to the multidimensional structure of the Boltzmann collision operator.

After the early work of Carleman ([11, 10]), Discrete Velocity Models - DVMs has been developed as a class of deterministic algorithms to resolve the Boltzmann equation numerically ([60, 7, 8, 5, 57, 6, 49, 9, 37]). They are based on a Cartesian grid in velocity and a discrete collision operator, which is a nonlinear system of conservation laws. In order to guarantee the convergence, the mesh size needs to be small and the truncated domain needs to be large. DVMs are then expensive. The models were proved to be consistent ([50, 28]), i.e. the discrete collision term could be seen as an approximation of the real collision operator. In [44, 51, 21] the approximate solutions are proved to converge weakly to the solution of the main equation by DiPerna-Lions theory ([24]). However, it is not easy to obtain an accuracy estimate of errors between the approximate solutions and the global solution on the entire non-truncated space.

The second deterministic approximation is the Fourier Spectral Methods - FSMs, which were first introduced in [53] inspired by spectral methods in fluid mechanics. The methods were later developed in several works, where a new way of accelerating the algorithms was also introduced ([54, 46, 56, 32, 47, 32, 46, 55, 52, 29, 33, 30, 39]). The analysis of the methods was provided in [31]. The idea of the methods is to truncate the Boltzmann equation on the velocity space and periodize the solution on this new bounded domain. To illustrate this idea, we consider the equation

$$\frac{\partial f}{\partial t} = Q^R(f, f), (x, v, t) \in \Omega \times (-R, R)^3 \times \mathbb{R}, \tag{1.6}$$

where Q^R is the truncated collision operator and f is periodic on $(-R,R)^3$. Since f is periodic on $(-R,R)^3$, we can write an approximation f_N of f in terms of Fourier series

$$f_N = \sum_{k=-(N,N,N)}^{(N,N,N)} \hat{f}_k \exp\left(-i\frac{\pi}{R}k.v\right),$$

which leads to a system of ODEs

$$\partial_t f_N = P_N Q^R(f_N, f_N).$$

The major problem with deterministic methods like DVMs and FSMs that use a fixed discretization in the velocity space is that the velocity space is approximated by a finite region. Physically, the velocity space is $\mathbb{R}^3$ and even if the initial condition is compactly supported, the collision operator does not preserve this property. The collision operator indeed spreads out the supports by a factor $\sqrt{2}$ (see [59]). Therefore in order to use both DVMs and FSMs, we have to impose nonphysical conditions to keep the supports of the solutions in the velocity space uniformly compact. For DVMs, we have to remove binary collisions which spread outside the bounded velocity space. This truncation breaks down the convolution structure of the collision operators. For FSMs, the convolution structure is perfectly preserved however we need to add nonphysical binary collisions by a periodized process. Notice that in [35], [36], Gamba and Tharkabhushanam proposed another class of FSMs, called Spectral-Lagrangian Methods (SLMs), to preserve the conservation of mass, momentum and energy on the numerical schemes. The method works very well and preserves several important properties of the solutions.
In order to be able to construct numerical schemes, it is natural that we require the computation domain to be bounded. The main idea of our new adaptive wavelet spectral method is the following: Consider the following change of variables φ from $\mathbb{R}^3$ to $(-1,1)^3$

$$v \to \bar{v} = \frac{v}{1+|v|}.$$

Apply this change of variables to the Boltzmann equation, we get a new formulation where the equation is considered on a bounded domain. The price that we need to pay after using this change of variable is its Jacobian, which is $\frac{1}{(1+|v|)^4}$. Notice that $(\sqrt{1+|v|^2})^{-4}$ is the momentum with order -4, which goes naturally into the physics of the equation. *This new formulation of the Boltzmann equation is discussed in details in Section* 2.
After having an equation on a bounded domain through a change of variables technique, we can construct a spectral algorithm similar as in [53]. However, different from [53], we do not use Fourier basis. We recall some quantitative properties of the Boltzmann equation that we want to preserve on the numerical schemes. Notice that these properties could not be preserved with previous strategies.

- **Maxwellian lower bounds** ([11, 59]): if the initial condition f_0 satisfies

$$\int_{\mathbb{R}^3} f_0(v)(1+|v|^2)dv < +\infty,$$

then

$$\forall t_0 > 0, \exists K_0 > 0, \exists A_0 > 0; t \geq t_0 \Longrightarrow \forall v \in \mathbb{R}^3, f(t,v) \geq K_0 \exp(-A_0|v|^2), \tag{1.7}$$

or

$$\forall t_0 > 0, \exists K_0 > 0, \exists A_0 > 0; \qquad t \geq t_0 \Longrightarrow \forall \bar{v} \in (-1,1)^3, f(t,\bar{v}) \geq K_0 \exp\left(-A_0 \left|\frac{\bar{v}}{1-|\bar{v}|}\right|^2\right).$$

- **Production of polynomial moments** ([58, 20, 64, 45]): if the initial condition f_0 satisfies
$$\int_{\mathbb{R}^3} f_0(v)(1+|v|^2)dv < +\infty,$$
then
$$\forall s \geq 2, \forall t_0 > 0, \sup_{t \geq t_0} \int_{\mathbb{R}^3} f(t,v)(1+|v|^s) < +\infty. \tag{1.8}$$

- **Propagation of exponential moments** ([4, 34, 1]): Assume that the initial data satisfies for some $s \in [\gamma, 2]$
$$\int_{\mathbb{R}^3} f_0(v)\exp(a_0|v|^s)dv \leq C_0,$$
then there are some constants $C, a > 0$ such that
$$\int_{\mathbb{R}^3} f(t,v)\exp(a|v|^s)dv < C. \tag{1.9}$$

Suppose that we approximate f by its truncated Fourier series
$$f_N = \sum_{k_1,k_2,k_3=(-N,-N,-N)}^{(N,N,N)} \hat{f}_k \exp(i\pi k.\bar{v}),$$
with
$$\hat{f}_k = \frac{1}{8}\int_{(-1,1)^3} f(\bar{v})\exp(-i\pi k.\bar{v})d\bar{v}.$$
We can see that the approximate solution f_N will not satisfy the properties that we mention above. The reason is that all components of the Fourier basis, i.e. the sin and cos functions are globally and smoothly defined on the whole interval $[-1,1]$ and they encounter singular problems at the extremes -1 and 1. This raises the need for a compactly supported wavelet basis and a wavelet filtering technique. In this paper, we only focus on the preservation of the Maxwellian lower bound. The wavelet filtering technique, whose role is to preserve the propagation of polynomial and exponential moments, will be presented in the second part of our work [61].
We preserve the good properties of both DVMs and FSMs: we are able to keep the convolution structure of the collision operators and do not have to impose a periodic boundary condition on the equation. The wavelet basis and the spectral method will be presented in section 3. More precisely, our spectral equation is defined in (3.12).
In order to understand better the mechanism of our nonlinear, adaptive spectral method, we now provide a different point of view based on Nonlinear Approximation Theory ([22, 19, 23]). The fundamental problem of approximation theory is to resolve a complicated function, by simpler, easier to compute functions called "the approximants". The main

idea of nonlinear approximation is that the approximants do not come from linear spaces but rather from nonlinear manifolds. An important application of nonlinear approximation is the adaptive finite element methods for elliptic equations originated in [3] and developed in [14, 13, 16]. These methods are based on the idea that fine meshes are put where the solutions are 'bad' and coarse meshes are set where the solutions are 'good'. Coming back to the Boltzmann equation, suppose that we use the Haar wavelet to solve the Boltzmann equation with the new variable $\bar{v}$ on $(-1,1)^3$. As we see later from (3.16) and (3.17), solving the Boltzmann equation with $\bar{v}$ on $(-1,1)^3$ means that we need construct a mesh by dividing $(-1,1)^3$ into 2^{3N} small cubes. To explain better our idea, suppose that we are in one dimension and we need to approximate the solution in a space spanned by the following orthogonal basis

$$\begin{cases} \phi_{N,k}(\bar{v}) = \chi_{(2^{-N}(2k-1),2^{-N}(2k+1))} \text{ for } k = 0, \pm 1, \dots, \pm(2^{N-1}-1), \\ \phi_{N,2^{N-1}}(\bar{v}) = \chi_{(-1,-1+2^{-N})\cup(1-2^{-N},1)}. \end{cases}$$

Let us make the change of variable $\bar{v} \to v = \frac{\bar{v}}{1-|\bar{v}|}$.

$$\begin{cases} \phi_{N,k}(v) = \chi_{\left(\min\left\{\frac{2k-1}{2^N-|2k-1|},\frac{2k+1}{2^N-|2k+1|}\right\},\max\left\{\frac{2k-1}{2^N-|2k-1|},\frac{2k+1}{2^N-|2k+1|}\right\}\right)} \text{ for } k = 0, \pm 1, \dots, \pm(2^{N-1}-1), \\ \phi_{N,2^{N-1}}(v) = \chi_{(-\infty,2^N-1)\cup(2^N-1,+\infty)}. \end{cases}$$

We can see that solving the Boltzmann equation in $\bar{v}$ on a uniform mesh in $(-1,1)$ is equivalent with solving the Boltzmann equation in v on a non-uniform mesh in $\mathbb{R}$. In other words, the role of the change of variables $v \to \bar{v}$ is to construct a new non-uniform mesh to approximate the Boltzmann equation. The non-uniform mesh has the following interesting property: the larger $|v|$ is the coarser the mesh is, and the smaller $|v|$ is the finer the mesh is. This is crucial, since properties (1.7), (1.8) and (1.9) play the role of a preconditioning analysis in our nonlinear approximation theory: the solution f of the Boltzmann equation behaves like a Maxwellian as $|v|$ large, which means that if $|v|$ is large, we only need a coarse mesh to represent the value of f. This is also the main difference between our approximation and classical ones. We can see from the spectral equations (3.12) and (3.13) that the mapping φ has a "support-stretching" effect: it maps the wavelet basis $\{\Phi_{N,k}\}$ supported in $(-1,1)^3$ to a new "nonlinear basis" $\{\Phi_{N,k}(\varphi)\}$ supported in the whole space, which are "the approximants" of our nonlinear approximation. Our method therefore gives a general frame work for solving kinetic integral equations (for example, the coagulation models [26], the quantum Boltzmann equations [27]) numerically: Suppose that we need to solve the following problem

$$\partial_t f(t,v) = \mathbb{Q}(f,f)(t,v), \text{ on } (0,T)\times\mathbb{R}^3,$$

$$f(0,v) = f_0(v) \text{ on } \mathbb{R}^3,$$

where $\mathbb{Q}$ is some bilinear form. We approximate f as

$$f_N(v) = \sum_{k=(-N,-N,-N)}^{(N,N,N)} a_k \Phi_{N,k}(\varphi(v)),$$

and get the approximate equation on the unknown $(a_k(t))_{k=(-N,-N,-N)}^{(N,N,N)}$

$$\frac{\partial a_k}{\partial t} = \sum_{i,j=(-N,-N,-N)}^{(N,N,N)} a_i a_j \langle \mathbb{Q}\left(\Phi_{N,i}(\varphi(v)), \Phi_{N,j}(\varphi(v))\right), \Phi_{N,k}(\varphi(v))\rangle.$$

Moreover, our approximation also provides a general view point for both DVMs and FSMs: FSMs and DVMs are special cases of our approximation using Fourier and Haar wavelet basis. If we take Haar wavelet basis as the spectral basis, our algorithm in this special case then gives an nonlinear, adaptive DVMs for Boltzmann equation, where no direct truncation is imposed and the convolution structure of the collision operator is preserved. Our new adaptive DVMs is then not expensive and it has a spectral accuracy. Therefore, both classical DVMs and FSMs could be seen as special linear and non-adaptive approximations in our theory. We will come back to this discussion at the end of subsection 3.2.
We also introduce a full new analysis to study theoretically our algorithm. Different with the periodized case ([31]) where the truncated Boltzmann collision operator is a bounded bilinear form and the projection of the collision operator onto the subspaces $P_N Q^R$ could be considered as a perturbation of Q^R with a small term $(Id - P_N)Q^R$, in our case, the collision operator is unbounded. Since $P_N Q$ does not preserve the symmetry of Q, the first problem is how we could preserve the conservation laws with this approximation

$$\int_{\mathbb{R}^3} P_N Q(f,f)dv = \int_{\mathbb{R}^3} P_N Q(f,f)v_i dv = \int_{\mathbb{R}^3} P_N Q(f,f)|v|^2 dv = 0.$$

Another problem is the preservation the "coercivity" property of the gain part of the collision operator

$$\int_{\mathbb{R}^3} P_N Q_+(f,f) f dv \geq \int_{\mathbb{R}^3} |v|^\gamma f^2 dv.$$

Notice that this is one of the main advantages of our approximation: preserve the coercivity property of the gain part of the collision operator. Approximation strategies using Fourier basis could not preserve this coercivity structure because of the effect of the Gibbs phenomenon. We construct the following scheme to study our algorithm theoretically.

- We approximate the projected operator $P_N Q$ by bounded operators $Q_{N,\lambda}$ and prove that the solutions $f_{N,\lambda}$ produced by the bounded operators are uniformly bounded in L^1 and L^2, moreover they are bounded from below by a Maxwellian.
- We prove that $f_{N,\lambda}$ converges to f_N as λ tends to infinity. Moreover f_N are uniformly bounded in L^1, L^2 and they are bounded from below by a Maxwellian.
- We perform a detailed analysis to prove that f_N converges to f which guarantees the convergence of the algorithm.

The structure of the paper is the following: In Section 2, we introduce the change of variable mapping and the new formulation of the Boltzmann equation. Section 3 is devoted to the contruction of the adaptive wavelet basis and the spectral method. We also provide some assumptions on the multiresolution analysis. Our main results are represented in Section 4. We prove that the algorithm converges; the energy, mass and momentum of the approximate solution converge to that of the original equation; moreover the approximate solution is bounded from below by a Maxwellian. These are the results of theorems 4.1 and 4.2.

2 A reformulation of the Boltzmann equation

Formally, Boltzmann collision operator has the properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^3} Q(f,f)dv = 0, \quad \int_{\mathbb{R}^3} Q(f,f)v dv = 0, \quad \int_{\mathbb{R}^3} Q(f,f)|v|^2 dv = 0,$$

and it satisfies the Boltzmann's H-theorem

$$-\frac{d}{dt}\int_{\mathbb{R}^3} f \log f dv = -\int_{\mathbb{R}^3} Q(f,f) log f dv \geq 0,$$

in which $-\int f \log f$ is defined as the *entropy* of the solution. A consequence of the Boltzmann's H-theorem is that any equilibrium distribution function has the form of a locally Maxwellian distribution

$$M(\rho,u,T) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|u-v|^2}{2T}\right),$$

where ρ, u, T are the *density, macroscopic velocity* and *temperature* of the gas

$$\rho = \int_{\mathbb{R}^3} f(v)dv, \quad u = \frac{1}{\rho}\int_{\mathbb{R}^3} vf(v)dv, \quad T = \frac{1}{3\rho}\int_{\mathbb{R}^3} |u-v|^2 f(v)dv.$$

We suppose that the initial datum f_0 satisfies $f_0(x,v) \geq 0$ on $\mathbb{R}^6$ and

$$\int_{\mathbb{R}^3} f_0(v)(1+|v|^2)dv < +\infty.$$

We refer to [12] and [63] for further details and discussions on the Boltzmann equation. In this work, we only consider the equation in $\mathbb{R}^3$ but the methodology would be exactly the same for other dimensions.
Different from [53], where a truncation technique is introduced in order to reduce the Boltzmann equation defined on the whole domain into an equation on a bounded domain, we introduce in this section a new formulation of the Boltzmann equation defined on $(-1,1)^3$ based on a change of variables technique. Let us define the following change of variables mapping

$$\varphi : \mathbb{R}^3 \to (-1,1)^3,$$

$$\varphi(v) = (\varphi_1(v_1), \varphi_2(v_2), \varphi_3(v_3)) = \left(\frac{v_1}{1+|v|}, \frac{v_2}{1+|v|}, \frac{v_3}{1+|v|}\right), \tag{2.1}$$

where we restrict our attention to the norm $|v| = \max\{|v_1|,|v_2|,|v_3|\}$ with $v = (v_1,v_2,v_3) \in \mathbb{R}^3$. The inverse mapping φ^{-1} of φ reads

$$\varphi^{-1} : (-1,1)^3 \to \mathbb{R}^3,$$

$$\varphi^{-1}(\bar{v}) = (\varphi_1(\bar{v}_1), \varphi_2(\bar{v}_2), \varphi_3(\bar{v}_3)) = \left(\frac{\bar{v}_1}{1-|\bar{v}|}, \frac{\bar{v}_2}{1-|\bar{v}|}, \frac{\bar{v}_3}{1-|\bar{v}|}\right).$$

The idea of our technique is to replace the variable v in $\mathbb{R}^3$ by a new variable in $(-1,1)^3$ through the mapping φ. Based on this idea, we define the new density function

$$g(t,\bar{v}) = f(t,\varphi^{-1}(\bar{v})),$$

where $\bar{v}$ is the new variable in $(-1,1)^3$.
With the notice that the Jacobian of the change of variable $\bar{v} \to v$ is $\frac{1}{(1+|v|)^4}$, we have

$$\int_{(-1,1)^3} |g(\bar{v})|^p (1-|\bar{v}|)^{-s-4} d\bar{v} = \int_{(-1,1)^3} |f(\varphi^{-1}(\bar{v}))|^p (1-|\bar{v}|)^{-s-4} d\bar{v}$$

$$= \int_{\mathbb{R}^3} |f(v)|^p (1+|v|)^{s+4} d(\varphi(v)) = \int_{\mathbb{R}^3} |f(v)|^p (1+|v|)^s dv.$$

Therefore if $f(v)$ belongs to L^1 with the weight $(1+|v|)^s$, then $g(\bar{v})$ belongs to L^1 with the weight $(1-|\bar{v}|)^{-s-4}$. Notice that there are several one-to-one mappings that map $\mathbb{R}^3$ to $(-1,1)^3$ however the above property makes us choose to work on φ.
We now define

$$L^p_s = \left\{ f \mid \int_{\mathbb{R}^3} |f(v)|^p (1+|v|)^{sp} dv < +\infty \right\},\ \mathcal{L}^p_s = \left\{ f \mid \int_{(-1,1)^3} |f(\bar{v})|^p (1-|\bar{v}|)^{-sp} d\bar{v} < +\infty \right\},$$

where p, s are real numbers. For further use, we also need

$$L^p(W) = \left\{ f \quad | \quad \int_{\mathbb{R}^3} |f(v)|^p W^p(v) dv < +\infty \right\},$$

$$\mathcal{L}^p(W') = \left\{ f \quad | \quad \int_{(-1,1)^3} |f(\bar{v})|^p (W'(\bar{v}))^p d\bar{v} < +\infty \right\},$$

where W, W' are some positive weights.
Moreover, we also need the notation

$$\langle v \rangle = \sqrt{1+|v|^2}, \quad \forall v \in \mathbb{R}^3.$$

The Boltzmann equation for g is now

$$\partial_t g(t,\bar{v}) = \int_{(-1,1)^3} \int_{\mathbb{S}^2} \frac{B(|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|, \sigma)}{(1-|\bar{v}_*|)^4} \tag{2.2}$$
$$\times \left[g\left(\varphi\left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \right.$$
$$\left. \times g\left(\varphi\left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) - g(\bar{v})g(\bar{v}_*) \right] d\sigma d\bar{v}_*.$$

Now define

$$h(t,\bar{v}) = g(t,\bar{v})(1-|\bar{v}|)^{-4},$$

which implies

$$\int_{(-1,1)^3} |h(\bar{v})| (1-|\bar{v}|)^{-s} d\bar{v} = \int_{\mathbb{R}^3} |f(v)| (1+|v|)^s dv.$$

This means if f belongs to L^1_s then h belongs to $\mathcal{L}^1_s$. Notice that we define $h(t,\bar{v}) = g(t,\bar{v})(1-|\bar{v}|)^{-4}$ to make our proof simpler, however the theoretical results remain the

same if $h(t,\bar{v}) = g(t,\bar{v})(1-|\bar{v}|)^{-n}$, with n being any constant in $\mathbb{R}$, n could be 0.
The Boltzmann equation for h then reads

$$\begin{aligned}
&\partial_t h(t,\bar{v}) = \int_{(-1,1)^3}\int_{\mathbb{S}^2} B(|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|,\sigma)\\
&\times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
&\left.\times h\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h(\bar{v})h(\bar{v}_*)\right]d\sigma d\bar{v}_*,
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
\mathcal{C}(\bar{v},\bar{v}_*,\sigma) \quad = \quad & \left[1-\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right]^4\\
&\times\left[1-\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right]^4\\
&\times(1-|\bar{v}|)^{-4}(1-|\bar{v}_*|)^{-4}.
\end{aligned} \tag{2.4}$$

Define

$$\mathcal{B}(\bar{v},\bar{v}_*,\sigma) = B(|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|,\sigma), \tag{2.5}$$

we get

$$\begin{aligned}
&\partial_t h(t,x,\bar{v}) = \int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\\
&\times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
&\left.\times h\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h(\bar{v})h(\bar{v}_*)\right]d\sigma d\bar{v}_*.
\end{aligned} \tag{2.6}$$

The initial datum is now defined

$$h_0(\bar{v}) = (1-|\bar{v}|)^{-4}f_0(\varphi^{-1}(\bar{v})),$$

then

$$\int_{(-1,1)^3} h_0(\bar{v})\left(1+\frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)d\bar{v} < +\infty.$$

Let us mention that though the two new formulations seem to be complicated, we only use them for theoretical purposes. Our spectral equation (3.12) is based on the former formulation of the equation.

3 Approximating the homogeneous Boltzmann equation: an adaptive spectral method

We will construct a wavelet basis for $L^2((-1,1)^3)$ in subsection 3.1. Our new spectral algorithm is defined in equation (3.12) of subsection 3.2. In subsection 3.3 we discuss about the assumption that we need for the multiresolution analysis and the wavelet filtering technique.

3.1 Wavelets for $L^2((-1,1)^3)$

We first construct a wavelet multiresolution analysis for $L^2((-1,1))$. Let ϕ be a *positive scaling function* which defines a *multiresolution analysis*, i.e., a ladder of embedded approximation subspaces of $L^2(\mathbb{R})$

$$\{0\} \to \dots V_1 \subset V_0 \subset V_{-1} \dots \to L^2(\mathbb{R})$$

such that $\phi_{j,k} = \{2^{-j/2}\phi(2^{-j}y-k)\}_{k\in\mathbb{Z}}$ constitutes an orthonormal basis for V_j. The *wavelet* ψ is built to characterize the missing details between two adjacent levels of approximation. More concretely, $\{\psi_{j,k}\}_{k\in\mathbb{Z}} = \{2^{-j/2}\psi(2^{-j}y-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j where

$$V_{j-1} = V_j \oplus W_j.$$

Multiresolution analysis is a frame work developed by Mallat [38] and Meyer [41], we refer to these two pioneering works or the books [18], [43] for more details, examples and proofs. We now follow exactly the construction in [18, Section 9.3] to build the same "periodized wavelets" for $L^2(-1,1)$. Notice that there are other ways besides this way (see [42], [15]). Suppose that the scaling function ϕ and the wavelet ψ have reasonable decays, for example $|\phi(y)|, |\psi(y)| \le C(1+|y|)^{-2-\epsilon}, \epsilon > 0$. Define

$$\phi^{per}_{j,k}(y) = \sum_{l\in\mathbb{Z}} \phi_{j,k}\left(\frac{y}{2}+l\right); \qquad \psi^{per}_{j.k}(y) = \sum_{l\in\mathbb{Z}} \psi_{j,k}\left(\frac{y}{2}+l\right);$$

and

$$V^{per}_j = \overline{Span\{\phi^{per}_{j,k}, k\in\mathbb{Z}\}}; \qquad W^{per}_j = \overline{Span\{\psi^{per}_{j,k}, k\in\mathbb{Z}\}}.$$

Similar as [18, Note 6, Chapter 9] we have

$$\sum_{l\in\mathbb{Z}} \phi\left(\frac{x}{2}+l\right) = 1,$$

which implies

$$\phi^{per}_{j,k} = 2^{-j/2}\sum_{l\in\mathbb{Z}} \phi(2^{-j-1}x - k + 2^{-j}l) = 2^{j/2} \text{ for } j \ge 0.$$

These facts mean V^{per}_j for $j \ge 0$ are one dimensional spaces of constant functions. Moreover, similar as [18, Note 7, Chapter 9] we have

$$\sum_{l\in\mathbb{Z}} \psi\left(\frac{x}{2}+l\right) = 1,$$

and $W^{per}_j = \{0\}$ for $j \ge 0$. As a consequence, we only need to consider the spaces V^{per}_j and W^{per}_j with $j \le 0$. According to the property of the multiresolution analysis V_j, $W_j \subset V_{j-1}$, then V^{per}_j, $W^{per}_j \subset V^{per}_{j-1}$. We also have that W^{per}_j and V^{per}_j are orthogonal

$$\begin{aligned} & \int_{-1}^{1} \psi^{per}_{j,k}(y)\phi^{per}_{j,k}(y)dy \\ = & \sum_{l,l'\in\mathbb{Z}} 2^{|j|}\int_{-1}^{1} \psi(2^{-j-1}y + 2^{-j}l - k)\phi(2^{-j-1}y + 2^{-j}l' - k')dy \end{aligned}$$

$$= \sum_{l,l'\in\mathbb{Z}} 2^{|j|}\int_{-1}^{1}\psi(2^{|j|-1}y+2^{|j|}(l-l')-k)\phi(2^{|j|-1}y-k')dy$$

$$= \sum_{r\in\mathbb{Z}} 2^{|j|}\int_{-1}^{1}\psi(2^{|j|-1}y+2^{|j|}r-k)\phi(2^{|j|-1}y-k')dy = 0.$$

Similarly, in W_j^{per}, we have also that $\psi_{j,k}^{per}$ and $\psi_{j,k'}^{per}$ are orthogonal. Since $\phi_{j,k+m2^{|j|}}^{per} = \phi_{j,k}^{per}$ $\forall m \in \mathbb{Z}$, then the spaces V_j^{per}, W_j^{per} are spanned by the $2^{|j|}$ functions obtained from $k = 0, 1, \dots, 2^{|j|-1}$.

We therefore have a ladder of multiresolution spaces

$$V_0^{per} \subset V_{-1}^{per} \subset V_{-2}^{per} \subset \dots \to L^2(-1,1)$$

with

$$W_0^{per} \oplus V_0^{per} = V_{-1}^{per} \dots$$

and $\{\phi_{0,0}^{per}\} \cup \{\psi_{j,k}^{per}; j \in -\mathbb{N}, k = 0, \dots, 2^{|j|}-1\}$ is an orthonormal basis of $L^2(-1,1)$.

Define by $S_j\varkappa$ the orthogonal projection of a function $\varkappa$ in $L^1(-1,1)$ onto V_j, similar as in [18, Section 9.3] we then have the following remarkable property, which is not true with a Fourier basis

$$\|S_j\varkappa\|_{L^\infty(-1,1)} \le C_S\|\varkappa\|_{L^\infty(-1,1)},$$

where C_S is a constant not depending on j and $\varkappa$.

We now construct a multiresolution analysis for $L^2((-1,1)^3)$. Define

$$\Psi_{\bar{j},k}^{per}(\bar{y}) = \psi_{j_1,k_1}^{per}(\bar{y}_1)\psi_{j_2,k_2}^{per}(\bar{y}_2)\psi_{j_3,k_3}^{per}(\bar{y}_3),$$

and

$$\Phi_{\bar{j},k}^{per}(\bar{y}) = \phi_{j_1,k_1}^{per}(\bar{y}_1)\phi_{j_2,k_2}^{per}(\bar{y}_2)\phi_{j_3,k_3}^{per}(\bar{y}_3),$$

where $\bar{j} = (j_1, j_2, j_3) \in (-\mathbb{N})^3$, $k = (k_1, k_2, k_3) \in \{0, \dots, 2^{|j|}-1\}^3$, $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in (-1,1)^3$. Then $\{\Phi_{0,0}^{per}\} \cup \{\Psi_{\bar{j},k}^{per}\}$ is an orthonormal basis of $L^2((-1,1)^3)$.

Set $j \in -\mathbb{N}$ and put

$$\mathcal{V}_{|j|} = Span\{\Phi_{|j|,k}(\bar{y}) = \Phi_{(j,j,j),k}^{per}(\bar{y}), k = (k_1,k_2,k_3) \in \{0, \dots, 2^{|j|}-1\}^3\}.$$

then

$$\overline{\cup_{|j|\in\mathbb{N}}\mathcal{V}_{|j|}} = L^2((-1,1)^3),$$

which is the ladder of multiresolution spaces for $L^2((-1,1)^3)$ we need.

Define by $P_{|j|}\varrho$ the orthogonal project of a function ϱ in $L^2((-1,1)^3)$ onto $\mathcal{V}_{|j|}$, we also have the following property

$$\|P_{|j|}\varrho\|_{L^\infty((-1,1)^3)} \le C_P\|\varrho\|_{L^\infty((-1,1)^3)}, \tag{3.1}$$

if $f \in L^2((-1,1)^3) \cap L^\infty((-1,1)^3)$ where C_P is a constant not depending on j or ϱ.

We also the assume that

$$\|P_{|j|}\varrho\|_{L^1((-1,1)^3)} \le C_P\|\varrho\|_{L^1((-1,1)^3)}, \tag{3.2}$$

which is true for some basis like Haar basis. Notice that since ϕ is a positive function, the following property is true

$$\varrho \ge 0 \Rightarrow P_{|j|}\varrho \ge 0. \tag{3.3}$$

3.2 The nonlinear approximation for the homogeneous Boltzmann equation

Definition 3.1. *Let ς be a function in $\mathcal{V}_N$, $N \in \mathbb{N}$ and*

$$\varsigma = \sum_{k=(0,0,0)}^{(2^N-1,2^N-1,2^N-1)} \varsigma_{N,k}\Phi_{N,k},$$

where

$$\varsigma_{N,k} = \int_{(-1,1)^3} \varsigma\Phi_{N,k}d\bar{v}.$$

Set $\mathfrak{A}_N$ to be the set of indices $\{k = (k_1, k_2, k_3) \quad | \quad 0 \le k_1, k_2, k_3 \le 2^N - 1\}$, and suppose that $\mathfrak{B}_N$ is a the set of indices k, such that the distance between the support of $\Phi_{N,k}$ and the boundary of $(-1,1)^3$ is 0. Define

$$F_N\varsigma = \sum_{k\in\mathfrak{A}_N\setminus\mathfrak{B}_N} \varsigma_{N,k}\Phi_{N,k}.$$

Since F_N is to remove wavelets containing the extreme points of $(-1,1)^3$, we can assume that $F_N\varsigma$ is supported in $(-\zeta_N,\zeta_N)^3$ with $0 < \zeta_N < 1$ and $F_N 1$ is the characteristic function of $(-\zeta_N,\zeta_N)^3$. Notice that if $\bar{v}$ belongs to $(-\zeta_N,\zeta_N)^3$, then $v = \varphi^{-1}(\bar{v})$ belongs to $\left(-\frac{\zeta_N}{1-\zeta_N}, \frac{\zeta_N}{1-\zeta_N}\right)^3$. For the sake of simplicity, we denote

$$\sum_{k\in\mathfrak{A}_N\setminus\mathfrak{B}_N} = \sum_{k=0}^{2^N-1}. \tag{3.4}$$

We also suppose that there exist a positive constant ϵ^* and an open bounded set $\mathcal{D} \subset \left(-\frac{\zeta_N}{1-\zeta_N}, \frac{\zeta_N}{1-\zeta_N}\right)^3$ (for N large enough) such that

$$f_0 > \epsilon^* \text{ in } \mathcal{D}, \tag{3.5}$$

where we could assume that $\mathcal{D}$.
Let N be a positive integer and define

$$h_N = \left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)^{-1} F_N P_N\left(\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)h\right),$$

where P_N is the orthogonal project onto the space $\mathcal{V}_N$ and F_N is defined in Definition 3.1. The reason that we multiply

$$\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)$$

with h before taking the projection P_N is that

$$\begin{aligned}\int_{(-1,1)^3} P_N\left[\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)h\right]d\bar{v} &= \int_{(-1,1)^3}\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)h(\bar{v})P_N(1)d\bar{v} \\ = \int_{(-1,1)^3}\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2}\right)h(\bar{v})d\bar{v} &= \int_{\mathbb{R}^3} f(1+|v|^2)dv, \end{aligned} \tag{3.6}$$

which means that we want to preserve the energy of the solution through the projection. We also denote

$$\tilde{h}_N = F_N P_N \left(\left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2} \right) h \right), \ \mathcal{P}_N = F_N P_N, \ \eta(\bar{v}) = \left(1 + \frac{|\bar{v}|^2}{(1-|\bar{v}|)^2} \right)^{-1}.$$

We therefore have

$$\begin{aligned}
& \partial_t \tilde{h}_N(t,\bar{v}) \\
= \;& Q_N(\tilde{h}_N,\tilde{h}_N) = Q_N^+(\tilde{h}_N,\tilde{h}_N) - Q_N^-(\tilde{h}_N,\tilde{h}_N) \\
:= \;& \mathcal{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma) \right. \\
& \times \left[\eta(\bar{v})^{-1} \mathcal{C}(\bar{v},\bar{v}_*,\sigma) \tilde{h}_N \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \right. \\
& \times \tilde{h}_N \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \\
& \times \eta \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \\
& \left. \left. \times \eta \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) - \tilde{h}_N(\bar{v}) \tilde{h}_N(\bar{v}_*) \eta(\bar{v}_*) \right] d\sigma d\bar{v}_* \right\},
\end{aligned} \tag{3.7}$$

or equivalently

$$\begin{aligned}
& \partial_t h_N(t,\bar{v}) \\
= \;& Q_N(\tilde{h}_N,\tilde{h}_N) = Q_N^+(\tilde{h}_N,\tilde{h}_N) - Q_N^-(\tilde{h}_N,\tilde{h}_N) \\
:= \;& \mathbb{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma) \right. \\
& \times \left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma) h_N \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \right. \\
& \left. \left. \times h_N \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) - h_N(\bar{v}) h_N(\bar{v}_*) \right] d\sigma d\bar{v}_* \right\},
\end{aligned} \tag{3.8}$$

where $\mathbb{P}_N$ is defined

$$\mathbb{P}_N(\varrho) = \eta \mathcal{P}_N(\eta^{-1}\varrho),$$

with some function ϱ, and

$$h_{0_N} = \mathbb{P}_N(h_0), \quad h_N = \eta \tilde{h}_N.$$

Suppose that

$$\tilde{h}_N = \sum_{k=0}^{2^N-1} a_{N,k} \Phi_{N,k}, \ \text{where } a_{N,k} = \int_{(-1,1)^3} \tilde{h}_N \Phi_{N,k} d\bar{v}.$$

Then (3.7) and (3.8) are equivalent with the following system of ODEs for $k \in \mathcal{A}_N \backslash \mathcal{B}_N$

$$\partial_t a_{N,k} = \int_{(-1,1)^3} \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma) \right.$$

$$
\begin{aligned}
&\times\Bigg[\mathcal{C}(\bar v,\bar v_*,\sigma)\left(\sum_{l=0}^{2^N-1} a_{N,l}\Phi_{N,l}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right)\\
&\times\left(\sum_{l'=0}^{2^N-1} a_{N,l'}\Phi_{N,l'}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right)\\
&\times\eta\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\\
&\times\eta\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\eta(\bar v)^{-1}\\
&-\left(\sum_{l=0}^{2^N-1} a_{N,l}\Phi_{N,l}(\bar v)\right)\left(\sum_{l'=0}^{2^N-1} a_{N,l'}\Phi_{N,l'}(\bar v_*)\right)\eta(\bar v_*)\Bigg]d\sigma d\bar v_*\Bigg\}\Phi_{N,k}d\bar v.
\end{aligned}
\tag{3.9}
$$

The resolution of this system gives an approximation of h. After solving the system (3.9), we can get a full solution in $\mathbb{R}^3$ by the following mapping

$$
f_N(v)=\tilde h_N(\varphi(v))(1+|v|)^{-4}\eta(v). \tag{3.10}
$$

However, system (3.9) is quite complicated and difficult to use in practical computations. We then introduce an equivalent form of it, which is easier to implement

$$
\begin{aligned}
\partial_t a_{N,k} \;=\; &\sum_{l,l'=0}^{2^N-1} a_{N,l}a_{N,l'}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)\times \qquad (3.11)\\
&\Bigg[\langle v\rangle^2\Phi_{N,l}(\varphi(v'_*))\frac{\langle v'_*\rangle^{-2}}{(1+|v'_*|)^4}\Phi_{N,l'}(\varphi(v'))\frac{\langle v'\rangle^{-2}}{(1+|v'|)^4}\\
&-\Phi_{N,l}(\varphi(v_*))\frac{\langle v_*\rangle^{-2}}{(1+|v_*|)^4}\Phi_{N,l'}(\varphi(v))\frac{1}{(1+|v|)^4}\Bigg]\Phi_{N,k}(\varphi(v))d\sigma dv_*dv,
\end{aligned}
$$

which gives an approximation of $f(v)(1+|v|)^4\langle v\rangle^2$. As we mention above, the weight $(1+|v|)^4$ is put just to make the proof simpler, therefore, in practical computations, we can drop it to get the following equivalent system

$$
\begin{aligned}
\partial_t a_{N,k} \;=\; &\sum_{l,l'=0}^{2^N-1} a_{N,l}a_{N,l'}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)\left[\frac{\Phi_{N,l}(\varphi(v'_*))}{\langle v'_*\rangle^2}\frac{\Phi_{N,l'}(\varphi(v'))}{\langle v'\rangle^2}\langle v\rangle^2\right. \qquad (3.12)\\
&\left.-\frac{\Phi_{N,l}(\varphi(v_*))}{\langle v_*\rangle^2}\Phi_{N,l'}(\varphi(v))\right]\Phi_{N,k}(\varphi(v))d\sigma dv_*dv,\quad \forall k\in\mathcal{A}_N\backslash\mathcal{B}_N,
\end{aligned}
$$

which is our **spectral equation** and numerical simulations could be done with this system. The resolution of this system gives us a direct approximation

$$
\sum_{k=0}^{2^N-1} a_{N,k}\Phi_{N,k}(\varphi(v)), \tag{3.13}
$$

of $f(v)\langle v\rangle^2$. This formulation also gives us a clearer understanding about the mapping φ: its role is to stretch the support of $\Phi_{N,l}$ from $(-1,1)^3$ to $\mathbb{R}^3$ to get a new "adaptive basis"

on the whole space, which are "the approximants" of our nonlinear approximation. Notice that the weight $\langle v\rangle^{-2}$ is put to preserve the energy of the solution due to (3.6).
As we mention in the introduction, if we choose ϕ to be the Haar scaling function, system (3.12) becomes a Discrete Velocity Model. However, different from classical ones, (3.12) has an adaptive mesh thanks to the mapping φ: the larger $|v|$ is, the coarser the mesh is, moreover it preserves the convolution structure of the collision operator. In other words, classical DVMs and Fourier-based spectral methods are in some sense non-adaptive cases of wavelet spectral approximations. *Notice that in* (3.9)*, we take the basis created by* ϕ*, but we can take the basis created by* ψ *as well and the analysis would remain the same.*
The existence and uniqueness of a solution of the equivalent systems (3.9), (3.11) and (3.12) is classical according to the theory of ODEs. The numerical resolution of (3.9) resolves the Boltzmann equation on the entire space with the same complexity with a normal truncated spectral method. Notice that one of the main advantages of adaptive, nonlinear approximations is that they are cheaper ([23, 22]).

Proposition 3.1. *The system* (3.9) *has a unique solution* $\{a_{N,k}\}$ *with* $a_{N,k}\in C^1(0,+\infty)$ $\forall k\in\mathcal{A}_N\backslash\mathcal{B}_N$.

3.3 Assumptions on the multiresolution analysis and the filter

3.3.1 Energy preserving property

Assumption 3.1. *Define* $\kappa=\eta(\bar v)^{-1}\mathcal{P}_N\chi_{(-1,1)^3}$*, where* $\chi_{(-1,1)^3}$ *is the characteristic function of* $(-1,1)^3$*. Set* $\varkappa(v)=\kappa(\varphi(v))$*, where* φ *is the change of variables mapping defined in* (2.1)*. In order to preserve the energy of the approximate solution, we impose the following assumption on* $\mathcal{P}_N$

$$\varkappa(v'_*)+\varkappa(v')-\varkappa(v)-\varkappa(v_*)\le 0,\quad \forall(v,v_*)\in\left(-\frac{\zeta_N}{1-\zeta_N},\frac{\zeta_N}{1-\zeta_N}\right)^6. \tag{3.14}$$

We now explain why this assumption is needed in order to preserve the energy of the the approximate solution. Take $\eta(\bar v)^{-1}$ as a test function for (3.7)

$$\begin{aligned}
&\int_{(-1,1)^3}\partial_t h_N(t,\bar v)\eta(\bar v)^{-1} \qquad\qquad (3.15)\\
=&\int_{(-1,1)^3}\eta(\bar v)^{-1}\mathbb{P}_N\Bigg\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar v,\bar v_*,\sigma)\\
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)-h_N(\bar v)h_N(\bar v_*)\right]d\sigma d\bar v_*\Bigg\}d\bar v\\
=&\int_{(-1,1)^3}\mathcal{P}_N\Bigg\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\eta(\bar v)^{-1}\mathcal{B}(\bar v,\bar v_*,\sigma)\\
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)-h_N(\bar v)h_N(\bar v_*)\right]d\sigma d\bar v_*\Bigg\}d\bar v
\end{aligned}$$

$$
\begin{aligned}
= & \int_{(-1,1)^3}\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right.\\
& \times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
& \times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\\
& \left.\left.-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\}\eta(\bar{v})^{-1}\mathcal{P}_N\chi(-1,1)^3d\bar{v}\\
= & \int_{(-1,1)^3}\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right.\\
& \times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right.\\
& \times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\\
& \left.\left.-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\}\kappa(\bar{v})d\bar{v}.
\end{aligned}
$$

Define

$$f_N(v)=h_N(\varphi(v))(1+|v|)^{-4},$$

then (3.15) is transformed into

$$
\begin{aligned}
& \frac{d}{dt}\int_{\mathbb{R}^3}(1+|v|^2)f_N dv\\
= & \int_{\mathbb{R}^6\times\mathbb{S}^2}B(|v-v_*|,\sigma)[f'_{N*}f'_N-f_{N*}f_N]\varkappa(v)d\sigma dv_*dv\\
= & \frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2}B(|v-v_*|,\sigma)f_{N*}f_N[\varkappa(v'_*)+\varkappa(v')-\varkappa(v_*)-\varkappa(v)]d\sigma dv_*dv\\
\leq & \ 0,
\end{aligned}
$$

if the assumption 3.1 is satisfied. Notice that we only need (3.14) on $\left(-\frac{\zeta_N}{1-\zeta_N},\frac{\zeta_N}{1-\zeta_N}\right)^6$ since if (v,v_*) lies outside this interval, $f_{N*}f_N$=0. A consequence of this inequality is that the energy of the approximate solution is decreasing

$$\int_{\mathbb{R}^3}(1+|v|^2)f_N(t)dv\leq\int_{\mathbb{R}^3}(1+|v|^2)f_N(0)dv.$$

Later, we will prove that the mass, momentum and energy of the approximate solution converge to the mass, momentum and energy of the exact solution.
We now point out an example which satisfies our assumption 3.1. Let us recall the simplest scaling function: Haar function (see [18])

$$\phi(y)=\begin{cases}1 \text{ for } -\frac{1}{2}\leq y\leq\frac{1}{2},\\ 0 \text{ otherwise }.\end{cases}\tag{3.16}$$

The corresponding $\phi^{per}_{j,k}$ are

$$\begin{cases}\phi^{per}_{j,k}(y)=2^{|j|-1}\chi_{(2^{-|j|}(2k-1),2^{-|j|}(2k+1))} \text{ for } k=0,\pm1,\dots,\pm(2^{|j|-1}-1),\\ \phi^{per}_{j,2^{|j|-1}}(y)=2^{|j|-1}\chi_{(-1,-1+2^{-|j|})\cup(1-2^{-|j|},1)}.\end{cases}\tag{3.17}$$

Then

$$\Phi_{|j|,k}(\bar{y}) = \Phi^{per}_{j,k}(\bar{y}) = \phi^{per}_{j,k_1}(\bar{y}_1)\phi^{per}_{j,k_2}(\bar{y}_2)\phi^{per}_{j,k_3}(\bar{y}_3),$$

where $k = (k_1, k_2, k_3) \in \{-2^{|j|-1}+1, \ldots, 2^{|j|-1}\}^3$, $j \in -\mathbb{N}$, and

$$\mathcal{V}_{|j|} = \{\Phi_{|j|,k}(\bar{y}), k = (k_1, k_2, k_3) \in \{-2^{|j|-1}+1, \ldots, 2^{|j|-1}\}^3\}.$$

Let $\hat{k}_{|j|}$ be $2^{|j|-1}-1$. Let ς be any function in $\mathcal{V}_{|j|}$, $j \in -\mathbb{N}$ and

$$\varsigma = \sum_{k=(-2^{|j|-1}+1,-2^{|j|-1}+1,-2^{|j|-1}+1)}^{(2^{|j|-1},2^{|j|-1},2^{|j|-1})} \varsigma_{|j|,k}\Phi_{|j|,k} =: \sum_{k=-2^{|j|-1}+1}^{2^{|j|-1}} \varsigma_{|j|,k}\Phi_{|j|,k},$$

where

$$\varsigma_{|j|,k} = \int_{(-1,1)^3} \varsigma\Phi_{|j|,k} d\bar{v},$$

define the

$$F_{|j|}\varsigma = \sum_{k=(-\hat{k}_{|j|},-\hat{k}_{|j|},-\hat{k}_{|j|})}^{(\hat{k}_{|j|},\hat{k}_{|j|},\hat{k}_{|j|})} \varsigma_{|j|,k}\Phi_{|j|,k} =: \sum_{k=-\hat{k}_{|j|}}^{\hat{k}_{|j|}} \varsigma_{|j|,k}\Phi_{|j|,k}. \tag{3.18}$$

In other words, the $F_{|j|}$ eliminates all of the components with indices $k = (k_1, k_2, k_3)$ where $\max\{|k_1|, |k_2|, |k_3|\} > \hat{k}_{|j|}$.

Proposition 3.2. *Let N be a positive integer. Suppose that we take the Haar function* (3.16) *as the scaling function for the multiresolution analysis, $\{\Phi_{N,k}\}$ is a basis for $\mathcal{V}_N$ and F_N is defined by* (3.18)*. Then $\mathcal{P}_N = F_N P_N$ satisfies assumption* 3.1.

Proof. First, we can see directly that

$$P_N\chi_{(-1,1)^3} = \chi_{(-1,1)^3},$$

and

$$\mathcal{P}_N\chi_{(-1,1)^3} = \chi_{(-2^{-N}(2\hat{k}_N+1),2^{-N}(2\hat{k}_N+1))^3},$$

which implies

$$\kappa(\bar{v}) = \eta(\bar{v})^{-1}\chi_{(-2^{-N}(2\hat{k}_N+1),2^{-N}(2\hat{k}_N+1))^3}.$$

Notice that if

$$|\bar{v}| = \left|\frac{v}{1+|v|}\right| \leq 2^{-N}(2\hat{k}_N+1),$$

then

$$|v| \leq \frac{2\hat{k}_N+1}{2^N - 2\hat{k}_N - 1}.$$

This leads to

$$\varkappa(v) = \kappa(\varphi(v)) = (1+|v|^2)\chi_{\left(-\frac{2\hat{k}_N+1}{2^N-2\hat{k}_N-1},\frac{2\hat{k}_N+1}{2^N-2\hat{k}_N-1}\right)^3},$$

where we recall that $|v| = |(v_1, v_2, v_3)| = \max\{|v_1|, |v_2|, |v_3|\}$. Inequality (3.14) follows directly from the above formula for $\varkappa(v)$. $\square$

3.3.2 Coercivity preserving property

Assumption 3.2. *Let N be a positive integer and ϑ, ϑ' be two positive functions in $L^2((-1,1)^3)$. Define*

$$\vartheta_N = \mathcal{P}_N\vartheta, \text{ and } \vartheta'_N = \mathcal{P}_N\vartheta'.$$

Let s be a constant. We impose the following assumption on the multiresolution analysis and F_N: There exist constants N_0, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ not depending on ϑ, ϑ' such that

$$\begin{aligned} \forall N > N_0, \qquad & \mathcal{K}_1(1-|\bar{v}|)^s \geq \mathcal{P}_N((1-|\bar{v}|)^s) \geq \mathcal{K}_2(1-|\bar{v}|)^s \text{ on } [-\zeta_N,\zeta_N]^3, \\ \text{and} \qquad & \mathcal{K}_3\vartheta_N\vartheta'_N \geq \mathcal{P}_N(\vartheta_N\vartheta') \geq \mathcal{K}_4\vartheta_N\vartheta'_N. \end{aligned} \tag{3.19}$$

We now explain the meaning of this assumption. Suppose we take $\eta(\bar{v})^{-1}h_N(t,\bar{v})(1-|\bar{v}|)^s$ as a test function for (3.8)

$$\begin{aligned} & \int_{(-1,1)^3} \eta(\bar{v})^{-1}(1-|\bar{v}|)^s\partial_t h_N h_N \\ = & \int_{(-1,1)^3} \eta(\bar{v})^{-1}(1-|\bar{v}|)^s h_N \mathbb{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right. \\ & \times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right. \\ & \left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\}d\bar{v} \\ = & \int_{(-1,1)^3} \mathcal{P}_N((1-|\bar{v}|)^s h_N)\left\{\eta(\bar{v})^{-1}\int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right. \\ & \times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right. \\ & \left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)-h_N(\bar{v})h_N(\bar{v}_*)\right]d\sigma d\bar{v}_*\right\}d\bar{v} \\ = & \int_{(-1,1)^3} \mathcal{P}_N((1-|\bar{v}|)^s h_N)\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2} \eta(\bar{v})^{-1}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right. \\ & \times\left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma)h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right. \\ & \left.\left.\times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\right]\right\}d\sigma d\bar{v}_* d\bar{v} \\ & -\int_{(-1,1)^6\times\mathbb{S}^2} \eta(\bar{v})^{-1}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)h_N(\bar{v})h_N(\bar{v}_*)\mathcal{P}_N((1-|\bar{v}|)^s h_N)d\sigma d\bar{v}_* d\bar{v}. \end{aligned}$$

By assumption 3.2, the last term of the above equation could be bounded in the following way

$$\begin{aligned} & \mathcal{K}_2\int_{(-1,1)^6\times\mathbb{S}^2} \eta(\bar{v})^{-1}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)h_N^2(\bar{v})h_N(\bar{v}_*)(1-|\bar{v}|)^s d\sigma d\bar{v}_* d\bar{v} \\ \leq & \int_{(-1,1)^6\times\mathbb{S}^2} \eta(\bar{v})^{-1}\mathcal{B}(\bar{v},\bar{v}_*,\sigma)h_N(\bar{v})h_N(\bar{v}_*)\mathcal{P}_N((1-|\bar{v}|)^s h_N)d\sigma d\bar{v}_* d\bar{v} \end{aligned}$$

$$\leq \quad \mathcal{K}_1 \int_{(-1,1)^6 \times \mathbb{S}^2} \eta(\bar{v})^{-1} \mathcal{B}(\bar{v}, \bar{v}_*, \sigma) h_N^2(\bar{v}) h_N(\bar{v}_*) (1 - |\bar{v}|)^s d\sigma d\bar{v}_* d\bar{v}.$$

Define f_N as in (3.10), we can transform the above equation into

$$\begin{aligned}
&\mathcal{K}_2 \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) f_N^2 f_{N*} (1 + |v|^2)(1 + |v|)^{4-s} d\sigma dv_* dv \\
\leq \quad & \int_{(-1,1)^6 \times \mathbb{S}^2} \eta(\bar{v})^{-1} \mathcal{B}(\bar{v}, \bar{v}_*, \sigma) h_N(\bar{v}) h_N(\bar{v}_*) \mathcal{P}_N((1 - |\bar{v}|)^s h_N) d\sigma d\bar{v}_* d\bar{v} \\
\leq \quad & \mathcal{K}_1 \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) f_N^2 f_{N*} (1 + |v|^2)(1 + |v|)^{4-s} d\sigma dv_* dv.
\end{aligned}$$

This estimate is crucial in our $\mathcal{L}^2$ estimate for h_N, since it preserves the following property of the Boltzmann equation

$$\begin{aligned}
&\int_{\mathbb{R}^6} \int_{\mathbb{S}^2} B(|v - v_*|, \sigma) f_N^2 f_{N*} (1 + |v|^2)(1 + |v|)^{4-s} d\sigma dv_* dv \\
&\qquad \geq C \int_{\mathbb{R}^3} f_N^2 (1 + |v|^2)(1 + |v|)^{4+\gamma-s} dv.
\end{aligned}$$

We will discuss about this in more details in the convergence theory of the algorithm.

Proposition 3.3. *Let N be a positive integer. Suppose that we take the Haar function* (3.16) *as the scaling function for the multiresolution analysis, $\{\Phi_{N,k}\}$ is a basis for $\mathcal{V}_N$ and F_N is the filter defined by* (3.18)*. Then $\mathcal{P}_N = F_N P_N$ satisfies assumption* 3.2.

Remark 3.1. *In both propositions 3.2 and 3.3, we can always take $\hat{k}_N$ to be $2^{N-1} - 1$, and the filter F_N only removes the components containing $\phi^{per}_{-N,2^{N-1}}$.*

Proof. Since the supports of $\Phi_{N,k}$ are disjoint and $\Phi_{N,k}\Phi_{N,k} = \Phi_{N,k}$,then

$$\vartheta_N \vartheta'_N = \mathcal{P}_N(\vartheta_N \vartheta').$$

Equation (3.19) is now equivalent with

$$\mathcal{K}_1(1 - |\bar{v}|)^s \geq \mathcal{P}_N((1 - |\bar{v}|)^s) \geq \mathcal{K}_2(1 - |\bar{v}|)^s \text{ on } [-2^{-N}(2\hat{k}_N + 1), 2^{-N}(2\hat{k}_N + 1)]^3. \quad (3.20)$$

Set

$$\mathcal{P}_N[(1 - |\bar{v}|)^s] = \sum_{k=-\hat{k}_N}^{\hat{k}_N} d_k \Phi_{N,k},$$

where

$$d_k = \int_{(-1,1)^3} (1 - |\bar{v}|)^s \Phi_{N,k} d\bar{v},$$

we consider the coefficient $d_k \neq 0$ of $\mathcal{P}_N[(1 - |\bar{v}|)^s]$. Suppose that

$$\Phi_{N,k}(\bar{v}) = \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3),$$

with $|k_1| \geq |k_2| \geq |k_3|$. Hence, $|\bar{v}| = \max\{|\bar{v}_1|, |\bar{v}_2|, |\bar{v}_3|\} \in [2^{-N}(2|k_1| - 1), 2^{-N}(2|k_1| + 1)]$ if $k_1 \neq 2^{N-1}$ and $|\bar{v}| \in [0, 2^{-N}]$ if $k_1 = 2^{N-1}$. Therefore $1 - |\bar{v}| \in [1 - 2^{-N}(2|k_1| + 1), 1 -$

$2^{-N}(2|k_1|-1)]$ if $k_1 \neq 2^{N-1}$ and $1-|\bar{v}| \in [1-2^{-N}, 1]$ if $k_1 = 2^{N-1}$.
Since $k_1 \neq 2^{N-1}$ and $(1-|\bar{v}|) \in [(1-2^{-N}(2|k_1|+1)), (1-2^{-N}(2|k_1|-1))]$.

$$\begin{aligned} 1 &\leq \frac{\max_{|\bar{v}|\in[2^{-N}(2|k_1|-1),2^{-N}(2|k_1|+1)]}(1-|\bar{v}|)^s}{\min_{|\bar{v}|\in[2^{-N}(2|k_1|-1),2^{-N}(2|k_1|+1)]}(1-|\bar{v}|)^s} \\ &\leq \left(\frac{2^N-2|k_1|+1}{2^N-2|k_1|-1}\right)^{|s|} \leq 3^{|s|}, \end{aligned} \tag{3.21}$$

which imples

$$\begin{aligned} & \int_{(-1,1)^3} (1-|\bar{v}|)^s \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3)d\bar{v} \\ \geq & \int_{(-1,1)^3} \frac{1}{3^{|s|}} \max_{1-|\bar{v}|\in[1-2^{-N}(2|k_1|+1),1-2^{-N}(2|k_1|-1)]} (1-|\bar{v}|)^s \times \\ & \times \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3)d\bar{v} \\ \geq & \frac{1}{3^{|s|}} \max_{1-|\bar{v}|\in[1-2^{-N}(2|k_1|+1),1-2^{-N}(2|k_1|-1)]} (1-|\bar{v}|)^s \geq \frac{1}{3^{|s|}}(1-|\bar{v}|)^s, \end{aligned}$$

for all $\bar{v}$ in the support of $\Phi_{N,k}$. We deduce from this inequality that

$$d_k\Phi_{N,k} \geq \frac{1}{3^{|s|}}(1-|\bar{v}|)^s, \tag{3.22}$$

for all $\bar{v}$ in the support of $\Phi_{N,k}$. Similarly, we also get

$$\begin{aligned} & \int_{(-1,1)^3} (1-|\bar{v}|)^s \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3)d\bar{v} \\ \leq & \int_{(-1,1)^3} 3^{|s|} \min_{1-|\bar{v}|\in[1-2^{-N}(2|k_1|+1),1-2^{-N}(2|k_1|-1)]} (1-|\bar{v}|)^s \times \\ & \times \phi^{per}_{-N,k_1}(\bar{v}_1)\phi^{per}_{-N,k_2}(\bar{v}_2)\phi^{per}_{-N,k_3}(\bar{v}_3)d\bar{v} \\ \leq & 3^{|s|} \min_{1-|\bar{v}|\in[1-2^{-N}(2|k_1|+1),1-2^{-N}(2|k_1|-1)]} (1-|\bar{v}|)^s \leq 3^{|s|}(1-|\bar{v}|)^s, \end{aligned}$$

for all $\bar{v}$ in the support of $\Phi_{N,k}$, and

$$d_k\Phi_{N,k} \leq 3^{|s|}(1-|\bar{v}|)^s, \tag{3.23}$$

for all $\bar{v}$ in the support of $\Phi_{-N,k}$. We deduce from inequality (3.22) and (3.23) that

$$3^{|s|}(1-|\bar{v}|)^s \geq \mathcal{P}_N((1-|\bar{v}|)^s) \geq \frac{1}{3^{|s|}}(1-|\bar{v}|)^s.$$

□

4 Convergence theory of the adaptive spectral method

Consider again equation (3.8)

$$\partial_t h_N(t,\bar{v}) = \mathbb{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\right.$$

$$
\times \left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma) h_N \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \right. \tag{4.1}
$$
$$
\left. \times h_N \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) - h_N(\bar{v}) h_N(\bar{v}_*) \right] d\sigma d\bar{v}_* \Bigg\}.
$$

Below is the plan for the proof of the convergence of h_N to the solution h of (2.3) as N tends to infinity

- The solution h_N of (4.1) is positive and uniformly bounded with respect to N in $\mathcal{L}^1_2$ norm.
- h_N has a Maxellian lower bound: for all $t_0 > 0$, such that for all N large enough, there exist $\hat{C}_1$, $\hat{C}_2 > 0$, and for all $\bar{v}$ in the support of h_N

$$
h_N(t,\bar{v}) \geq \hat{C}_1 \exp\left(-\hat{C}_2 \left| \frac{|\bar{v}|}{1-|\bar{v}|} \right|^2 \right), \quad \forall t > t_0.
$$

- h_N is uniformly bounded with respect to N in $\mathcal{L}^2_{-4}$ norm.
- The approximate solution h_N converges to the solution h of (2.3) as N tends to infinity.

In order to prove the positivity and boundedness of h_N in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ norms, we consider the approximate Boltzmann equation with a bounded collision kernel as in [2] and [24]

$$
\begin{aligned}
\partial_t h_{N,\lambda}(t,\bar{v}) &= Q_{N,\lambda}(h_{N,\lambda}, h_{N,\lambda}) = Q^+_{N,\lambda}(h_{N,\lambda}, h_{N,\lambda}) - Q^-_{N,\lambda}(h_{N,\lambda}, h_{N,\lambda}) \\
&:= \mathbb{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} B_\lambda(|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|, \sigma) \right. \qquad (4.2) \\
&\quad \times \left[\mathcal{C}(\bar{v},\bar{v}_*,\sigma) h_{N,\lambda} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \right. \\
&\quad \times h_{N,\lambda} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right) \right) \\
&\quad \left. \left. - h_{N,\lambda}(\bar{v}) h_{N,\lambda}(\bar{v}_*) \right] d\sigma d\bar{v}_* \right\},
\end{aligned}
$$

with

$$
B_\lambda(|u|,\sigma) := |(u \wedge \lambda)|^\gamma b(\cos\theta) = |\min\{u,\lambda\}|^\gamma b(\cos\theta),
$$

where λ is a positive constant. For the sake of simplicity, we denote

$$
\mathcal{B}_\lambda(\bar{v},\bar{v}_*,\sigma) = B_\lambda\left(|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|, \sigma \right).
$$

Since (4.2) is a system of ODEs, it admits a unique strong solution which is continuous in time. In this section we always assume that N and λ are sufficiently large. We will prove that $h_{N,\lambda}$ is bounded in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ and bounded from below by a Maxwellian uniformly with respect to λ. By Nagumo's criterion, Dunford-Pettis theorem and Smulian theorem (see [25] and [40]), h_N is bounded in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ and bounded from below by a Maxwellian. The convergence of the algorithm then follows after some technical computations.

4.1 Positivity and $\mathcal{L}^1$ estimate of $h_{N,\lambda}$

Proposition 4.1. *The solution $h_{N,\lambda}(t)$ of (4.2) is positive for all time t in $\mathbb{R}_+$, moreover*

$$\|h_{N,\lambda}(t)\|_{\mathcal{L}^1(\eta^{-1})} \leq \|h_{0_N}\|_{\mathcal{L}^1(\eta^{-1})}, \forall t \in \mathbb{R}_+.$$

Proof. First, equation (4.2) implies

$$\begin{aligned}
&\int_{(-1,1)^3} \partial_t |h_{N,\lambda}| \eta^{-1} d\bar{v} \\
\leq\ & \int_{(-1,1)^3} \eta^{-1} \mathbb{P}_N \Bigg\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar{v}, \bar{v}_*, \sigma) \\
& \times \left[\mathcal{C}(\bar{v}, \bar{v}_*, \sigma) \left| h_{N,\lambda} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right)\right) \right| \right. \\
& \times \left| h_{N,\lambda} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right)\right) \right| \\
& \left. + |h_{N,\lambda}(\bar{v})||h_{N,\lambda}(\bar{v}_*)| \right] d\sigma d\bar{v}_* \Bigg\} d\bar{v} \\
\leq\ & \int_{(-1,1)^3} \eta^{-1} \mathbb{P}_N \Bigg\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar{v}, \bar{v}_*, \sigma) \\
& \times \left[\mathcal{C}(\bar{v}, \bar{v}_*, \sigma) \left| h_{N,\lambda} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right)\right) \right| \right. \\
& \times \left| h_{N,\lambda} \left(\varphi \left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2} \right)\right) \right| \\
& \left. - |h_{N,\lambda}(\bar{v})||h_{N,\lambda}(\bar{v}_*)| \right] d\sigma d\bar{v}_* \Bigg\} d\bar{v} \\
& + 2 \int_{(-1,1)^3} \eta^{-1} \mathbb{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar{v}, \bar{v}_*, \sigma) |h_{N,\lambda}(\bar{v})||h_{N,\lambda}(\bar{v}_*)| d\sigma d\bar{v}_* \right\} d\bar{v} \\
\leq\ & 2 \int_{(-1,1)^3} \eta^{-1} \mathbb{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar{v}, \bar{v}_*, \sigma) |h_{N,\lambda}(\bar{v})||h_{N,\lambda}(\bar{v}_*)| d\sigma d\bar{v}_* \right\} d\bar{v},
\end{aligned}$$

where the last inequality follows from assumption 3.1 and (3.15).
We deduce from the above equation and (3.2) that

$$\begin{aligned}
&\frac{d}{dt} \int_{(-1,1)^3} |h_{N,\lambda}| \eta^{-1} d\bar{v} \\
\leq\ & C \int_{(-1,1)^3} \mathcal{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \eta^{-1} \mathcal{B}_\lambda(\bar{v}, \bar{v}_*, \sigma) |h_{N,\lambda}(\bar{v})||h_{N,\lambda}(\bar{v}_*)| d\sigma d\bar{v}_* \right\} d\bar{v} \\
\leq\ & C \int_{(-1,1)^3} \int_{(-1,1)^3} \int_{\mathbb{S}^2} \eta^{-1} \mathcal{B}_\lambda(\bar{v}, \bar{v}_*, \sigma) |h_{N,\lambda}(\bar{v})||h_{N,\lambda}(\bar{v}_*)| d\sigma d\bar{v}_* d\bar{v} \\
\leq\ & C \left[\int_{(-1,1)^3} |h_{N,\lambda}| \eta^{-1} d\bar{v} \right]^2,
\end{aligned}$$

where the last constant C depends on λ, which implies

$$\|h_{N,\lambda}\|_{\mathcal{L}^1(\eta^{-1})} \leq \frac{\|h_{0_N}\|_{\mathcal{L}^1(\eta^{-1})}}{1 - C\|h_{0_N}\|_{\mathcal{L}^1(\eta^{-1})} t}. \tag{4.3}$$

Set $M = 2\|h_{0_N}\|_{\mathcal{L}^1(\eta^{-1})}$, then put $\tau < \frac{1}{2C\|h_{0_N}\|_{\mathcal{L}^1(\eta^{-1})}}$, we get

$$\forall t \in [0,\tau] \qquad \|h_{N,\lambda}\|_{\mathcal{L}^1(\eta^{-1})} \leq M.$$

We now prove that on $[0,\tau]$, $h_{N,\lambda}$ is positive. Split $h_{N,\lambda}$ as $h_{N,\lambda} = h_{N,\lambda,+} - h_{N,\lambda,-}$ where $h_{N,\lambda,+} = \max\{h_{N,\lambda}, 0\}$ and $h_{N,\lambda,-} = \max\{-h_{N,\lambda}, 0\}$, we get

$$\begin{aligned} Q^+_{N,\lambda}(h_{N,\lambda}, h_{N,\lambda}) &= Q^+_{N,\lambda}(h_{N,\lambda,+} - h_{N,\lambda,-}, h_{N,\lambda,+} - h_{N,\lambda,-}) \\ &\geq -Q^+_{N,\lambda}(h_{N,\lambda,+}, h_{N,\lambda,-}) - Q^+_{N,\lambda}(h_{N,\lambda,-}, h_{N,\lambda,+}). \end{aligned} \tag{4.4}$$

We consider the term

$$\begin{aligned}
&\left\| Q^+_{N,\lambda}(h_{N,\lambda,+}, h_{N,\lambda,-})\eta^{-1} \right\|_{\mathcal{L}^\infty_{-4}} \\
= \ & \Bigg\| (1-|\bar v|)^4 \mathcal{P}_N \Bigg\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar v, \bar v_*, \sigma)\eta^{-1} \\
& \times \mathcal{C}(\bar v, \bar v_*, \sigma) \left| h_{N,\lambda,+}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| \\
& \times \left| h_{N,\lambda,-}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| d\sigma d\bar v_* \Bigg\} \Bigg\|_{\mathcal{L}^\infty} \\
\leq \ & C \Bigg\| \mathcal{P}_N[(1-|\bar v|)^4] \mathcal{P}_N \Bigg\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar v, \bar v_*, \sigma)\eta^{-1} \\
& \times \mathcal{C}(\bar v, \bar v_*, \sigma) \left| h_{N,\lambda,+}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| \\
& \times \left| h_{N,\lambda,-}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| d\sigma d\bar v_* \Bigg\} \Bigg\|_{\mathcal{L}^\infty} \\
\leq \ & C \Bigg\| \mathcal{P}_N \Bigg\{ \mathcal{P}_N[(1-|\bar v|)^4] \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar v, \bar v_*, \sigma)\eta^{-1} \\
& \times \mathcal{C}(\bar v, \bar v_*, \sigma) \left| h_{N,\lambda,+}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| \\
& \times \left| h_{N,\lambda,-}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| d\sigma d\bar v_* \Bigg\} \Bigg\|_{\mathcal{L}^\infty} \\
\leq \ & C \Bigg\| (1-|\bar v|)^4 \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}_\lambda(\bar v, \bar v_*, \sigma)\eta^{-1} \\
& \times \mathcal{C}(\bar v, \bar v_*, \sigma) \left| h_{N,\lambda,+}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} - \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| \\
& \times \left| h_{N,\lambda,-}\left(\varphi\left(\frac{\varphi^{-1}(\bar v) + \varphi^{-1}(\bar v_*)}{2} + \sigma \frac{|\varphi^{-1}(\bar v) - \varphi^{-1}(\bar v_*)|}{2}\right)\right)\right| d\sigma d\bar v_* \Bigg\|_{\mathcal{L}^\infty},
\end{aligned}$$

where we use assumption 3.2 and (3.1). Notice that the norms $\mathcal{L}^\infty$ and L^∞ are taken on the support of the projection.
Similar as (3.10), set

$$f_{N,\lambda,-}(v) = h_{N,\lambda,-}(\varphi(v))(1+|v|)^{-4}, \quad f_{N,\lambda,+}(v) = h_{N,\lambda,+}(\varphi(v))(1+|v|)^{-4},$$

then the above equation is now

$$\begin{aligned}
&\left\|Q^+_{N,\lambda}(h_{N,\lambda,+},h_{N,\lambda,-})\eta^{-1}\right\|_{\mathcal{L}^\infty_{-4}} \\
\leq\ &\left\|\int_{\mathbb{R}^3\times\mathbb{S}^2} B_\lambda(|v-v_*|,\sigma)f'_{N,\lambda,+*}f'_{N,\lambda,-}(1+|v|^2)d\sigma dv_*\right\|_{L^\infty}.
\end{aligned} \tag{4.5}$$

By Remark 3 of Theorem 2.1 [48], we have

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^3\times\mathbb{S}^2} B_\lambda(|v-v_*|,\sigma)f'_{N,\lambda,+*}f'_{N,\lambda,-}(1+|v|^2)d\sigma dv_*\right\|_{L^\infty} \\
\leq\ & C\|f_{N,\lambda,+}\|_{L^1_2}\|f_{N,\lambda,-}\|_{L^\infty_2},
\end{aligned}$$

which is equivalent with

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^3\times\mathbb{S}^2} B_\lambda(|v-v_*|,\sigma)f'_{N,\lambda,+*}f'_{N,\lambda,-}(1+|v|^2)d\sigma dv_*\right\|_{L^\infty} \\
\leq\ & C\|h_{N,\lambda,+}\|_{\mathcal{L}^1(\eta)}\|h_{N,\lambda,-}\|_{\mathcal{L}^\infty_{-2}},
\end{aligned} \tag{4.6}$$

where C is some positive constant.
Inequalities (4.5) and (4.6) lead to

$$\left\|Q^+_{N,\lambda}(h_{N,\lambda,+},h_{N,\lambda,-})\eta^{-1}\right\|_{\mathcal{L}^\infty_{-4}} \leq C\|h_{N,\lambda,+}\|_{\mathcal{L}^1(\eta)}\|h_{N,\lambda,-}\|_{\mathcal{L}^\infty_{-2}}. \tag{4.7}$$

Due to assumption 1.5, we can permute $h_{N,\lambda,+}$ and $h_{N,\lambda,-}$ to get

$$\left\|Q^+_{N,\lambda}(h_{N,\lambda,-},h_{N,\lambda,+})\eta^{-1}\right\|_{\mathcal{L}^\infty_{-4}} \leq C\|h_{N,\lambda,-}\|_{\mathcal{L}^\infty_{-2}}\|h_{N,\lambda,+}\|_{\mathcal{L}^1(\eta)}. \tag{4.8}$$

Inequalities (4.3), (4.4), (4.7) and (4.8) lead to

$$\begin{aligned}
&\left\|Q^+_{N,\lambda}(h_{N,\lambda,-},h_{N,\lambda,+})\eta^{-1}\right\|_{\mathcal{L}^\infty_{-4}} + \left\|Q^+_{N,\lambda}(h_{N,\lambda,+},h_{N,\lambda,-})\eta^{-1}\right\|_{\mathcal{L}^\infty_{-4}} \\
\leq\ & C(M)\|h_{N,\lambda,-}\|_{\mathcal{L}^\infty_{-2}} \text{ on } [0,\tau],
\end{aligned} \tag{4.9}$$

where $C(M)$ is a constant depending on M.
Equation (4.4) implies the following inequality holds pointwisely

$$\begin{aligned}
-\partial_t h_{N,\lambda,-}\eta^{-1} &= Q^+_{N,\lambda}(h_{N,\lambda},h_{N,\lambda})\eta^{-1} - Q^-_{N,\lambda}(h_{N,\lambda},h_{N,\lambda,-})\eta^{-1} \\
&\geq -Q^+_{N,\lambda}(h_{N,\lambda,+},h_{N,\lambda,-})\eta^{-1} - Q^+_{N,\lambda}(h_{N,\lambda,-},h_{N,\lambda,+})\eta^{-1} - Q^-_{N,\lambda}(h_{N,\lambda},h_{N,\lambda,-})\eta^{-1},
\end{aligned}$$

which means

$$\begin{aligned}
\partial_t h_{N,\lambda,-}\eta^{-1} \leq\ & Q^+_{N,\lambda}(h_{N,\lambda,+},h_{N,\lambda,-})\eta^{-1} + Q^+_{N,\lambda}(h_{N,\lambda,-},h_{N,\lambda,+})\eta^{-1} \\
&+Q^-_{N,\lambda}(h_{N,\lambda},h_{N,\lambda,-})\eta^{-1}.
\end{aligned} \tag{4.10}$$

Since

$$\left\|Q^-_{N,\lambda}(h_{N,\lambda},h_{N,\lambda,-})\eta^{-1}\right\|_{\mathcal{L}^\infty_{-4}} \leq C(M)\|h_{N,\lambda,-}\|_{\mathcal{L}^\infty_{-2}},$$

where $C(M)$ is some constant depending on M. Inequalities (4.9) and (4.10) lead to

$$\frac{d}{dt}\|h_{N,\lambda,-}\eta^{-1}\|_{\mathcal{L}^\infty_{-4}} \leq C'(M)\|h_{N,\lambda,-}\|_{\mathcal{L}^\infty_{-2}} \text{ on } [0,\tau],$$

which implies

$$\|h_{N,\lambda,-}(t)\eta^{-1}\|_{\mathcal{L}^\infty_{-4}} \leq \exp(C'(M)t)\|h_{0_N,-}\eta^{-1}\|_{\mathcal{L}^\infty_{-4}} = 0, \text{ on } [0,\tau].$$

Hence $h_{N,\lambda,-} = 0$ on $[0,\tau]$, which means $h_{N,\lambda} \geq 0$ on $[0,\tau]$. As a consequence, assumption 3.1 and (3.15) imply

$$\int_{(-1,1)^3} h_{N,\lambda}(t)\eta^{-1}d\bar{v} \leq \int_{(-1,1)^3} h_{0_N}\eta^{-1}d\bar{v} \text{ on } [0,\tau].$$

By repeating the argument for $[\tau,2\tau]$, $[2\tau,3\tau]$... we conclude that $h_{N,\lambda}$ is positive and $\|h_{N,\lambda}\|_{\mathcal{L}^1(\eta^{-1})}$ is bounded at all time. □

4.2 Maxwellian lower bound for $h_{N,\lambda}$

In this section, we establish a Maxwellian lower bound for the solution $h_{N,\lambda}$ of (4.2). We first formulate some inequalities of Duhamel's type that will be the base of our estimates to obtain a Maxwellian lower bound for $h_{N,\lambda}$. Notice that the results in this subsection still hold for the case $\lambda = \infty$. Consider equation (4.2) on $h_{N,\lambda}$, by assumption 3.2 we have

$$\begin{aligned}
& \partial_t h_{N,\lambda}(t,\bar{v}) \\
= \; & Q_{N,\lambda}(h_{N,\lambda},h_{N,\lambda}) = Q^+_{N,\lambda}(h_{N,\lambda},h_{N,\lambda}) - Q^-_{N,\lambda}(h_{N,\lambda},h_{N,\lambda}) \\
= \; & Q^+_{N,\lambda}(h_{N,\lambda},h_{N,\lambda}) \\
& -\eta\mathcal{P}_N\left\{\eta^{-1}\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar{v},\bar{v}_*,\sigma)h_{N,\lambda}(\bar{v})h_{N,\lambda}(\bar{v}_*)d\sigma d\bar{v}_*\right\} \\
\geq \; & Q^+_{N,\lambda}(h_{N,\lambda},h_{N,\lambda}) \\
& -C\eta[\eta^{-1}h_{N,\lambda}(\bar{v})]\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar{v},\bar{v}_*,\sigma)h_{N,\lambda}(\bar{v}_*)d\sigma d\bar{v}_*\right\} \\
\geq \; & Q^+_{N,\lambda}(h_{N,\lambda},h_{N,\lambda}) \\
& -Ch_{N,\lambda}(\bar{v})\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar{v},\bar{v}_*,\sigma)h_{N,\lambda}(\bar{v}_*)d\sigma d\bar{v}_*\right\},
\end{aligned} \tag{4.11}$$

notice that $P_N(\eta^{-1}h_{N,\lambda}) = \eta^{-1}h_{N,\lambda}$. Set

$$\mathcal{H}(\bar{v}) = \mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar{v},\bar{v}_*,\sigma)h_{N,\lambda}(\bar{v}_*)d\sigma d\bar{v}_*\right\},$$

and

$$G^{t_2}_{t_1}(\bar{v}) = \exp\left(-\int_{t_1}^{t_2}\mathcal{H}(t,\bar{v})dt\right), \quad \forall t_1,t_2 > 0.$$

Apply Duhamel's representation to inequality (4.11), we get

$$h_{N,\lambda}(t,\bar v) \geq h_{0_N}(\bar v)G_0^t(\bar v) + \int_0^t G_\tau^t(\bar v)Q^+_{N,\lambda}(h_{N,\lambda}(\tau,.),h_{N,\lambda}(\tau,.))(\bar v)d\tau. \tag{4.12}$$

In order to come back to the original formulation of the Boltzmann equation, we define

$$f_{N,\lambda}(t,v) = h_{N,\lambda}(t,\varphi(v))(1+|v|)^{-4}, \tag{4.13}$$

and accordingly

$$f_{0_N}(v) = h_{0_N}(\varphi(v))(1+|v|)^{-4}.$$

With this new function, $\mathcal{H}$ becomes

$$\mathcal{P}_N\left\{\int_{\mathbb{R}^3\times\mathbb{S}^2} B_\lambda(|v-v_*|,\sigma)f_{N,\lambda}(v_*)d\sigma dv_*\right\},$$

where for the sake of simplicity, we still denote by $\mathcal{P}_N$ the orthogonal project from $L^2((-1,1)^3)$ onto $\mathcal{V}_N$ but with the new variable v.
By proposition 4.1, we can see that

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(|v-v_*|,\sigma)f_{N,\lambda}(v_*)d\sigma d\bar v_* \leq C(1+|v|)^\gamma,$$

then

$$\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)h_{N,\lambda}(\bar v_*)d\sigma d\bar v_* \leq C(1-|\bar v|)^{-\gamma},$$

which means

$$\mathcal{H} = \mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)h_{N,\lambda}(\bar v_*)d\sigma d\bar v_*\right\} \leq C\mathcal{P}_N\left\{(1-|\bar v|)^{-\gamma}\right\},$$

with the notice that $\mathcal{P}_N$ is a positive projection and C is a constant not depending on N and λ.
Define

$$\tilde G_{t_1}^{t_2}(\bar v) = \exp\left(-C(t_2-t_1)\mathcal{P}_N[(1-|\bar v|)^{-\gamma}]\right),$$

we get

$$G_{t_1}^{t_2}(\bar v) \geq \tilde G_{t_1}^{t_2}(\bar v).$$

Using this inequality in (4.12), we obtain

$$h_{N,\lambda}(t,\bar v) \geq h_{0_N}(\bar v)\tilde G_0^t(\bar v) + \int_0^t \tilde G_\tau^t(\bar v)Q^+_{N,\lambda}(h_{N,\lambda}(\tau,.),h_{N,\lambda}(\tau,.))(\bar v)d\tau. \tag{4.14}$$

From (4.14), we deduce the following two inequalities, which will be used several times in the rest of this subsection

$$h_{N,\lambda}(t,\bar v) \geq \int_0^t \tilde G_\tau^t(\bar v)Q^+_{N,\lambda}(h_{0_N}\tilde G_0^\tau, h_{0_N}\tilde G_0^\tau)(\bar v)d\tau, \tag{4.15}$$

and

$$h_{N,\lambda}(t,\bar v) \tag{4.16}$$

$$\geq \int_0^t \tilde{G}_\tau^t(\bar{v}) Q_{N,\lambda}^+ \left(\int_0^\tau \tilde{G}_{\tau_1}^\tau(\bar{v}) Q_{N,\lambda}^+ (h_{0_N} \tilde{G}_0^{\tau_1}, h_{0_N} \tilde{G}_0^{\tau_1})(\bar{v}) d\tau_1, h_{0_N} \tilde{G}_0^\tau \right)(\bar{v}) d\tau.$$

With the notation

$$\hat{G}_\tau^t(v) = \tilde{G}_\tau^t(\varphi(v)),$$

where φ is defined in (2.1) and

$$\mathcal{Q}_\lambda^+(F_2, F_1)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B_\lambda(|v - v_*|, \cos\theta) F_1{}' F_2{}'_* d\sigma dv_*,$$

for all measurable functions F_1 and F_2, we have

$$f_{N,\lambda}(t,v) \geq \int_0^t \hat{G}_\tau^t(v) \mathcal{P}_N \mathcal{Q}_\lambda^+ (f_{0_N} \hat{G}_0^\tau, f_{0_N} \hat{G}_0^\tau)(v) d\tau, \tag{4.17}$$

and

$$\begin{aligned} & f_{N,\lambda}(t,v) \qquad\qquad (4.18) \\ \geq & \int_0^t \hat{G}_\tau^t(v) \mathcal{P}_N \mathcal{Q}_\lambda^+ \left(\int_0^\tau \hat{G}_{\tau_1}^\tau(v) \mathcal{P}_N \mathcal{Q}_\lambda^+ (f_{0_N} \hat{G}_0^{\tau_1}, f_{0_N} \hat{G}_0^{\tau_1})(v) d\tau_1, f_{0_N} \hat{G}_0^\tau \right)(v) d\tau, \end{aligned}$$

where (4.18) follows from assumption 3.2.

Lemma 4.1. *There are constants R, α, ϵ_0 and $\bar{\mathcal{O}} \in \mathbb{R}^3$ such that for N, λ sufficiently large, we have $f_{N,\lambda}(t,v) > \epsilon_0$ for all $|v - \bar{\mathcal{O}}| < \alpha$, $|\bar{\mathcal{O}}| < R$. Moreover $\epsilon_0 = O(t^2)$ for small t.*

Proof. We suppose that $\int_{\mathbb{R}^3} f_0 = 1$. Let R be a positive constant and divide $K_R = (-R,R)^3$ into $\left(\frac{2R}{r}\right)^3$ cubes, centred at $\mathcal{O}_i$ and of length r. If R is large enough, we have

$$\int_{K_R} f_0 dv > \frac{1}{2}.$$

Since $f_{0_N} = \mathbb{P}_N(f_0)$

$$\lim_{N\to\infty} \int_{K_R} |f_{0_N} - f_0| dv = 0,$$

there exists N_0 such that for $N > N_0$

$$\int_{K_R} f_{0_N} dv > \frac{1}{2}.$$

Since$\int_{\mathbb{R}^3} f_0 = 1$, we can infer that for r sufficiently small depending on f_0

$$\int_{K_i} f_0 dv < \frac{1}{4.3^3},$$

for all i. Therefore, no set of 27 subcubes can contain more than half of the mass contained in K_R, which means there exist two subcubes K_1, K_2 with $|\mathcal{O}_1 - \mathcal{O}_2| \geq 2\sqrt{3}r$ satisfying

$$\int_{K_i} f_0 \geq \frac{1}{4(2R/r)^3}, \quad i = 1,2.$$

Since

$$\lim_{N\to\infty}\int_{K_i}|f_{0_N}-f_0|dv=0,\forall i,$$

there exists a constant, still denoted by N_0 such that for all $N>N_0$

$$\int_{K_i}f_{0_N}dv<\frac{1}{4.3^3},\text{ and }\int_{K_i}f_{0_N}\geq\frac{1}{8(2R/r)^3},\quad i=1,2.$$

We define $\bar{\mathcal{O}}=(\mathcal{O}_1+\mathcal{O}_2)/2$ and $\alpha=(|\mathcal{O}_1-\mathcal{O}_2|-\sqrt{6}r)/(4\sqrt{2})$ then $\alpha>(2\sqrt{3}-\sqrt{6})r/(4\sqrt{2})$. Let w_1, w_2 be in K_1 and K_2 and define S_{w_1,w_2} to be the sphere taking the segment $\overline{w_1,w_2}$ as its diagonal. We can see that the ball with center $\bar{\mathcal{O}}$ and radius 2α lies entirely inside S_{w_1,w_2}. Define χ_1, χ_2 and χ_R to be the characteristic functions of K_1, K_2 and K_R. Set

$$F_1=f_{0_N}\chi_1;\quad F_2=f_{0_N}\chi_2;\quad F_3=f_{0_N}\chi_R,$$

and use these functions in (4.18), we get

$$\begin{aligned}
&f_{N,\lambda}(t,v) &&(4.19)\\
\geq\ &\int_0^t\hat{G}_\tau^t(v)\mathcal{P}_N\mathcal{Q}_\lambda^+\left(\int_0^\tau\hat{G}_{\tau_1}^\tau(v)\mathcal{P}_N\mathcal{Q}_\lambda^+(f_{0_N}\hat{G}_0^{\tau_1},f_{0_N}\hat{G}_0^{\tau_1})(v)d\tau_1,f_{0_N}\hat{G}_0^\tau\right)(v)d\tau\\
\geq\ &\int_0^t\hat{G}_\tau^t(v)\mathcal{P}_N\mathcal{Q}_\lambda^+\left(\int_0^\tau\hat{G}_{\tau_1}^\tau(v)\mathcal{P}_N\mathcal{Q}_\lambda^+(F_2\hat{G}_0^{\tau_1},F_1\hat{G}_0^{\tau_1})(v)d\tau_1,F_3\hat{G}_0^\tau\right)(v)d\tau.
\end{aligned}$$

Since F_1, F_2, F_3 are all supported in K_R, then $\hat{G}_{t_1}^{t_2}(v)$ could be considered as being supported in $\{v\ :\ |v|<2R\}$ and

$$\hat{G}_{t_1}^{t_2}(v)=\exp\left(-C(t_2-t_1)\mathcal{P}_N((1+|v|)^\gamma)\geq\exp\left(-C(t_2-t_1)\mathcal{P}_N((1+2R)^\gamma)\right)\right),$$

then

$$F_1\hat{G}_0^{\tau_1}\geq\exp\left(-C\tau_1((1+2R)^\gamma)\right)f_{0_N}\chi_1,\quad F_2\hat{G}_0^{\tau_1}\geq\exp\left(-C\tau_1((1+2R)^\gamma)\right)f_{0_N}\chi_2,$$

$$F_3\hat{G}_0^\tau\geq\exp\left(-C\tau((1+2R)^\gamma)\right)f_{0_N}\chi_R,$$

which, together with (4.19) implies

$$\begin{aligned}
f_{N,\lambda}(t,v)\ \geq\ &\int_0^t\int_0^\tau\exp(-C((1+2R)^\gamma)(t+\tau+\tau_1))d\tau_1d\tau &&(4.20)\\
&\times\mathcal{P}_N\mathcal{Q}_\lambda^+\left(\mathcal{P}_N\mathcal{Q}_\lambda^+(F_2,F_1),F_3\right)(v),
\end{aligned}$$

where C is some constant not depending on N and λ.
We assume without loss of generality that $b(\cos\theta)$ is bounded from below by a constant b_0. By Carleman's representation,

$$\begin{aligned}
&\mathcal{Q}_\lambda^+(\mathcal{P}_N\mathcal{Q}_\lambda^+(F_2,F_1),F_3)(v) &&(4.21)\\
=\ &\int_{\mathbb{R}^3}F_3(v')\frac{(|v-v'|\wedge\lambda)^\gamma}{|v-v'|^2}\int_{E_{v,v'}}\mathcal{P}_N\mathcal{Q}_\lambda^+(F_2,F_1)(v'_*)b(\cos\theta)dE(v'_*)dv',
\end{aligned}$$

where $E_{v,v'}$ is the plane containing v and perpendicular to $v'-v$ and $dE(v'_*)$ is the Lebesgue measure on $E_{v,v'}$. Since

$$F_1(v)=\int_{\mathbb{R}^3}F_1(w)\delta(v-w)dw \text{ and } F_2(v)=\int_{\mathbb{R}^3}F_2(w)\delta(v-w)dw,$$

denote v' and v'_* by u and w we have

$$\begin{aligned}
&\int_{E_{v,v'}}\mathcal{P}_N\mathcal{Q}_\lambda^+(F_2,F_1)(v'_*)b(\cos\theta)dE(v'_*)\\
\geq\ &\int_{E_{v,u}}\mathcal{P}_N[\mathcal{Q}_\lambda^+(F_2,F_1)(w)b_0]dE(w)\\
\geq\ &\int_{E_{v,u}}\mathcal{P}_N\left[\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2F_1'F_2{}'_*d\sigma dw_*\right]dE(w)\\
\geq\ &\int_{E_{v,u}}\mathcal{P}_N[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta(w'-w_1)\delta(w'_*-w_2)d\sigma dw_*dw_1dw_2]dE(w)\\
\geq\ &\int_{E_{v,u}}\mathcal{P}_N[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]dE(w)
\end{aligned}\tag{4.22}$$

where

$$\delta_1(v')=\delta(v'-w_1),\ \text{and}\ \delta_2(v'_*)=\delta_2(v'_*-w_2).$$

Let χ_ϵ be the characteristic function of $\{w|dist(w,E_{v,u})<\epsilon\}$, then

$$\begin{aligned}
&\int_{E_{v,u}}\mathcal{P}_N[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]dE(w)\\
=\ &\lim_{\epsilon\to0}\frac{1}{2\epsilon}\int_{\mathbb{R}^3}\mathcal{P}_N[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]\chi_\epsilon dw\\
=\ &\lim_{\epsilon\to0}\frac{1}{2\epsilon}\int_{\mathbb{R}^3}[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]\mathcal{P}_N(\chi_\epsilon(1+|w|)^4)(1+|w|)^{-4}dw\\
\geq\ &\lim_{\epsilon\to0}\frac{1}{2\epsilon}\int_{\mathbb{R}^3}[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]\mathcal{P}_N(\chi_\epsilon\mathcal{P}_N(1+|w|)^4)(1+|w|)^{-4}dw\\
\geq\ &\lim_{\epsilon\to0}\frac{C}{2\epsilon}\int_{\mathbb{R}^3}[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]\mathcal{P}_N\chi_\epsilon\mathcal{P}_N(1+|w|)^4)(1+|w|)^{-4}dw\\
\geq\ &\lim_{\epsilon\to0}\frac{C}{2\epsilon}\int_{\mathbb{R}^3}[\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\\
&\times\delta_1(w')\delta_2(w'_*)d\sigma dw_*dw_1dw_2]\mathcal{P}_N\chi_\epsilon dw,
\end{aligned}\tag{4.23}$$

where the last inequalities follow from assumption 3.2. Moreover, we have that

$$\begin{aligned}
& \lim_{\epsilon\to 0} C\frac{1}{2\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\delta_1(w')\delta_2(w'_*)\mathcal{P}_N\chi_\epsilon(w)d\sigma dw_* dw \\
= & \lim_{\epsilon\to 0} C\frac{1}{2\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathbb{S}^2}(|w-w_*|\wedge\lambda)^\gamma {b_0}^2\delta_1(w)\delta_2(w_*)\mathcal{P}_N\chi_\epsilon(w')d\sigma dw_* dw \\
= & \lim_{\epsilon\to 0} C\frac{1}{2\epsilon}\frac{(|w_1-w_2|\wedge\lambda)^\gamma}{|w_1-w_2|^2}\int_{S_{w_1,w_2}}\frac{b_0^2}{\cos\theta}\mathcal{P}_N\chi_\epsilon(w')d\tilde{n},
\end{aligned} \tag{4.24}$$

where the first equality follows from the change of variables $dw_*dw \to dw'_*dw'$ and the second one is Carleman's change of variables (see [10], [59]), $\tilde{n}$ denotes the measure on the surface of the sphere. Since for $\lambda > 2R$

$$\begin{aligned}
& \lim_{\epsilon\to 0} C\frac{1}{2\epsilon}\frac{(|w_1-w_2|\wedge\lambda)^\gamma}{|w_1-w_2|^2}\int_{S_{w_1,w_2}}\frac{b_0^2}{\cos\theta}\chi_\epsilon(w')d\tilde{n} \\
= & \lim_{\epsilon\to 0} C\frac{1}{2\epsilon}|w_1-w_2|^{\gamma-2}\int_{S_{w_1,w_2}}\frac{b_0^2}{\cos\theta}\chi_\epsilon(w')d\tilde{n} \\
\geq & C\pi|w_1-w_2|^{\gamma-1}b_0^2 \\
. \geq & C\pi\min\{(2R)^{\gamma-1},(2r)^{1-\gamma}\}b_0^2,
\end{aligned}$$

then we can have for $\lambda > 2R$, N sufficiently large

$$\lim_{\epsilon\to 0} C\frac{1}{2\epsilon}\frac{(|w_1-w_2|\wedge\lambda)^\gamma}{|w_1-w_2|^2}\int_{S_{w_1,w_2}}\frac{b_0^2}{\cos\theta}\mathcal{P}_N\chi_\epsilon(w')d\tilde{n} \geq C.$$

Combine (4.21), (4.22), (4.23), (4.24) and (4.25), we get

$$\begin{aligned}
f_{N,\lambda}(t,v) \geq & \bar{C}\int_0^t\int_0^\tau \exp(-C((1+2R)^\gamma)(t+\tau+\tau_1))d\tau_1 d\tau \\
& \times\int_{\mathbb{R}^3}F_3(u)\frac{(|v-u|\wedge\lambda)^\gamma}{|v-u|^2}\int_{\mathbb{R}^6}F_1(w_1)F_2(w_2)dw_1dw_2du \\
\geq & \int_0^t\int_0^\tau \exp(-C((1+2R)^\gamma)(t+\tau+\tau_1))d\tau_1 d\tau \\
& \times\bar{C}(2R)^{\gamma-2}\frac{1}{2}\left(\frac{1}{8(2R/r)^3}\right)^2,
\end{aligned} \tag{4.25}$$

which leads to the conclusion of the lemma. □

Lemma 4.2. *Suppose that there is $\bar{\mathcal{O}}\in\mathbb{R}^3$, such that $F(v) > \epsilon$ for $|v-\bar{\mathcal{O}}| < \alpha$, then there exist $C, \varsigma, \epsilon > 0$, which do not depend on λ, such that*

$$\mathcal{Q}^+_\lambda(F,F)(v) > C\alpha^{3+\gamma}\varsigma^{5/2}\epsilon^2, \tag{4.26}$$

for all v, $|v-\bar{\mathcal{O}}| < \alpha\sqrt{2}(1-\varsigma)$.

Proof. Without loss of generality, we can assume that $\bar{\mathcal{O}}$ is the origin. According to Carleman's representation, we have the following scaling property

$$\mathcal{Q}^+_\lambda(F,F)(\beta v)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} F(u)\frac{(|u-\beta v|\wedge\lambda)^\gamma}{|u-\beta v|^2}\int_{E_{\beta v,u}} b(\cos\theta)F(w)dE(w)du \\
&= \beta^3\int_{\mathbb{R}^3} F(\beta u)\frac{\beta^{\gamma-2}(|u-v|\wedge(\lambda/\beta))^\gamma}{|u-v|^2}\int_{E_{\beta v,\beta u}} b(\cos\theta)F(w)dE(w)du \\
&= \beta^3\int_{\mathbb{R}^3} F(\beta u)\frac{\beta^{\gamma-2}(|u-v|\wedge(\lambda/\beta))^\gamma}{|u-v|^2}\beta^2\int_{E_{v,u}} b(\cos\theta)F(\beta w)dE(w)du \\
&= \beta^{\gamma+3}\int_{\mathbb{R}^3} F(\beta u)\frac{(|u-v|\wedge(\lambda/\beta))^\gamma}{|u-v|^2}\int_{E_{v,u}} b(\cos\theta)F(\beta w)dE(w)du,
\end{aligned}
$$

where we still use the notation of the previous lemma $u = v'$ and $w = v'_*$. This scaling property means that we can suppose α to be 1 and since we only consider λ sufficiently large, we can still keep λ instead of changing it into λ/β. Suppose without loss of generality that F is the characteristic function of the ball $\{w \mid |w| < 1\}$ and assume by a rotation of the coordinate that $v = (0, 0, z)$, $1 \le z < \sqrt{2}$. Use polar coordinates for u with v to be the origin, i.e. $u - v = (r\sin\varpi\cos\varpi', r\sin\varpi\sin\varpi', r\cos\varpi')$ then $du = r^2 d\sigma = r^2\sin\varpi d\varpi d\varpi' dr$ and

$$
\begin{aligned}
\mathcal{Q}^+_\lambda(F,F)(v) &= \int_{\mathbb{R}^3} F(u)\frac{(|u-v|\wedge\lambda)^\gamma}{|u-v|^2}\int_{E_{v,u}} b(\cos\theta)F(w)dE(w)du \\
&\ge b_0 2\pi\int_0^\pi\int_0^\infty F(u)\frac{(r\wedge\lambda)^\gamma}{r^2}\int_{E_{v,u}} F(w)dE(w)\sin\varpi d\varpi r^2 dr. \\
&\ge b_0 2\pi\int_0^\pi\int_0^\infty F(u)(r\wedge\lambda)^\gamma\int_{E_{v,u}} F(w)dE(w)\sin\varpi d\varpi dr.
\end{aligned}
$$

Since F is the characteristic function of $\{|w| < 1\}$, we can suppose that $|u| \le 1$ and $|w| \le 1$. Then $|u|^2 = r^2\sin^2\varpi + |z + r\cos\varpi|^2 < 1$, which leads to $z\cos\varpi - \sqrt{1 - z^2\sin^2\varpi} \le r \le z\cos\varpi + \sqrt{1 - z^2\sin^2\varpi}$ and $z\cos\varpi > 1$ or $\arccos(1/z) \le \varpi$. Moreover the fact that $1 \ge z^2\sin^2\varpi$ implies $\varpi \le \arcsin(1/z)$. Applying the change of variables $y = z\cos\varpi$ with $dy = z\sin\varpi d\varpi$ gives

$$
\begin{aligned}
\mathcal{Q}^+_\lambda(F,F)(v) &\ge \frac{2\pi^2 b_0}{z}\int_{\sqrt{z^2-1}}^1\int_{y-\sqrt{1-z^2+y^2}}^{y+\sqrt{1-z^2+y^2}} |r\wedge\lambda|^\gamma dr(1-y^2)dy \qquad (4.27)\\
&\ge \frac{2\pi^2 b_0}{z}\int_{\sqrt{z^2-1}}^1\int_{y-\sqrt{1-z^2+y^2}}^{y+\sqrt{1-z^2+y^2}} |r|^\gamma dr(1-y^2)dy,
\end{aligned}
$$

the last inequality follows when we take $\lambda > 10 > y + \sqrt{1 - z^2 + y^2}$. Notice that we need to prove (4.26) for $|v - \bar{\mathcal{O}}| < \alpha\sqrt{2}(1-\varsigma)$, we now need to estimate the integral near $z = \sqrt{2}$. Put $y' = 1 - y$ and $z = \sqrt{2}(1-\varsigma)$, then the integral of r becomes $\sqrt{1 - z^2 + y^2} = \sqrt{4\varsigma - 2\varsigma^2 - 2y' + |y'|^2} \backsim 2\sqrt{4\varsigma - 2y'} + O(\varsigma^{3/2})$. Moreover $\frac{1}{z} = 1 + O(\varsigma)$, $\sqrt{z^2-1} \backsim 2\varsigma + O(\varsigma^2)$ and the right hand side of (4.27) could be bounded from below by

$$
\begin{aligned}
&8\pi^2 b_0(1+O(\varsigma))\int_0^{2\varsigma+O(\varsigma^2)}\left(\sqrt{2\varsigma - y'} + O(\varsigma^{3/2})\right)y'(1+O(\varsigma))dy' \\
=\ &8\pi^2 b_0(1+O(\varsigma))(2\varsigma)^{5/2}\int_0^{1+O(\varsigma)}\left(\sqrt{1-y''} + O(\varsigma)\right)(y''+O(\varsigma))dy''
\end{aligned}
$$

$$= \frac{64\sqrt{2}\pi^2}{15}\varsigma^{5/2} + O(\varsigma^{7/2}),$$

where the first equality follows from the change of variables $y' = 2y''$. □

Proposition 4.2. *There exist positive constants $\tilde{C}_1$, $\tilde{C}_2$ independent of N and λ, such that for all $\bar{v}$ in the support of $h_{N,\lambda}$*

$$h_{N,\lambda}(\bar{v}, t) \geq \tilde{C}_1 \exp\left(-\tilde{C}_2 \left|\frac{|\bar{v}|}{1-|\bar{v}|}\right|^2\right).$$

The constants $\tilde{C}_1$ and $\tilde{C}_2$ depend on t, however, they could be chosen uniformly for all $t > t_0$, where t_0 is an arbitrary positive time.

Proof. We now proceed the proof by a classical iteration process as in [59]. By lemma 4.1, there exists a ball $|v - \bar{\mathcal{O}}| < \alpha$ such that $f_{N,\lambda}(t_0, v) > \epsilon_0$. By (4.17)

$$f_{N,\lambda}(t_0+t_1, v) \geq \int_{t_0}^{t_0+t_1} \hat{G}_\tau^{t_0+t_1}(v)\mathcal{P}_N \mathcal{Q}_\lambda^+(f_{N,\lambda}(t_0)\hat{G}_{t_0}^\tau, f_{N,\lambda}(t_0)\hat{G}_{t_0}^\tau)(v)d\tau. \tag{4.28}$$

Now, for v near the given ball and lies in the support of $f_{N,\lambda}$,

$$\hat{G}_{\tau_1}^{\tau_2} \geq \exp(-(\tau_2-\tau_1)c(1+2|\bar{\mathcal{O}}|^\gamma + 2^{1+\gamma/2}\alpha^\gamma)).$$

Plug this inequality into (4.28) and use lemma 4.2

$$\begin{aligned} f_{N,\lambda}(t_0+t_1, v) &\geq t_1 \exp(-t_1 C(1+2|\bar{\mathcal{O}}|^\beta + 2^{1+\gamma/2}\alpha^\gamma))\alpha^{3+\gamma}\varsigma_1^{5/2}\epsilon_0^2 \\ &\geq t_1 \exp(-Ct_1 2^{1+\gamma/2}\alpha^\gamma)\alpha^{3+\gamma}\varsigma_1^{5/2}\epsilon_0^2, \end{aligned}$$

and this holds with $|v - \bar{\mathcal{O}}| < \sqrt{2}(1-\varsigma_1)\alpha$ and $v \in (-\frac{\zeta_N}{1-\zeta_N}, \frac{\zeta_N}{1-\zeta_N})^3$. Now, we take the iteration

$$\begin{aligned} f_{N,\lambda}(t_0+t_1+t_2, v) &\geq (t_1 \exp(-t_1 C 2^{1+\gamma/2}\alpha^\gamma)\alpha^{3+\gamma}\varsigma_1^{5/2}\epsilon_0^2)^2 \\ &\quad \times t_2 \exp(-t_2 C 2^{1+2\gamma/2}\alpha^\gamma)(2^{1/2}(1-\varsigma_1)\alpha)^{3+\gamma}\varsigma_2^{5/2}, \end{aligned}$$

for $|v - \bar{\mathcal{O}}| < 2(1-\varsigma_1)(1-\varsigma_2)\alpha$ and $v \in (-\frac{\zeta_N}{1-\zeta_N}, \frac{\zeta_N}{1-\zeta_N})^3$. At the $n-th$ step

$$\begin{aligned} & f_{N,\lambda}(t_0+t_1+\dots+t_n, v) \\ > \; & \epsilon_0^{2^n}(C\alpha^{3+\gamma})^{2^n-1}(2^{1/2}(1-\varsigma_1))^{(3+\gamma)2^{n-1-1}} \dots (2^{k/2}(1-\varsigma_1)\dots(1-\varsigma_k))^{(3+\gamma)2^{n-1-k}} \\ & \dots (t_1\varsigma_1^{5/2})^{2^{n-1}} \dots (t_k\varsigma_k^{5/2})^{2^{n-k}} \dots (t_n\varsigma_n^{5/2})^{2^0} \\ & \exp(-Ct_1\alpha^\gamma 2^{1+\gamma/2}2^{n-1}) \dots \exp(-Ct_k\alpha^\gamma 2^{1+\gamma k/2}2^{n-k}) \dots \exp(-Ct_n\alpha^\gamma 2^{1+\gamma n/2}2^{n-n}), \end{aligned}$$

for $|v - \bar{\mathcal{O}}| < 2^{n/2}(1-\varsigma_1)\dots(1-\varsigma_n)\alpha$ and $v \in (-\frac{\zeta_N}{1-\zeta_N}, \frac{\zeta_N}{1-\zeta_N})^3$, which leads to the conclusion of the proposition. □

4.3 $\mathcal{L}^2_{-4}$ estimate for $h_{N,\lambda}$

Define
$$\Upsilon_\lambda(v) = [1 + (|v| \wedge \lambda)]^\gamma, \tag{4.29}$$
we now prove a technical lemma on
$$\mathcal{Q}^+_\lambda(F,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\lambda(|v-v_*|,\cos\theta)F'_*F'd\sigma dv_*,$$
before going to the $\mathcal{L}^2_{-4}$ estimate for $h_{N,\lambda}$.

Lemma 4.3. *Let ν, δ and k be three constants satisfying $\nu,\delta \geq -\gamma$ and $k > \gamma$. There exist positive constants C and ι, such that the following estimate holds for all $\epsilon > 0$ and all measurable function F*
$$\begin{aligned}&\left|\int_{\mathbb{R}^3} \mathcal{Q}^+_\lambda(F,F)F dv\right| \leq C\epsilon^{-\iota}\|F\|_{L^{10/7}_\delta}\|F\|_{L^2_{-\delta}}\|F\|_{L^1_{2|\delta|}} \\ &+\epsilon\|F\|_{L^2(\Upsilon_\lambda^{\gamma+\nu})}\|F\|_{L^2(\Upsilon_\lambda^{-\nu})}\|F\|_{L^1_{|k+\nu|+|\nu|}}.\end{aligned} \tag{4.30}$$
In particular, if we take $\delta = 0$ and $\nu = -\gamma/2$
$$\left|\int_{\mathbb{R}^3} \mathcal{Q}^+_\lambda(F,F)F dv\right| \leq C\epsilon^{-\iota}\|F\|_{L^2}\|F\|_{L^{10/7}}\|F\|_{L^1} + \epsilon\|F\|^2_{L^2(\Upsilon_\lambda^{1/2})}\|F\|_{L^1_{|k|}}. \tag{4.31}$$

Remark 4.1. *Notice that the lemma is still valid for the case $\lambda = \infty$.*

Proof. By similar arguments as in [48], we can suppose that $b \in C^\infty_c(-1,1)$. Let $\Theta : \mathbb{R}^3 \to \mathbb{R}$ be a radial C^∞ function such that $supp\Theta \subset B(0,1)$ and $\int_{\mathbb{R}^3}\Theta = 1$. Let μ be a constant smaller than λ and define the regularizing function
$$\Theta_\mu(x) = \mu^3\Theta(\mu x) \quad (x \in \mathbb{R}^3).$$
Define
$$\Phi_S = \Phi * (\Theta 1_{\mathbb{A}_\mu}), \quad \Phi_R = \Phi - \Phi_S,$$
where $\mathbb{A}_\mu$ is the annulus $\mathbb{A}_\mu = \{x \in \mathbb{R}^3; \frac{2}{\mu} \leq |x| \leq \mu\}$.
Set
$$B_\lambda(|v|,\sigma) = B^S(|v|,\sigma) + B^R(|v|,\sigma),$$
where
$$B^S(|v|,\sigma) := \Phi_S(v)b(\cos\theta).$$
Set
$$\mathcal{Q}^+_\lambda = Q^+_S + Q^+_R,$$
with
$$Q^+_S(F,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^S(|v-v_*|,\cos\theta)F'_*F'd\sigma dv_*.$$
By Corollary 3.2 [48], the first term Q^+_S could be bounded in the following way
$$\left|\int_{\mathbb{R}^3} Q^+_S(F,F)F dv\right| \leq C(\delta,b)\|F\|_{L^{10/7}_\delta}\|F\|_{L^2_{-\delta}}\|F\|_{L^1_{2|\delta|}}. \tag{4.32}$$

Now, we will estimate the second term Q_R^+. For all test function ϱ the following equality holds

$$\int_{\mathbb{R}^3} Q_R^+(F,F)\varrho dv = \int_{\mathbb{R}^6} F(v_*)F(v)\left[\int_{\mathbb{S}^2} B^R(|v-v_*|,\sigma)\varrho(v')d\sigma\right] dv_* dv.$$

By defining

$$\mathcal{S}\varrho(v) = \int_{\mathbb{S}^2} B^R(|v|,\sigma)\varrho\left(\frac{v+|v|\sigma}{2}\right) d\sigma,$$

we have

$$\int_{\mathbb{R}^3} Q_R^+(F,F)\varrho dv = \int_{\mathbb{R}^3} F(v_*)\left(\int_{\mathbb{R}^3} F(v)(T_{v_*}\mathcal{S}(T_{-v_*}\varrho))(v)dv\right) dv_*,$$

where $T_h f(v) = f(v-h)$. Let ξ_1, ξ_2 be two non-negative constants. Consider the weighted L^∞ norm of $\mathcal{S}\varrho$, since $|v^+| \leq |v|$

$$\|\mathcal{S}\varrho\|_{L^\infty(\Upsilon_\lambda^{-\xi_1-\xi_2})} \leq C\|b\|_{L^1(\mathbb{S}^2)}\|\varrho\|_{L^\infty(\Upsilon_\lambda^{-\xi_2})}\|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\xi_1})},$$

where C is some positive constant.
Now, consider the weighted L^1 norm of $\mathcal{S}_2\varrho$

$$\begin{aligned}
\|\mathcal{S}\varrho\|_{L^1(\Upsilon_\lambda^{-\xi_1-\xi_2})} &\leq \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \Phi_R(v)\Upsilon_\lambda^{-\xi_1-\xi_2-1}(v)b(\cos\theta)\left|\varrho\left(\frac{v+|v|\sigma}{2}\right)\right| d\sigma dv \\
&\leq \|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\xi_1})}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \Upsilon_\lambda^{-\xi_2}(v)b(\cos\theta)\left|\varrho\left(\frac{v+|v|\sigma}{2}\right)\right| d\sigma dv \\
&\leq C\|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\xi_1})}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\Upsilon_\lambda^{-\xi_2}(v^+)|\varrho(v^+)|d\sigma dv.
\end{aligned}$$

The last inequality follows from the fact that $|v^+| \leq |v|$ and $\xi_2 \geq 0$.
Apply the change of variables $v \to v^+$, whose Jacobian is $\frac{1}{8}\left(1+\left\langle\frac{v}{|v|},\sigma\right\rangle\right) = \frac{\cos^2(\theta/2)}{4}$ we obtain

$$\begin{aligned}
\|\mathcal{S}\varrho\|_{L^1(\Upsilon_\lambda^{-\xi_1-\xi_2})} &\leq \|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\xi_1})} \times \\
&\quad \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\frac{4b(\cos\theta)}{\cos^2(\theta/2)}\Upsilon_\lambda^{-\xi_2}(v^+)|\varrho(v^+)|d\sigma dv^+ \\
&\leq C(\theta_b,\xi)\|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\xi_1})}\|b\|_{L^1(\mathbb{S}^2)}\|\varrho\|_{L^1(\Upsilon_\lambda^{-\xi_2})},
\end{aligned}$$

with the notice that $\theta \in [0,\theta_b]$. By the Riesz-Thorin interpolation theorem, the above estimates on the weighted L^1 and L^∞ norms of $\mathcal{S}\varrho$ lead to

$$\|\mathcal{S}\varrho\|_{L^2(\Upsilon_\lambda^{-\xi_1-\xi_2})} \leq C(\theta_b,\xi)\|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\xi_1})}\|b\|_{L^1(\mathbb{S}^2)}\|\varrho\|_{L^2(\Upsilon_\lambda^{-\xi_2})}.$$

Now, we will estimate the term

$$\int_{\mathbb{R}^3} Q_R^+(F,F)F dv,$$

by using the above bound on $\|\mathcal{S}_1\varrho\|_{L^2(\Upsilon_\lambda^{-\xi_1-\xi_2-1})}$. In order to do this, we separate F into large and small velocities:

$$F = F_r + F_r^c, \text{ with } r < \lambda,$$

$$F_r = F\chi_{\{|v|\leq r\}} \text{ and } F_r^c = F\chi_{\{|v|>r\}},$$

where $\chi_{\{|v|\leq r\}}$ and $\chi_{\{|v|>r\}}$ are the characteristic functions of the sets $\{|v|\leq r\}$ and $\{|v|>r\}$. Let ν be a positive constant. We make the following separation

$$\int_{\mathbb{R}^3} Q_R^+(F,F)F dv = \int_{\mathbb{R}^3} Q_R^+(F,F_r^c)F dv + \int_{\mathbb{R}^3} Q_R^+(F,F_r)F dv. \tag{4.33}$$

Estimating the first term on the right hand side of (4.33), we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q_R^+(F,F_r^c)F dv\\
\leq\ & \int_{\mathbb{R}^3} |F_r^c(v_*)| \int_{\mathbb{R}^3} |F(v)||T_{-v_*}\mathcal{S}(T_{v_*}F)(v)| dv dv_*\\
\leq\ & \int_{\mathbb{R}^3} |F_r^c(v_*)| \|F\|_{L^2(\Upsilon_\lambda^{\gamma+\nu})} \|T_{-v_*}\mathcal{S}(T_{v_*}F)\|_{L^2(\Upsilon_\lambda^{-\gamma-\nu})} dv_*\\
\leq\ & \int_{\mathbb{R}^3} |F_r^c(v_*)| \|F\|_{L^2(\Upsilon_\lambda^{\gamma+\nu})} < v_* >^{|\gamma+\nu|} \|\mathcal{S}(T_{v_*}F)\|_{L^2(\Upsilon_\lambda^{-\gamma-\nu})} dv_*\\
\leq\ & C\int_{\mathbb{R}^3} |F_r^c(v_*)| \|F\|_{L^2(\Upsilon_\lambda^{\gamma+\nu})} < v_* >^{|\gamma+\nu|} \|T_{v_*}F\|_{L^2(\Upsilon_\lambda^{-\nu})} dv_* \|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-\gamma})}\\
\leq\ & C\int_{\mathbb{R}^3} |F_r^c(v_*)| \|F\|_{L^2(\Upsilon_\lambda^{\gamma+\nu})} < v_* >^{|\gamma+\nu|+|\nu|} \|F\|_{L^2(\Upsilon_\lambda^{-\nu})} dv_* \qquad (4.34)\\
\leq\ & Cr^{\gamma-k} \|F\|_{L^2(\Upsilon_\lambda^{\gamma+\nu})} \|F\|_{L^2(\Upsilon_\lambda^{-\nu})} \|F\|_{L^1((1+|v|)^{|k+\nu|+|\nu|})},
\end{aligned}$$

with $k>\gamma$.
We estimate the second term on the right hand side of (4.33)

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q_R^+(F,F_r)F dv\\
\leq\ & \int_{\mathbb{R}^3} |F(v_*)| \int_{\mathbb{R}^3} |F_r(v)||T_{-v_*}\mathcal{S}(T_{v_*}F)(v)| dv dv_*\\
\leq\ & \int_{\mathbb{R}^3} |F(v_*)| \|F_r\|_{L^2(\Upsilon_\lambda^{(k+\nu)})} \|T_{-v_*}\mathcal{S}(T_{v_*}F)\|_{L^2(\Upsilon_\lambda^{-(k+\nu)})} dv_*\\
\leq\ & \int_{\mathbb{R}^3} |F(v_*)| \|F_r\|_{L^2(\Upsilon_\lambda^{(k+\nu)})} < v_* >^{|k+\nu|} \|\mathcal{S}(T_{v_*}F)\|_{L^2(\Upsilon_\lambda^{-(k+\nu)})} dv_*\\
\leq\ & C\int_{\mathbb{R}^3} |F(v_*)| \|F_r\|_{L^2(\Upsilon_\lambda^{(k+\nu)/\gamma})} < v_* >^{|k+\nu|} \|T_{v_*}F\|_{L^2(\Upsilon_\lambda^{-\nu})} dv_*\\
& \times \|\Phi_R\|_{L^\infty(\Upsilon_\lambda^{-k})} \qquad (4.35)\\
\leq\ & C\left(\frac{1}{\mu}\right)^{\min\{\gamma,k-\gamma\}} \int_{\mathbb{R}^3} |F(v_*)| \|F_r\|_{L^2(\Upsilon_\lambda^{(k+\nu)})} < v_* >^{|k+\nu|} \|T_{v_*}F\|_{L^2(\Upsilon_\lambda^{-\nu})} dv_*\\
\leq\ & C\left(\frac{1}{\mu}\right)^{\min\{\gamma,k-\gamma\}} \|F_r\|_{L^2(\Upsilon_\lambda^{(k+\nu)})} \|F\|_{L^2(\Upsilon_\lambda^{-\nu})} \|F\|_{L^1_{|k+\nu|+|\nu|}}\\
\leq\ & Cr^{k-\gamma}\left(\frac{1}{\mu}\right)^{\min\{\gamma,k-\gamma\}} \|F\|_{L^2(\Upsilon_\lambda^{(\gamma+\nu)})} \|F\|_{L^2(\Upsilon_\lambda^{-\nu})} \|F\|_{L^1_{|k+\nu|+|\nu|}},
\end{aligned}$$

with $k>\gamma$. Combine (4.33), (4.34) and (4.35), we get

$$\int_{\mathbb{R}^3} Q_R^+(F,F)F dv \tag{4.36}$$

$$\leq \left(r^{\gamma-k} + Cr^{k-\gamma}\left(\frac{1}{\mu}\right)^{\min\{\gamma,k-\gamma\}}\right)\|F\|_{L^2(\Upsilon_\lambda^{(\gamma+\nu)/\gamma})}\|F\|_{L^2(\Upsilon_\lambda^{-\nu/\gamma})}\|F\|_{L^1_{|k+\nu|+|\nu|}}.$$

We deduce from (4.32) and (4.36) that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\mathcal{Q}^+_\lambda(F,F)Fdv\right|\\
\leq\ & C(\delta,b)\|F\|_{L^{10/7}_\delta}\|F\|_{L^2_{-\delta}}\|F\|_{L^1_{2|\delta|}} \qquad (4.37)\\
&+\left(r^{\gamma-k} + Cr^{k-\gamma}\left(\frac{1}{\mu}\right)^{\min\{\gamma,k-\gamma\}}\right)\|F\|_{L^2(\Upsilon_\lambda^{(\gamma+\nu)/\gamma})}\|F\|_{L^2(\Upsilon_\lambda^{-\nu/\gamma})}\|F\|_{L^1_{|k+\nu|+|\nu|}}.
\end{aligned}$$

For suitable choices of r and μ, we have the conclusions of the lemma . □

Proposition 4.3. *For all $t_0 > 0$, there exist constants C, N_0, λ_0 such that the solution $h_{N,\lambda}$ of (4.2) is globally bounded in the following sense*

$$\forall N \in \mathbb{N}, N > N_0, \forall\lambda > \lambda_0 \quad \sup_{t\geq t_0}\|h_{N,\lambda}\|_{\mathcal{L}^2_{-4}} < C. \qquad (4.38)$$

Moreover, if $h_{0_N} \in \mathcal{L}^2_{-4}$, then there exist constants C', λ_0 such that

$$\forall\lambda > \lambda_0 \quad \sup_{t\geq 0}\|h_{N,\lambda}\|_{\mathcal{L}^2_{-4}} < C'.$$

Proof. Use $(1-|\bar v|)^6\eta^{-1}h_{N,\lambda}$ as a test function for (4.2), we get

$$\begin{aligned}
&\int_{(-1,1)^3}(1-|\bar v|)^6\eta^{-1}\partial_t h_{N,\lambda}h_{N,\lambda}d\bar v\\
=\ & \int_{(-1,1)^3}\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)\eta^{-1}\right. \qquad (4.39)\\
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2} - \sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\times h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2} + \sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\\
&\left.\left. -h_{N,\lambda}(\bar v)h_{N,\lambda}(\bar v_*)\right]d\sigma d\bar v_*\right\}(1-|\bar v|)^6 h_{N,\lambda}d\bar v.
\end{aligned}$$

Define as in (4.13)

$$f_{N,\lambda}(v) = h_{N,\lambda}(\varphi(v))(1+|v|)^{-4}, \quad v \in \mathbb{R}^3,$$

the left hand side of (4.39) becomes

$$\begin{aligned}
\int_{(-1,1)^3}(1-|\bar v|)^6\eta^{-1}\partial_t h_{N,\lambda}h_{N,\lambda}d\bar v &= \int_{\mathbb{R}^3}\partial_t f_{N,\lambda}(1+|v|^2)f_{N,\lambda}(1+|v|)^{-2}dv\\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|f_{N,\lambda}|^2(1+|v|^2)(1+|v|)^{-2}dv. \quad (4.40)
\end{aligned}$$

Consider the right hand side of (4.39)

$$\int_{(-1,1)^3}\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)\eta^{-1}\right.$$

$$
\begin{aligned}
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\times h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\\
&\left.-h_{N,\lambda}(\bar v)h_{N,\lambda}(\bar v_*)\right]d\sigma d\bar v_*\Big\}(1-|\bar v|)^6h_{N,\lambda}(\bar v)d\bar v\\
=\ &\int_{(-1,1)^3}\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)\eta^{-1}\right. \qquad (4.41)\\
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\times h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\\
&\left.-h_{N,\lambda}(\bar v)h_{N,\lambda}(\bar v_*)\right]d\sigma d\bar v_*\Big\}\mathcal{P}_N[(1-|\bar v|)^6h_{N,\lambda}(\bar v)]d\bar v\\
\leq\ &C_1\int_{(-1,1)^6\times\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)\mathcal{C}(\bar v,\bar v_*,\sigma)\eta^{-1}(1-|\bar v|)^6h_{N,\lambda}(\bar v)\\
&\times h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\\
&\times h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)d\sigma d\bar v_*d\bar v\\
&-C_2\int_{(-1,1)^6\times\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)\eta^{-1}(1-|\bar v|)^6|h_{N,\lambda}(\bar v)|^2h_{N,\lambda}(\bar v_*)d\sigma d\bar v_*d\bar v,
\end{aligned}
$$

where the last inequality follows from assumption 3.2 and C_1, C_2 are some positive constants. We deduce from (4.41) that

$$
\begin{aligned}
&\int_{(-1,1)^3}\mathcal{P}_N\left\{\int_{(-1,1)^3}\int_{\mathbb{S}^2}\mathcal{B}_\lambda(\bar v,\bar v_*,\sigma)\eta^{-1}\right.\\
&\times\left[\mathcal{C}(\bar v,\bar v_*,\sigma)h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}-\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right)\right.\\
&\times h_{N,\lambda}\left(\varphi\left(\frac{\varphi^{-1}(\bar v)+\varphi^{-1}(\bar v_*)}{2}+\sigma\frac{|\varphi^{-1}(\bar v)-\varphi^{-1}(\bar v_*)|}{2}\right)\right) \qquad (4.42)\\
&\left.-h_{N,\lambda}(\bar v)h_{N,\lambda}(\bar v_*)\right]d\sigma dv_*\Big\}(1-|\bar v|)^6h_{N,\lambda}(\bar v)d\bar v\\
\leq\ &C_1\int_{\mathbb{R}^6\times\mathbb{S}^2}B_\lambda(|v-v_*|,\sigma)f_{N,\lambda}(v_*')f_{N,\lambda}(v')f_{N,\lambda}(v)(1+|v|^2)(1+|v|)^{-2}d\sigma dv_*dv\\
&-C_2\int_{\mathbb{R}^6\times\mathbb{S}^2}B_\lambda(|v-v_*|,\sigma)f_{N,\lambda}(v_*)|f_{N,\lambda}(v)|^2(1+|v|^2)(1+|v|)^{-2}d\sigma dv_*dv.
\end{aligned}
$$

Combine (4.40) and (4.42), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|f_{N,\lambda}|^2(1+|v|^2)(1+|v|)^{-2}dv \qquad (4.43)\\
\leq\ &C_1\int_{\mathbb{R}^6\times\mathbb{S}^2}B_\lambda(|v-v_*|,\sigma)f_{N,\lambda}(v_*')f_{N,\lambda}(v')f_{N,\lambda}(v)(1+|v|^2)(1+|v|)^{-2}d\sigma dv_*dv\\
&-C_2\int_{\mathbb{R}^6\times\mathbb{S}^2}B_\lambda(|v-v_*|,\sigma)f_{N,\lambda}(v_*)|f_{N,\lambda}(v)|^2(1+|v|^2)(1+|v|)^{-2}d\sigma dv_*dv.
\end{aligned}
$$

According to lemma 4.3, the first term on the right hand side of (4.43) could be bounded by

$$\begin{aligned} & C_1 \int_{\mathbb{R}^6\times\mathbb{S}^2} B_\lambda(|v-v_*|,\sigma) f_{N,\lambda}(v'_*) f_{N,\lambda}(v') f_{N,\lambda}(v)(1+|v|^2)(1+|v|)^{-2} d\sigma dv_* dv \\ \leq\ & C\epsilon^{-\iota}\|f_{N,\lambda}\|_{L^2}\|f_{N,\lambda}\|_{L^{10/7}}\|f_{N,\lambda}\|_{L^1} + \epsilon\|f_{N,\lambda}\|^2_{L^2(\Upsilon_\lambda^{1/2})}\|f_{N,\lambda}\|_{L^1_{|k|}}, \end{aligned} \tag{4.44}$$

where C is some positive constant.
By the inequality

$$(|v-v_*|\wedge\lambda)^\gamma \geq \frac{1}{4}(|v|\wedge\lambda)^\gamma - |v_*|^\gamma,$$

we have

$$\begin{aligned} & \int_{\mathbb{R}^3\times\mathbb{S}^2} B_\lambda(|v-v_*|,\sigma) f_{N,\lambda}(v_*) d\sigma dv_* \\ =\ & C\int_{\mathbb{R}^3\times\mathbb{S}^2} (|v-v_*|\wedge\lambda)^\gamma b(\cos(\theta)) f_{N,\lambda}(v_*) d\sigma dv_* \\ \geq\ & C\int_{\mathbb{R}^3\times\mathbb{S}^2} b(\cos(\theta))\left(\frac{1}{4}(|v|\wedge\lambda)^\gamma - |v_*|^\gamma\right) f_{N,\lambda}(v_*) d\sigma dv_* \\ \geq\ & C(|v|\wedge\lambda)^\gamma - C\|f_{N,\lambda}\|_{L^1_\gamma} \geq C(|v|\wedge\lambda)^\gamma - C\|f_0\|_{L^1_2}, \end{aligned} \tag{4.45}$$

where the last inequality follows from the L^1_2 boundedness of $f_{N,\lambda}$.
Combine (4.43), (4.44) and (4.45), and choose ϵ small enough, we get

$$\frac{d}{dt}\|f_{N,\lambda}\|^2_{L^2} \leq C\epsilon^{-\iota}\|f_{N,\lambda}\|_{L^2}\|f_{N,\lambda}\|_{L^{10/7}}\|f_{N,\lambda}\|_{L^1} - (C-\epsilon)\|f_{N,\lambda}\|^2_{L^2(\Upsilon_\lambda^{1/2})}, \tag{4.46}$$

where C is some positive constant varying from lines to lines. By a classical argument as [48], there exist constants C, N_0 such that

$$\forall s\geq 0, \forall t_0>0, \forall N>N_0 \quad \sup_{t\geq t_0}\|f_{N,\lambda}\|_{L^2} < C.$$

□

4.4 The convergence analysis

Theorem 4.1. *Suppose that assumptions 3.1 and 3.2 are satisfied. The solution h_N of (3.8) is positive and uniformly bounded with respect to N in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ norms, i.e. for all $t_0>0$ there exist constants C, N_0 such that*

$$\forall N\in\mathbb{N}, N>N_0 \quad \sup_{t\geq t_0}\|h_N\|_{\mathcal{L}^1_2} < C,$$

and

$$\forall t_0>0, \forall N\in\mathbb{N}, N>N_0 \quad \sup_{t\geq t_0}\|h_N\|_{\mathcal{L}^2_{-4}} < C.$$

Moreover there are positive constants $\hat{C}_1$, $\hat{C}_2$, such that for all $\bar{v}$ in the support of h_N

$$h_N(\bar{v},t) \geq \hat{C}_1 \exp\left(-\hat{C}_2\left|\frac{|\bar{v}|}{1-|\bar{v}|}\right|^2\right).$$

Proof. Since the sequence $\{h_{N,\lambda}\}$ is uniformly bounded with respect to N and λ in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ norms, the proof is direct and similar to the proofs of classical cases (for example theorem 3.2 [2]). First we observe that since h_{0_N} is a sum of finite compactly supported wavelets not containing the extreme points of -1 and 1, then h_{0_N} belongs to $\mathcal{L}^2_{-4}$. Hence the sequence $\{h_{N,\lambda}\}$ is uniformly bounded with respect to λ (but not N) in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ norms for all time. By Nagumo's criterion, Dunford-Pettis theorem and Smulian theorem (see [25] and [40]) there exists a subsequence $\{h_{N,\lambda_j}\}_{j=1}^{\infty}$ converging weakly to a positive function $\check{h}_N$ in $\mathcal{L}^1$, which is a solution of (3.8). According to proposition 3.1, the linear ODEs (3.8) has a unique solution, then $h_N \equiv \check{h}_N \geq 0$. Since the proofs of propositions 4.3 and 4.1 are still valid when $\lambda = +\infty$, we infer that h_N is uniformly bounded with respect to N in $\mathcal{L}^1_2$ and $\mathcal{L}^2_{-4}$ norms and it is also bounded from below by a Maxwellian truncated in its support. □

Theorem 4.2. *Suppose that assumptions 3.1 and 3.2 are satisfied. If $f_0 \in L^1_{2+\gamma}$, the solution of* (3.8) *tends to the solution of* (2.6) *in the energy sense*

$$\sup_{t\in[0,T]} \lim_{N\to\infty} \|h_N(t) - h(t)\|_{\mathcal{L}^1_2} = 0, \forall T \in \mathbb{R},$$

which implies the limits of the mass and momentum

$$\sup_{t\in[0,T]} \lim_{N\to\infty} \|h_N(t) - h(t)\|_{\mathcal{L}^1} = 0, \forall T \in \mathbb{R},$$

$$\sup_{t\in[0,T]} \lim_{N\to\infty} \|h_N(t) - h(t)\|_{\mathcal{L}^1_1} = 0, \forall T \in \mathbb{R}.$$

Proof. Take the difference between (3.8) and (2.6), multiply both sides with η^{-1}, we get

$$\begin{aligned}
& \partial_t(h_N - \mathcal{P}_N h)\eta^{-1} \qquad (4.47)\\
= \; & \mathcal{P}_N \left\{ \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v}, \bar{v}_*, \sigma)\mathcal{C}(\bar{v}, \bar{v}_*, \sigma)\eta^{-1} \right.\\
& \times h_N \left(\varphi\left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma\frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\\
& \times h_N \left(\varphi\left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma\frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2}\right)\right) d\sigma d\bar{v}_*\\
& - \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v}, \bar{v}_*, \sigma)\eta^{-1} h_N(\bar{v}) h_N(\bar{v}_*) d\sigma d\bar{v}_*\\
& - \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v}, \bar{v}_*, \sigma)\mathcal{C}(\bar{v}, \bar{v}_*, \sigma)\eta^{-1}\\
& \times h \left(\varphi\left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} - \sigma\frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2}\right)\right)\\
& \times h \left(\varphi\left(\frac{\varphi^{-1}(\bar{v}) + \varphi^{-1}(\bar{v}_*)}{2} + \sigma\frac{|\varphi^{-1}(\bar{v}) - \varphi^{-1}(\bar{v}_*)|}{2}\right)\right) d\sigma d\bar{v}_*\\
& \left. + \int_{(-1,1)^3} \int_{\mathbb{S}^2} \mathcal{B}(\bar{v}, \bar{v}_*, \sigma)\eta^{-1} h(\bar{v}) h(\bar{v}_*) d\sigma d\bar{v}_* \right\}.
\end{aligned}$$

We deduce from (4.47) that

$$
\begin{aligned}
& \frac{d}{dt}\int_{(-1,1)^3} |h_N - \mathcal{P}_N h|\eta^{-1} d\bar{v} \\
= & \int_{(-1,1)^3} \Bigg\{ \int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\mathcal{C}(\bar{v},\bar{v}_*,\sigma)\eta^{-1} \qquad (4.48) \\
& \times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2} - \sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right) \\
& \times h_N\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2} + \sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right) d\sigma d\bar{v}_* \\
& - \int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\eta^{-1} h_N(\bar{v}) h_N(\bar{v}_*) d\sigma d\bar{v}_* \\
& - \int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\mathcal{C}(\bar{v},\bar{v}_*,\sigma)\eta^{-1} \\
& \times h\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2} - \sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right) \\
& \times h\left(\varphi\left(\frac{\varphi^{-1}(\bar{v})+\varphi^{-1}(\bar{v}_*)}{2} + \sigma\frac{|\varphi^{-1}(\bar{v})-\varphi^{-1}(\bar{v}_*)|}{2}\right)\right) d\sigma d\bar{v}_* \\
& + \int_{(-1,1)^3}\int_{\mathbb{S}^2} \mathcal{B}(\bar{v},\bar{v}_*,\sigma)\eta^{-1} h(\bar{v}) h(\bar{v}_*) d\sigma d\bar{v}_* \Bigg\} \mathcal{P}_N[sign(h_N - \mathcal{P}_N h)] d\bar{v},
\end{aligned}
$$

where $sign(h_N - \mathcal{P}_N h) = 1$ if $h_N - \mathcal{P}_N h > 0$, $sign(h_N - \mathcal{P}_N h) = -1$ if $h_N - \mathcal{P}_N h < 0$ and $sign(h_N - \mathcal{P}_N h) = 0$ if $h_N - \mathcal{P}_N h = 0$. Set

$$
f_N(v) = h_N(\varphi(v))(1+|v|)^{-4},
$$

$$
\vartheta(v) = (1+|v|^2)\mathcal{P}_N(sign(f_N - \tilde{f}_N)(v)) := (1+|v|^2)\mathcal{P}_N(sign(f_N - \mathcal{P}_N f)(v)),
$$

we transform equation (4.48) into

$$
\begin{aligned}
& \frac{d}{dt}\int_{\mathbb{R}^3} |f_N - \tilde{f}_N|(1+|v|^2) dv \\
= & \int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f'_{N_*}f'_N - f_{N_*}f_N]\vartheta(v) d\sigma dv_* dv \qquad (4.49) \\
& - \int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f'_*f' - f_*f]\vartheta(v) d\sigma dv_* dv \\
= & \frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f_{N_*}f_N - f_*f][\vartheta(v'_*)+\vartheta(v')-\vartheta(v_*)-\vartheta(v)] d\sigma dv_* dv \\
= & \frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[f_{N_*}f_N - \tilde{f}_{N_*}\tilde{f}_N][\vartheta(v'_*)+\vartheta(v')-\vartheta(v_*)-\vartheta(v)] d\sigma dv_* dv \\
& + \frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[\tilde{f}_{N_*}\tilde{f}_N - f_*f][\vartheta(v'_*)+\vartheta(v')-\vartheta(v_*)-\vartheta(v)] d\sigma dv_* dv \\
= & \frac{1}{2}\int_{\{(f_N-\tilde{f}_N)(f_{N_*}-\tilde{f}_{N_*})>0\}} B(|v-v_*|,\sigma)[f_{N_*}f_N - \tilde{f}_{N_*}\tilde{f}_N] \\
& \times[\vartheta(v'_*)+\vartheta(v')-\vartheta(v_*)-\vartheta(v)] d\sigma dv_* dv
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\{(f_N-\tilde f_N)>0>(f_{N_*}-\tilde f_{N_*})\}} B(|v-v_*|,\sigma)[f_{N_*}f_N-\tilde f_{N_*}\tilde f_N]\\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\\
&+\int_{\{(f_N-\tilde f_N)>0=(f_{N_*}-\tilde f_{N_*})\}} B(|v-v_*|,\sigma)[f_{N_*}f_N-\tilde f_{N_*}\tilde f_N]\\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\\
&+\int_{\{(f_N-\tilde f_N)=0>(f_{N_*}-\tilde f_{N_*})\}} B(|v-v_*|,\sigma)[f_{N_*}f_N-\tilde f_{N_*}\tilde f_N]\\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\\
&+\frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[\tilde f_{N_*}\tilde f_N-f_*f][\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv.
\end{aligned}
$$

On the set $\mathcal{I}_1=\{(f_N-\tilde f_N)(f_{N_*}-\tilde f_{N_*})>0\}$, then $\vartheta(v)=1+|v|^2$ and $\vartheta(v_*)=1+|v_*|^2$ or $\vartheta(v)=-1-|v|^2$ and $\vartheta(v_*)=-1-|v_*|^2$; we can see that

$$
[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]sign(f_N-\tilde f_N)\le 0 \text{ on } \mathcal{I}_1.
$$

Therefore

$$
\begin{aligned}
&\int_{\{(f_N-\tilde f_N)(f_{N_*}-\tilde f_{N_*})>0\}} B(|v-v_*|,\sigma) \qquad (4.50)\\
&\times[f_{N_*}f_N-\tilde f_{N_*}\tilde f_N][\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\le 0.
\end{aligned}
$$

On the set $\mathcal{I}_2=\{(f_N-\tilde f_N)>0>(f_{N_*}-\tilde f_{N_*})\}$, $\vartheta(v_*)=-(1+|v_*|^2)$ and $\vartheta(v)=(1+|v|^2)$. Hence

$$
-2(1+|v|^2)\le\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)\le 2(1+|v_*|^2) \text{ on } \mathcal{I}_2,
$$

which leads to

$$
\begin{aligned}
&\int_{\{(f_N-\tilde f_N)>0>(f_{N_*}-\tilde f_{N_*})\}} B(|v-v_*|,\sigma)[f_{N_*}f_N-\tilde f_{N_*}\tilde f_N] \qquad (4.51)\\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\\
=\;&\int_{\{(f_N-\tilde f_N)>0>(f_{N_*}-\tilde f_{N_*})\}} B(|v-v_*|,\sigma)[f_N-\tilde f_N]f_{N_*}\\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\\
&+\int_{\{(f_N-\tilde f_N)>0>(f_{N_*}-\tilde f_{N_*})\}} B(|v-v_*|,\sigma)[f_{N_*}-\tilde f_{N_*}]\tilde f_N\\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv\\
\le\;& C\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)|f_N-\tilde f_N|\tilde f_{N_*}(1+|v_*|)^2 d\sigma dv_* dv\\
&+C\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)|f_{N_*}-\tilde f_{N_*}|\tilde f_N(1+|v|)^2 d\sigma dv_* dv\\
\le\;& C\|f_N-\tilde f_N\|_{L^1_2}.
\end{aligned}
$$

On the set $\mathcal{I}_3=\{(f_N-\tilde f_N)>0=(f_{N_*}-\tilde f_{N_*})\}$, $\vartheta(v_*)=0$ and $\vartheta(v)=(1+|v|^2)$. Hence

$$
\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)\le(1+|v_*|^2) \text{ on } \mathcal{I}_3,
$$

which leads to

$$\begin{aligned}
&\int_{\{(f_N-\tilde f_N)>0=(f_{N*}-\tilde f_{N*})\}} B(|v-v_*|,\sigma)[f_{N*}f_N-\tilde f_{N*}\tilde f_N] \\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv \\
=\ &\int_{\{(f_N-\tilde f_N)>0=(f_{N*}-\tilde f_{N*})\}} B(|v-v_*|,\sigma)[f_N-\tilde f_N]\tilde f_{N*} \\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv \\
=\ &\int_{\{(f_N-\tilde f_N)>0=(f_{N*}-\tilde f_{N*})\}} B(|v-v_*|,\sigma)[f_N-\tilde f_N]\tilde f_{N*}(1+|v_*|^2)d\sigma dv_* dv \\
\leq\ & C\|f_N-\tilde f_N\|_{L^1_2}.
\end{aligned} \tag{4.52}$$

On the set $\mathcal{I}_4=\{(f_N-\tilde f_N)=0>(f_{N*}-\tilde f_{N*})\}$, $\vartheta(v_*)=-(1+|v_*|^2)$ and $\vartheta(v)=0$. Hence

$$-(1+|v|^2)\leq\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v) \text{ on } \mathcal{I}_4,$$

which leads to

$$\begin{aligned}
&\int_{\{(f_N-\tilde f_N)=0>(f_{N*}-\tilde f_{N*})\}} B(|v-v_*|,\sigma)[f_{N*}f_N-\tilde f_{N*}\tilde f_N] \\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv \\
=\ &\int_{\{(f_N-\tilde f_N)=0>(f_{N*}-\tilde f_{N*})\}} B(|v-v_*|,\sigma)[f_{N*}-\tilde f_{N*}]\tilde f_N \\
&\times[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv \\
\leq\ & C\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)|f_{N*}-\tilde f_{N*}|\tilde f_N(1+|v|)^2 d\sigma dv_* dv \\
\leq\ & C\|f_N-\tilde f_N\|_{L^1_2}.
\end{aligned} \tag{4.53}$$

Therefore (4.49), (4.50), (4.51), (4.52), (4.53) imply

$$\begin{aligned}
&\frac{d}{dt}\|f_N-\tilde f_N\|_{L^1_2} \\
\leq\ & C\|f_N-\tilde f_N\|_{L^1_2} \\
&+\frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} B(|v-v_*|,\sigma)[\tilde f_{N*}\tilde f_N-f_*f][\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv,
\end{aligned} \tag{4.54}$$

where C is a constant varying from lines to lines.
Apply Gronwall's inequality to (4.54), we get

$$\begin{aligned}
&\|f_N(T)-\tilde f_N(T)\|_{L^1_2} \\
\leq\ &\int_0^T\int_{\mathbb{R}^6\times\mathbb{S}^2}\frac{e^{C(T-t)}}{2}B(|v-v_*|,\sigma)[\tilde f_{N*}\tilde f_N-f_*f]\times \\
&[\vartheta(v_*')+\vartheta(v')-\vartheta(v_*)-\vartheta(v)]d\sigma dv_* dv dt+e^{CT}\|f_N(0)-\tilde f_N(0)\|_{L^1_2}.
\end{aligned} \tag{4.55}$$

Inequality (4.55) implies that the accuracy of the method is indeed the accuracy of the orthogonal projection onto the subspaces created by the wavelets. □

5 Conclusion

For the last two decades, nonlinear approximation based on wavelets has become one of the most important theories in scientific computing and the theory for elliptic equations has been fully developed ([23, 22, 19, 13, 17]). This paper is trying to make a bridge between the two important theories: kinetic and nonlinear approximation. The strategy is based on a new way of constructing an adaptive non-uniform mesh. The non-uniform mesh is created by a wavelet 'support-stretching' technique: we stretch supports of wavelets defined in a bounded domain to the entire space to get a new 'nonlinear basis', which are 'the approximants' of our nonlinear approximation and solve the problem on the whole space. In our approximation, the lower-upper Maxwellian bounds play the role of a preconditioning technique. We have provided a complete convergence theory for the method. Our nonlinear approximation solves the equation without having to impose non-physics conditions on the equation. In the second part [61], we introduce a filtering technique to preserve the propagation of polynomial and exponential moments of the approximate solution. Our wavelet filtering technique designed to preserve the properties of propagation of polynomial and exponential moments is inspired by Zuazua's Fourier filtering technique in Control Theory ([65, 66]). The third part of the work [62] is devoted to the practical and numerical aspects of the theory.

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