Vietnam Journal of Mathematics **37**:2&3(2009) 1-18

Vietnam Journal
of
MATHEMATICS

On Partially Elliptic and Coercive Boundary Problems

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Received February 2, 2009
Revised May 14, 2009

Abstract. Applying iteration method, we prove fixed point theorems for operators, which may neither be continuous nor monotone. Using these results and some considerations in sub-supersolution methods, we can partially relax the coercivity, ellipticity and compactness in some boundary problems.

2000 Mathematics Subject Classification: 47H07, 47H10, 35J55, 35J67.

Key words: Monotone operators, Fixed point theorems, Boundary value problems.

1. Introduction

Let X be a non-empty set, $\leq$ and d be a partially order and a metric on X respectively. We call $(X, d, \leq)$ an ordered metric space if $(X, d, \leq)$ satisfies the following condition

(C) $x \leq y$ (resp. $y \leq x$) for any x and y in X such that x is the limit of an increasing (resp. decreasing) sequence $\{x_n\}$ and $x_n \leq y$ (resp. $y \leq x_n$) for any integer n.

We say $x \geq y$ (resp. $x < y$; $x > y$) if $y \leq x$ (resp. $x \leq y$ and $x \neq y$; $y \leq x$ and $x \neq y$).

The continuity and monotonicity of mappings and their modified versions play essential roles of fixed point theorems in ordered metric spaces (see [2, 3, 5-7, 10-13, 16-18]). The motivation of our paper is the following example: let $f(t) = t$ if t is a rational number in the interval $(0, 1]$ and $f(t) = \frac{1}{2} + \frac{1}{2}t$ if t is a irrational number in the interval $(0, 1]$. We see that f has many fixed points in $(0, 1]$, but it is neither continuous nor monotone in $(0, 1]$. We point out that the relation between x and $f(x)$ can give us the fixed points of f by using iteration methods. We obtain the following result.

Theorem 1.1. *Let A be a non-empty subset of an ordered metric space $(X, d, \leq)$, and f be an operator from X into itself. Suppose that*

(i) *$f(A) \subset A$ and $x \leq f(x)$ for any x in A,*

(ii) *each increasing sequence of A has a limit in X and an upper bound in A.*
Then f has a fixed point in A.

Applying this result we solve a class of elliptic equations in the last section.

2. Proof of Theorem 1.1

We will prove the theorem by using the lemmas, what follow.

Lemma 2.1. *Let W be a non-empty subset of an ordered metric space $(X, d, \leq)$, and g be a mapping from W into W. Suppose that*

(i) *$x \leq g(x)$ for any x in W, and*

(ii) *$\{g(x_n)\}$ has a limit in X and an upper bound in W for any increasing sequence $\{x_n\}$ in W.*
Then W has a maximal element y, i.e. $a = y$ whenever a is in W and $y \leq a$.

Proof. By Hausdorff's principle, there exists a maximal chain B of W. Now we prove that B has the greatest element. Let x_0 be an arbitrary element of B. We shall show that there is a sequence $\{x_n\}$ in B having the following property

$$x_n \geq x_{n-1} \text{ and } d(g(x), g(x_n)) < \frac{1}{n}, \forall\, x \in \{z \in B : z \geq x_n\}, n \in \mathbb{N}. \tag{1}$$

Suppose by contradiction that we only can find a finite family $\{x_0, \ldots, x_{m-1}\}$ satisfying (1), where m is a positive integer. In this case, for each x in $\{z \in B : z \geq x_{m-1}\}$, we can find y_x in B such that $y_x > x$ and $d(g(x), g(y_x)) \geq \frac{1}{m}$. Hence we can construct an increasing sequence $\{y_k\}$ such that $y_0 = x_{m-1}$ and $d(g(y_{k+1}), g(y_k)) \geq \frac{1}{m}$ for any non-negative integer k. Since $\{y_k\}$ is increasing, $\{g(y_k)\}$ has a limit. This is a contradiction and we get such a sequence $\{x_n\}$.

Since $\{x_n\}$ is increasing, then $\{g(x_n)\}$ has a limit x in X and an upper bound y in W. Because $x_n \leq g(x_n)$ for any non-negative integer n, y is also an upper bound of $\{x_n\}$. Since $(X, d, \leq)$ is an ordered metric space, we have $x \leq y$. Let z be in B, we prove that $z \leq y$. If $z \leq x_n$ for some positive integer n, then $z \leq y$. Otherwise, $z > x_n$ for any positive integer n. Hence $d(g(z), g(x_n)) < \frac{1}{n}$, for any positive integer n, which implies $z \leq g(z) = x \leq y$. Since B is a maximal chain, then $y \in B$ and y is the greatest element of B.

Finally, we show that y is a maximal element of W. Suppose by contradiction that there exists a in W such that $a > y$. Then $B \cup \{a\}$ is a chain containing B and B is not a maximal chain. This contradiction yields the lemma. ∎

Lemma 2.2. *Let W be a non-empty set in an ordered metric space $(X, d, \leq)$. Suppose that each increasing sequence of W has a limit in X and an upper bound in W. Then W has a maximal element.*

Proof. Apply Lemma 2.1 for the case $g(x) \equiv x$, we get the lemma. ∎

Lemma 2.3. *Let U be a non-empty ordered set and f be an operator from U into U such that $x \leq f(x)$ for any x in U. Suppose that α is a maximal element of U. Then α is a fixed point of f.*

Proof. We have $\alpha \leq f(\alpha)$ and $f(\alpha)$ is in U. Thus $\alpha = f(\alpha)$.

Combining Lemmas 2.2 and 2.3, we get the theorem. ∎

Remark 2.4. Our results relax the monotonicity in [2, 3, 5-7, 10-12, 16-18]. In next sections, using this idea, we can solve some equations involving with operators which may not be monotone.

3. Applications to Elliptic Equations with Discontinuity

Let N be a positive integer, Ω be a smooth bounded open subset of R^N and p and r be in $(1, \infty)$. We denote by $L^s(\Omega)$ and $W_0^{1,s}(\Omega)$ the usual Lebesgue space and Sobolev space as in [1] for any s in $[1, \infty)$. Let $a_1, \ldots, a_N$ be real functions on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, f be a real function on $\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ having the following properties.

(A0) The functions $a_1, \ldots, a_N$ satisfy the Caratheodory conditions on $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

(A1) There exist $k_0 \in L^{p/p-1}(\Omega)$, a non-negative real number C_0, and $\underline{u}$ and $\overline{u}$ in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$ such that for all (s, ζ) in $[\underline{u}(x), \overline{u}(x)] \times \mathbb{R}^N$ and for almost everywhere x in Ω, we have

$$|a_i(x, s, \zeta)| \leq k_0(x) + C_0(|s|^{\frac{r(p-1)}{p}} + |\zeta|^{p-1}) \quad \forall\ i = 0, \ldots, N.$$

(A2) For almost everywhere x in Ω, all s in $[\underline{u}(x), \overline{u}(x)]$ and any $\zeta \neq \zeta'$ in $\mathbb{R}^N$

$$\sum_{i=1}^{N} [a_i(x, s, \zeta) - a_i(x, s, \zeta')](\zeta_i - \zeta_i') > 0.$$

(A3) There exist $C_1 > 0$ and $k_1 \in L^1(\Omega)$ such that for all (s, ζ) in $[\underline{u}(x), \overline{u}(x)] \times \mathbb{R}^N$ and for almost everywhere x in Ω

$$\sum_{i=1}^{N} a_i(x,s,\zeta)\zeta_i \geq C_1|\zeta|^p - k_1(x).$$

(F1) There exist a function $k_2 \in L^{p/p-1}(\Omega)$ and a constant $C_2 \geq 0$ such that

$$|f(x,t,s,\zeta)| \leq k_2(x)+C_2(|s|^{\frac{r(p-1)}{p}}+|\zeta|^{p-1}) \text{ a.e.} x \in \Omega, \forall\, \zeta \in R^N, t,s \in [\underline{u}(x),\overline{u}(x)]$$

(F2) The function f satisfies the Caratheodory conditions on $\Omega \times \mathbb{R}^{N+2}$, and there exist a continuous real function a on $\mathbb{R}$ and a non-negative real number C_3 such that: the function $f(x,.,s,\zeta)+a(.)$ is increasing on $[\underline{u}(x),\overline{u}(x)]$ for almost everywhere x in Ω and for any $(s,\zeta) \in [\underline{u}(x),\overline{u}(x)] \times \mathbb{R}^N$, and

$$|a(t)| \leq C_3(1+|t|^{\frac{r(p-1)}{p}}) \text{ and } [a(t_1)-a(t_2)](t_1-t_2) \geq 0 \text{ for any } t \in \mathbb{R}.$$

Remark 3.1. For almost everywhere x in Ω, we only need the conditions (A1), (A2), (A3), (F1) and (F2) for any s in $[\underline{u}(x),\overline{u}(x)]$ instead of in the whole $\mathbb{R}$, therefore our results can be applied to the cases that we partially have the ellipticity, coercivity and compactness.

In this section we consider the following equation

$$\begin{cases} -\sum\limits_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,u,\nabla u) = f(x,u,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Let u be in $W_0^{1,p}(\Omega)$. Then u is called a solution (resp. subsolution, supersolution) of (2) if

$$\int_\Omega \sum_{i=1}^{N} a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx + \int_\Omega f(x,u,u,\nabla u)\varphi dx = 0 \text{ (resp. } \leq, \geq)$$

for all $v \in W_0^{1,p}(\Omega), v \geq 0$.

The main result of this section is the following theorem.

Theorem 3.2. *Suppose that the conditions* (A0), (A1)-(A3), (F1) *and* (F2) *are satisfied,* $\underline{u}$ *and* $\overline{u}$ *are a subsolution and a supersolution of* (2) *respectively. Then* (2) *has a solution* u *in* $[\underline{u},\overline{u}]$.

In order to prove the theorem we need following lemmas.

Lemma 3.3. *For any u in $W_0^{1,p}(\Omega)$, we put*

$$T(u(x)) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \end{cases}$$

and we define $S_1(u)$ *in* $(W_0^{1,p}(\Omega))^*$ *as follows*

$$< S_1(u), \varphi >= \int_\Omega \sum_{i=1}^N a_i(x, T(u), \nabla u) \frac{\partial \varphi}{\partial x_i} dx \ \ \forall \varphi \in W^{1,p}(\Omega).$$

Then S_1 *is a* $(S)_+$ *operator on* $W^{1,p}(\Omega)$*, i.e. it has the following properties.*

(i) $\{S_1(u_n)\}$ *converges weakly to* $S_1(u)$ *in* $(W_0^{1,p}(\Omega))^*$ *for any sequence* $\{u_n\}$ *converging strongly to* u *in* $W_0^{1,p}(\Omega)$.

(ii) *Let* $\{u_n\}$ *be a sequence in* $W_0^{1,p}(\Omega)$ *such that* $\{u_n\}$ *converges weakly to* u *in* $W_0^{1,p}(\Omega)$. *Then* $\{u_n\}$ *converges strongly to* x *in* $W_0^{1,p}(\Omega)$ *if*

$$\limsup_{n\to\infty} < S_1(u_n), u_n - u >\leq 0.$$

Moreover S_1 *is pseudomonotone, i.e.*

(iii) *If* $\{u_n\}$ *weakly converges to* x *in* $W_0^{1,p}(\Omega)$ *and*

$$\limsup_{n\to\infty} < S_1(x_n), x_n - x >\leq 0,$$

then $\{S_1(x_n)\}$ *weakly converges to* $S_1(x)$ *in* $(W_0^{1,p}(\Omega))^*$ *and*

$$\lim_{n\to\infty} < S_1(x_n), x_n - x >= 0.$$

Proof. (i) We note that T is a bounded and continuous operator from $W_0^{1,p}(\Omega)$ into itself (see [8]). Let w be in $W_0^{1,p}(\Omega)$, we see that $|Tw(x)| \leq (|\overline{u}(x)| + |\underline{u}(x)|)$, therefore Tw belongs to $L^r(\Omega)$ by $(A1)$ and for all ζ in $\mathbb{R}^N$ and for almost everywhere x in Ω, we have

$$|a_i(x, Tw(x), \zeta)| \leq k_0(x) + C_0(|\overline{u}(x)| + |\underline{u}(x)|)^{\frac{r(p-1)}{p}} + C_0|\zeta|^{p-1} \,\forall\, i = 0, \ldots, N.$$

Applying a result on superposition operators (see [14, p. 30]), we get the continuity of the map $w \mapsto a_i(x, Tw(x), \nabla w)$ from $W_0^{1,p}(\Omega)$ into $L^{p/p-1}(\Omega)$, and (i).

(ii) and (iii) Let $\{u_n\}$ be a sequence weakly converging to u in $W_0^{1,p}(\Omega)$ such that

$$\limsup_{n\to\infty} < S_1 u_n, u_n - u >\leq 0.$$

We shall prove (ii) and (iii) by the following steps.

Step 1. We show that $\{\nabla u_n\}$ converges pointwise to ∇u almost everywhere in Ω.

Using (A2), we have

$$< S_1 u_n, u_n - u > = \int_\Omega \sum_{i=1}^N [a_i(x, T(u_n), \nabla u_n) - a_i(x, T(u_n), \nabla u)] \frac{\partial}{\partial x_i}(u_n - u) dx$$

$$+\int_{\Omega}\sum_{i=1}^{N} a_i(x,T(u_n),\nabla u)\frac{\partial}{\partial x_i}(u_n-u)dx$$
$$\geq \int_{\Omega}\sum_{i=1}^{N} a_i(x,T(u_n),\nabla u)\frac{\partial}{\partial x_i}(u_n-u)dx.$$

Note that the sequence $\left\{\frac{\partial}{\partial x_i}(u_n-u)\right\}$ converges weakly to 0 in $L^p(\Omega)$. By the Sobolev embedding theorem, $(A1)$ and the Lebesgue dominated convergence theorem, we see that $\{a_i(x,T(u_n),\nabla u)\}$ converges strongly to $a_i(x,T(u),\nabla u)$ in $L^q(\Omega)$. Therefore, we obtain

$$\lim_{n\to\infty}\int_{\Omega}\sum_{i=1}^{N} a_i(x,T(u_n),\nabla u)\frac{\partial}{\partial x_i}(u_n-u)dx = 0.$$

Since $\limsup_{n\to\infty} < S_1u_n, u_n-u > \leq 0$, it follows that

$$\lim_{n\to\infty} < S_1u_n, u_n-u >= 0. \tag{3}$$

Thus

$$\lim_{n\to\infty}\int_{\Omega}\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u)dx = 0.$$

By (A2), it implies the convergence in $L^1(\Omega)$ of the sequence of non-negative functions

$$\left\{\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u)\right\}.$$

By Theorem IV.9 in [4], we can assume that

$$\lim_{n\to\infty}\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u) = 0 \ a.e. \ in \ \Omega \tag{4}$$

and there is a non-negative integrable function h on Ω such that

$$\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u) \leq h(x) \ a.e. \ in \ \Omega. \tag{5}$$

Denote by Ω_0 the set of all x in Ω such that

$$\lim_{n\to\infty}\sum_{i=1}^{N}[a_i(x,T(u_n)(x),\nabla u_n(x))-a_i(x,T(u_n)(x),\nabla u(x))]\frac{\partial(u_n-u)}{\partial x_i}(x)=0 \tag{6}$$

and

$$\lim_{n\to\infty}T(u_n)(x)=T(u)(x). \tag{7}$$

We see that the measure of $\Omega\setminus\Omega_0$ is null. Let x be in Ω_0, we shall prove that $\{\nabla u_n(x)\}$ converges to $\nabla u(x)$. Assume by contradiction that there is a subsequence $\{\nabla u_{n_m}(x)\}$ of $\{\nabla u_n(x)\}$ such that $|\nabla u_{n_m}(x)-\nabla u(x)|>\epsilon$ for some positive real number ϵ and for every integer m. Denote $\nabla u(x)$, $\nabla u_{n_m}(x)$, $T(u_{n_m}(x))$ and $T(u(x))$ by ρ, ρ_m, s_m and s respectively. We can suppose that $\left\{\dfrac{\rho_m-\rho}{|\rho_m-\rho|}\right\}$ converges to ρ^* in $\mathbb{R}^N$. Note that $|\rho^*|=1$. Using (A2), we have

$$\begin{aligned}
&\sum_{i=1}^{N}[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})](\rho_{mi}-\rho_i)\\
&=\frac{|\rho_m-\rho|}{|\rho_m-\rho|-\epsilon}\sum_{i=1}^{N}\left[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})\right]\times\\
&\quad\times\left(1-\frac{\epsilon}{|\rho_m-\rho|}\right)(\rho_{mi}-\rho_i)\\
&\geq 0,
\end{aligned}\tag{8}$$

$$0\leq\sum_{i=1}^{N}[a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})-a_i(x,s_m,\rho)](\rho_{mi}-\rho_i) \tag{9}$$

$$\begin{aligned}
&=\sum_{i=1}^{N}[a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})-a_i(x,s_m,\rho_m)](\rho_{mi}-\rho_i)\\
&\quad+\sum_{i=1}^{N}[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho)](\rho_{mi}-\rho_i).
\end{aligned}$$

Combining (8) and (9), we get

$$\begin{aligned}
0&\leq\sum_{i=1}^{N}[a_i(x,s_m,\rho+\epsilon\frac{\rho_m-\rho}{|\rho_m-\rho|})-a_i(x,s_m,\rho)]\frac{\rho_{mi}-\rho_i}{|\rho_m-\rho|}\\
&\leq\frac{1}{|\rho_m-\rho|}\sum_{i=1}^{N}[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho)](\rho_{mi}-\rho_i).
\end{aligned}\tag{10}$$

Since $|\rho_m-\rho|>\epsilon$, by (6) and (A0), we have

$$\sum_{i=1}^{N}[a_i(x,s,\rho+\epsilon\rho^*)-a_i(x,s,\rho)]\rho_i^*=0.$$

Therefore, $\rho^* = 0$ by $(A2)$. This is a contradiction and the sequence $\{\nabla u_n(x)\}$ should converge to $\nabla u(x)$ and we get the first step.

Step 2. $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$.

Let E be a measurable subset of Ω, by (A1), (A3), we have

$$\begin{aligned} C_1 \int\limits_E |\nabla u_n|^p dx &\le \int\limits_E k_1(x)dx + \int\limits_E \sum_{i=1}^N a_i(x, T(u_n), \nabla u_n) \frac{\partial u_n}{\partial x_i} dx \\ &= \int\limits_E k_1(x)dx + \sum_{j=1}^4 I_j, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int\limits_E \sum_{i=1}^N [a_i(x, T(u_n), \nabla u_n) - a_i(x, T(u_n), \nabla u)] \frac{\partial (u_n - u)}{\partial x_i} dx \le \int\limits_E h(x)dx, \\ I_2 &= \int\limits_E \sum_{i=1}^N a_i(x, T(u_n), \nabla u_n) \frac{\partial u}{\partial x_i} dx \\ &\le \sum_{i=1}^N \left(\int\limits_E |a_i(x, T(u_n), \nabla u_n)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int\limits_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p} \\ &\le \sum_{i=1}^N \left\| k_0 + C_0 |T(u_n)|^{\frac{r(p-1)}{p}} + C_0 |\nabla u_n|^{p-1} \right\|_{L^{\frac{p}{p-1}}(E)} \left(\int\limits_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p} \\ &\le \sum_{i=1}^N \left\| k_0(x) + C_0 (|\underline{u}|^{\frac{r(p-1)}{p}} + |\overline{u}|^{\frac{r(p-1)}{p}}) + C_0 |\nabla u_n|^{p-1} \right\|_{L^{\frac{p}{p-1}}(E)} \times \\ &\quad \times \left(\int\limits_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p} \\ &\le \sum_{i=1}^N \left\{ ||k_0||_{L^q(E)} + C_0 ||\underline{u}||_{L^r(E)}^{\frac{r(p-1)}{p}} + C_0 ||\overline{u}||_{L^r(E)}^{\frac{r(p-1)}{p}} + C_0 ||\nabla u_n||_{L^p(E)}^{p-1} \right\} \times \\ &\quad \times \left(\int\limits_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p}, \\ I_3 &= \int\limits_E \sum_{i=1}^N a_i(x, T(u_n), \nabla u) \frac{\partial u_n}{\partial x_i} dx \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{N}\left[\int_E |a_i(x,T(u_n),\nabla u)|^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left(\int_E |\frac{\partial u_n}{\partial x_i}|^p dx\right)^{1/p}\\
&\le \sum_{i=1}^{N}\left\{\|k_0\|_{L^q(E)}+C_0\|\underline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\overline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\nabla u\|_{L^p(E)}^{p-1}\right\}\times\\
&\quad\times\left(\int_E |\frac{\partial u_n}{\partial x_i}|^p dx\right)^{1/p},
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= -\int_E \sum_{i=1}^{N} a_i(x,T(u_n),\nabla u)\frac{\partial u}{\partial x_i}dx\\
&\le \sum_{i=1}^{N}\left[\int_E |a_i(x,T(u_n),\nabla u)|^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left(\int_E |\frac{\partial u}{\partial x_i}|^p dx\right)^{1/p}\\
&\le \sum_{i=1}^{N}\left\{\|k_0\|_{L^q(E)}+C_0\|\underline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\overline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\nabla u\|_{L^p(E)}^{p-1}\right\}\times\\
&\quad\times\left(\int_E |\frac{\partial u}{\partial x_i}|^p dx\right)^{1/p}.
\end{aligned}
$$

Let ε be a positive real number. By the boundedness of $\{\|\nabla u_n\|_{L^p(\Omega)}\}$, the r-integrability of $\overline{u}$ and $\underline{u}$, and conditions (A1) and (A3), there is a positive real number δ such that for any measurable subset E of Ω with Lebesgue measure $m(E) < \delta$, we have

$$\int_E |\nabla u_n|^p dx \le \varepsilon \quad \forall\, n \in \mathbb{N}.$$

Thus the sequence $\{|\nabla u_n|^p\}$ is equi-integrable. It follows that $\{|\nabla u_n - \nabla u|^p\}$ is also equi-integrable. By Vitali's theorem (see [19]), $\{\nabla u_n\}$ converges to ∇u in $L^p(\Omega)$, which implies $\{u_n\}$ converges strongly to u in $W^{1,p}(\Omega)$.

Step 3. $\{S_1(u_n)\}$ weakly converges to $S_1(u)$ in $(W_0^{1,p}(\Omega))^*$.

By the previous steps, $\{T(u_n)\}$ and $\{\nabla u_n\}$ converge to $T(u)$ and ∇u in $L^p(\Omega)$ respectively. Thus we can find an integrable function k such that

$$|T(u_n)|^p + |\nabla u_n|^p \le k \quad \forall\, n \in \mathbb{N}.$$

Therefore, by $(A1)$ and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \sum_{i=1}^{N}\left[a_i(x,T(u_n),\nabla u_n) - a_i(x,T(u),\nabla u)\right]\frac{\partial\varphi}{\partial x_i}dx = 0 \;\forall\varphi \in W^{1,p}(\Omega).$$

Step 4. $\lim_{n\to\infty} < S_1(u_n), u_n - u >= 0$.

It is just (3). Thus we get the lemma. ∎

Lemma 3.4. *Let u, v and w be in $W^{1,p}(\Omega)$ such that $v \leq w$. We put*

$$\gamma_{v,w}(u)(x) = (u(x) - w(x))_+^{p-1} - (v(x) - u(x))_+^{p-1}.$$

We define an operator $B_{v,w}$ from $W_0^{1,p}(\Omega)$ into $(W_0^{1,p}(\Omega))^*$ as follows

$$< B_{v,w}u, \varphi >= \int_\Omega \gamma_{v,w}(u)\varphi dx \quad \forall\ u, \varphi \in W_0^{1,p}(\Omega).$$

Then we have

(i) *$B_{v,w}$ is bounded.*

(ii) *There exist two positive real numbers α and β such that*

$$\int_\Omega \gamma_{v,w}(u)udx \geq \alpha||u||_p^p - \beta \quad \forall\ u\ \in W_0^{1,p}(\Omega).$$

(iii) *$\{B_{v,w}u_n\}$ converges strongly to $B_{v,w}u$ in $(W_0^{1,p}(\Omega))^*$ for any sequence $\{u_n\}$ weakly converging to u in $W_0^{1,p}(\Omega)$.*

Proof. The proof of (i) and (ii) can be found in ([15, p. 791]). We prove (iii). Let $\{u_n\}$ be a sequence weakly converging to u in $W_0^{1,p}(\Omega)$. We can assume that $\{u_n\}$ converges strongly to u in $L^p(\Omega)$ and $\{u_n(x)\}$ converges to $u(x)$ for a.e. $x \in \Omega$, and there exists a nonnegative function h in $L^p(\Omega)$ such that $|u_n(x)| \leq h(x)$ for a.e. $x \in \Omega$. Hence $\{\gamma_{v,w}(u_n)(x)\}$ converges to $\gamma_{v,w}(u)(x)$ for a.e. $x \in \Omega$. We have

$$\begin{aligned} |\gamma_{v,w}(u_n)(x)| &\leq \left\{[|v(x)| + |u_n(x)|]^{p-1} + [|u_n(x)| + |w(x)|]^{p-1}\right\} \\ &\leq \left\{[|v(x)| + h(x)]^{p-1} + [|w(x)| + h(x)]^{p-1}\right\} \quad a.e.\ x \in \Omega. \end{aligned}$$

Since $[(|v| + h)^{p-1} + (|w| + h)^{p-1}]$ is in $L^q(\Omega)$, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \|\gamma_{v,w}(u_n) - \gamma_{v,w}(u)\|_q = 0, \tag{11}$$

$$\begin{aligned} | < B_{v,w}u_n - B_{v,w}u, \varphi > | &= \left| \int_\Omega \gamma_{v,w}(u_n)\varphi - \gamma_{v,w}(u)\varphi dx \right| \\ &\leq \|\gamma_{v,w}(u_n) - \gamma_{v,w}(u)\|_q \|\varphi\|_{1,p}\ \forall\ \varphi \in W_0^{1,p}(\Omega). \end{aligned} \tag{12}$$

Combining (11) and (12), we get the lemma. ∎

Lemma 3.5. *Let v be a subsolution of (2) such that $\underline{u} \leq v \leq \overline{u}$. We put*

$$a_v(x,u,\nabla u) = -f(x,v,u,\nabla u) + a(u(x)) - a(v(x)) \ \forall\ x \in \Omega,$$

Then the following equation has a solution w in $W_0^{1,p}(\Omega)$

$$\begin{cases} -\sum\limits_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,u,\nabla u) + a_v(x,u,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

such that $v \le w \le \overline{u}$. Moreover w is also a subsolution of (2).

Proof. We define the operator S_2, S_3 and S as follows

$$\begin{aligned} < S_2 u, \varphi > &= \int_\Omega a_0(x,Tu,\nabla Tu)\varphi dx, \\ < S_3 u, \varphi > &= M \int_\Omega \gamma(x,u)\varphi dx \\ < Su, \varphi > &= < (S_1 + S_2 + S_3)u, \varphi > \quad \forall u, \varphi \in W_0^{1,p}(\Omega). \end{aligned}$$

We prove the lemma by the following steps.

Step 1. S is bounded.

By (A1), we have

$$\begin{aligned} | < S_1 u, \varphi > | &= \left| \int_\Omega \sum_{i=1}^{N} a_i(x,Tu,\nabla u) \frac{\partial \varphi}{\partial x_i} dx \right| \\ &\le \int_\Omega \sum_{i=1}^{N} [k_0(x) + C_0(|Tu|^{\frac{r(p-1)}{p}} + |\nabla u|^{p-1})] |\frac{\partial \varphi}{\partial x_i}| dx \\ &\le N\|\varphi\|_{1,p}[||k_0||_q + C_0||\underline{u}||_r^{\frac{r(p-1)}{p}} + C_0||\overline{u}||_r^{\frac{r(p-1)}{p}} + C_0||\nabla u||_p^{p-1}], \\ | < S_2 u, \varphi > | &= | \int_\Omega a_0(x,Tu,\nabla Tu)\varphi dx| \\ &\le \int_\Omega [k_0(x) + C_0|Tu|^{\frac{r(p-1)}{p}} + C_0|\nabla Tu|^{p-1}]|\varphi| dx \\ &\le ||\varphi||_{1,p}[||k_0||_q + C_0||\nabla Tu||_p^{p-1} + C_0||\underline{u}||_r^{\frac{r(p-1)}{p}} + C_0||\overline{u}||_r^{\frac{r(p-1)}{p}}. \end{aligned}$$

According to Lemma 3.4, S_3 is bounded. Thus $S = S_1 + S_2 + S_3$ is bounded.

Step 2. S is pseudomonotone.

By Lemma 3.4, and Proposition 27.7 in [20], it is sufficient to prove that S_1+S_2 is a pseudomonotone operator on $W_0^{1,p}(\Omega)$. Let $\{u_n\}$ be a sequence converging weakly to u in $W_0^{1,p}(\Omega)$ such that $\limsup\limits_{n\to\infty} < S_1 u_n + S_2 u_n, u_n - u > \le 0$. Note that

$$\begin{aligned}| < S_2u_n, u_n - u > | &\leq \int_\Omega |a_0(x, Tu_n, \nabla Tu_n)(u_n - u)|dx \\ &\leq ||u_n - u||_p ||a_0(x, T(u_n), \nabla Tu_n)||_q,\end{aligned}$$

which implies

$$\lim_{n\to\infty} < S_2u_n, u_n - u > = \ 0. \tag{14}$$

Since $\limsup_{n\to\infty} < (S_1 + S_2)u_n, u_n - u > \leq \ 0$, then $\limsup_{n\to\infty} < S_1u_n, u_n - u > \leq \ 0$.

By Lemma 3.3, $\{S_1u_n\}$ converges weakly to S_1u in $(W_0^{1,p}(\Omega))^*$, $\{u_n\}$ converges to u in $W_0^{1,p}(\Omega)$ and $\lim_{n\to\infty} < S_1u_n, u_n > = < S_1u, u >$. Hence $\{S_2u_n\}$ weakly converges to S_2u in $(W_0^{1,p}(\Omega))^*$ and $\lim_{n\to\infty} < S_2u_n, u_n >=< S_2u, u >$. Consequently, $\{(S_1 + S_2)u_n\}$ weakly converges to $(S_1 + S_2)u$ in $(W_0^{1,p}(\Omega))^*$ and $\lim_{n\to\infty} < (S_1 + S_2)u_n, u_n >=< (S_1 + S_2)u, u >$. That means $S_1 + S_2$ is pseudomonotone. Therefore, S is pseudomonotone.

Step 3. S is coercive.

By (A3), we have

$$\begin{aligned}< S_1u, u > &= \int_\Omega \sum_{i=1}^N a_i(x, T(u), \nabla u) \frac{\partial}{\partial x_i} u dx \\ &\geq \int_\Omega [C_1 |\nabla u|^p - k_1(x)]dx \qquad (15) \\ &= C_1 ||\nabla u||_p^p - ||k_1||_1,\end{aligned}$$

$$\begin{aligned}\int_\Omega |\nabla Tu|^p dx &= \int_{\underline{u}\leq u\leq\overline{u}} |\nabla u|^p dx + \int_{u<\underline{u}} |\nabla \underline{u}|^p dx + \int_{u>\overline{u}} |\nabla \overline{u}|^p dx \qquad (16) \\ &\leq ||\nabla u||_p^p + ||\nabla \underline{u}||_p^p + ||\nabla \overline{u}||_p^p,\end{aligned}$$

$$\int_\Omega |Tu|^r dx \leq \int_\Omega (|\underline{u}| + |\overline{u}|)^r dx = M_0. \tag{17}$$

Combining (16), (17), using Young's inequality and the Sobolev embedding theorem, we can find a positive constant M_1 such that for any positive number ϵ

$$\begin{aligned}< S_2u, u > &= \int_\Omega a_0(x, Tu, \nabla Tu)u dx \\ &\geq \int_\Omega \left[-C_0 |Tu|^{r\frac{p-1}{p}} - C_0 |\nabla Tu|^{p-1} - k_0(x)\right] |u| dx \\ &\geq -C_0 ||Tu||_r^{r\frac{p-1}{p}} ||u||_p - C_0 ||\nabla Tu||_p^{p-1} ||u||_p - ||k_0||_q ||u||_p \\ &\geq -C_0 M_0^{\frac{p-1}{p}} ||u||_p - C_0 \left[\frac{||u||_p^p}{\epsilon^p p} + \frac{\epsilon^q ||\nabla Tu||_p^p}{q}\right]\end{aligned}$$

$$\geq -C_0 M_0^{\frac{p-1}{p}} ||u||_p - C_0 \left[\frac{||u||_p^p}{\epsilon^p p} + \frac{\epsilon^q ||\nabla u||_p^p}{q} \right]$$
$$- C_0 \frac{\epsilon^q [||\nabla \underline{u}||_p^p + ||\nabla \overline{u}||_p^p]}{q} - ||k_0||_q ||u||_p. \qquad (18)$$

Applying Lemma 3.4, we can find positive real numbers α, β such that

$$< S_3 u, u > \geq M(\alpha ||u||_p^p - \beta). \qquad (19)$$

Combining (15), (18) and (19), we obtain

$$< Su, u > \geq C_1 \|\nabla u\|_p^p - \|k_1\|_1 - C_0 M_0^{\frac{p-1}{p}} \|u\|_p - C_0 \left[\frac{||u||_p^p}{\epsilon^p p} + \frac{\epsilon^q ||\nabla u||_p^p}{q} \right]$$
$$- C_0 \frac{\epsilon^q [||\nabla \underline{u}||_p^p + ||\nabla \overline{u}||_p^p]}{q} - ||k_0||_q ||u||_p + M(\alpha ||u||_p^p - \beta). \qquad (20)$$

Choosing a sufficiently small positive real number ϵ and a sufficiently large positive real number M such that $C_1 > \dfrac{C_0 \epsilon^q}{q}$, $M\alpha > \dfrac{C_0}{\epsilon^p p}$, we see that

$$\lim_{||u||_{1,p} \to \infty} \frac{< Su, u >}{||u||_{1,p}} = \infty.$$

Therefore, S is coercive.

Step 4. There is a solution of (13) in $[v, \overline{u}]$.

By Theorem 27.A in [20], there is a solution w of $S(u, \varphi) = 0$ in $W_0^{1,p}(\Omega)$. We prove that w is in the interval $[v, \overline{u}]$. Choosing $\varphi = (w - \overline{u})_+$, we obtain

$$0 = \int_\Omega \sum_{i=1}^N a_i(x, Tw, \nabla w) \frac{\partial}{\partial x_i} (w - \overline{u})_+ dx + \int_\Omega a_0(x, T(w), \nabla T(w))(w - \overline{u})_+ dx$$
$$+ M \int_\Omega (w - \overline{u})_+^p dx$$
$$= \int_\Omega \sum_{i=1}^N a_i(x, \overline{u}, \nabla w) \frac{\partial}{\partial x_i} (w - \overline{u})_+ dx + \int_\Omega a_0(x, \overline{u}, \nabla \overline{u})(w - \overline{u})_+ dx$$
$$+ M \int_\Omega (w - \overline{u})_+^p dx. \qquad (21)$$

Since $\overline{u}$ is a supersolution of (2) and $(w - \overline{u})_+ \geq 0$, then

$$\int_\Omega \sum_{i=1}^N a_i(x, \overline{u}, \nabla \overline{u}) \frac{\partial}{\partial x_i} (w - \overline{u})_+ dx + \int_\Omega a_0(x, \overline{u}, \nabla \overline{u})(w - \overline{u})_+ dx \geq 0 \qquad (22)$$

Therefore,

$$\int_\Omega \sum_{i=1}^N [a_i(x,\overline{u},\nabla w)-a_i(x,\overline{u},\nabla \overline{u})]\frac{\partial}{\partial x_i}(w-\overline{u})_+dx+M\int_\Omega (w-\overline{u})_+^p dx \le 0. \quad (23)$$

It follows from (A2) that

$$\int_\Omega \sum_{i=1}^N [a_i(x,\overline{u},\nabla w)-a_i(x,\overline{u},\nabla \overline{u})]\frac{\partial}{\partial x_i}(w-\overline{u})_+dx \ge 0. \quad (24)$$

Combining (23) and (24), we have

$$M\int_\Omega (w-\overline{u})_+^p dx \le 0,$$

which implies that $(w-\overline{u})_+(x)=0$ for a.e. x in Ω. Thus $w(x)\le \overline{u}(x)$ for a.e. $x\in\Omega$. Similarly, we also have $w(x)\ge v(x)$ for a.e. $x\in\Omega$.

Step 5. w is a subsolution of (2).

By (F2), it follows that for any nonnegative function φ in $W_0^{1,p}(\Omega)$

$$\begin{aligned}\int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial \varphi}{\partial x_i}dx &= \int_\Omega [f(x,v,w,\nabla w)+a(v)-a(w)]\varphi dx\\ &\le \int_\Omega f(x,w,w,\nabla w)\varphi dx. \end{aligned} \quad (25)$$

Thus w is also a subsolution of (2). ■

Lemma 3.6. *There exists a positive real number M independent of v such that $||w||_{W_0^{1,p}(\Omega)} \le M$ for any w in Lemma* 3.5.

Proof. Replacing φ by w in (25), by (A3), (F1) and (F2), we get

$$\begin{aligned}C_1\|\nabla w\|_p^p-\|k_1\|_1 &= \int_\Omega [C_1|\nabla w|^p-k_1(x)]dx\\ &\le \int_\Omega \sum_{i=1}^N a_i(x,u,\nabla w)\frac{\partial w}{\partial x_i}dx\\ &= \int_\Omega [f(x,v,w,\nabla w)+a(v)-a(w)]udx\\ &\le \int_\Omega (k_2+C_2|\nabla w|^{p-1}+C_2|w|^{\frac{r(p-1)}{p}}+C_3|v|^{\frac{r(p-1)}{p}}\\ &\quad + C_3|w|^{\frac{r(p-1)}{p}}+2C_3)|w|dx\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega [k_2 + 2C_3 + C_2|\nabla w|^{p-1} + C_2(|\underline{u}| + |\overline{w}|)^{\frac{r(p-1)}{p}} \\
&\quad + 2C_3(|\underline{w}| + |\overline{w}|)^{\frac{r(p-1)}{p}}](|\underline{u}| + |\overline{u}|)dx \\
&\le \|k_2\|_q \|(|\underline{u}| + |\overline{u}|)\|_p + 2C_3||(|\underline{u}| + |\overline{u}|)||_1 \\
&\quad + (C_2 + 2C_3)||(|\underline{u}| + |\overline{u}|)||_r^{\frac{r(p-1)}{p}} ||(|\underline{u}| + |\overline{u}|)||_p \\
&\quad + C_2 \int_\Omega |\nabla u|^{p-1}(|\underline{u}| + |\overline{u}|) \\
&\le M_4 + C_2||\nabla u||_p^{p-1}||(|\underline{u}| + |\overline{u}|)||_p,
\end{aligned}
$$

Thus we have

$$
C_1||\nabla u||_p^p - ||k_1||_1 \le M_4 + M_5 + C_2||\nabla u||_p^{p-1}||(|\underline{u}| + |\overline{u}|)||_p,
$$

which yields the lemma. ∎

Proof of Theorem 3.2. Denote by $\mathfrak{S}_0$ the set of subsolutions u in $[\underline{u}, \overline{u}]$ of (2) such that there exists a subsolution v in $[\underline{u}, u]$ of (2) and u is a solution of (13). We see that $\mathfrak{S}_0$ is non-empty and bounded by Lemmas 3.5 and 3.6.

Let u be in $\mathfrak{S}_0$, by Lemma 3.5, there is a solution $u' \equiv H_0(u)$ in $[u, \overline{u}]$ of the following equation

$$
\begin{cases}
-\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u', \nabla u') + a(u') = f(x, u, u', \nabla u') + a(u) & \text{in } \Omega, \\
u' = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{26}
$$

It is easy to see that $H_0(\mathfrak{S}_0) \subset \mathfrak{S}_0$. Let $\{w_n\}$ be an increasing sequence in $\mathfrak{S}_0$. Since $\mathfrak{S}_0$ is bounded, then $\{w_n\}$ converges weakly to w. Since $w_n \in \mathfrak{S}_0$, there exists v_n being a subsolution of (2) such that $\underline{u} \le v_n \le w_n \le \overline{u}$ and for any nonnegative function φ in $W_0^{1,p}(\Omega)$ we have

$$
\begin{aligned}
\int_\Omega \sum_{i=1}^N a_i(x, w_n, \nabla w_n) \frac{\partial \varphi}{\partial x_i} dx &= \int_\Omega [f(x, v_n, w_n, \nabla w_n) + a(v_n) - a(w_n)]\varphi dx \\
&\ge \int_\Omega [f(x, \underline{u}, w_n, \nabla w_n) + a(\underline{u}) - a(w_n)]\varphi dx
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega \sum_{i=1}^N a_i(x, w_n, \nabla w_n) \frac{\partial}{\partial x_i}(w_n - w) dx \\
&\le \int_\Omega [f(x, \underline{u}, w_n, \nabla w_n) + a(\underline{u}) - a(w_n)](w_n - w) dx
\end{aligned}
$$

Thus

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N [a_i(x,w_n,\nabla w_n) - a_i(x,w_n,\nabla w)]\frac{\partial}{\partial x_i}(w_n - w)dx\\ &\le \int_\Omega \sum_{i=1}^N a_i(x,w_n,\nabla w)\frac{\partial}{\partial x_i}(w_n - w)dx\\ &\quad + \int_\Omega [f(x,\underline{u},w_n,\nabla w_n) + a(\underline{u}) - a(w_n)](w_n - w)dx.\end{aligned}$$

Using the same argument as in Lemma 3.3, we see that $\{w_n\}$ converges strongly to w in $W_0^{1,p}(\Omega)$. We can suppose that $\{w_n(x)\}$ and $\{\nabla w_n(x)\}$ converge to $w(x)$ and $\nabla w(x)$ for almost everywhere x in Ω. Now, we prove that $\{w_n\}$ has an upper bound v in $\mathfrak{S}_0$. Since $v_n \le w_n$ for any integer n, we have

$$v_n \le w \quad \forall\, n \in \mathbb{N}. \tag{27}$$

By (F2) and (27), for any nonnegative function φ in $W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N a_i(x,w_n,\nabla w_n)\frac{\partial \varphi}{\partial x_i}dx &= \int_\Omega [f(x,v_n,w_n,\nabla w_n) + a(v_n) - a(w_n)]\varphi dx\\ &\le \int_\Omega [f(x,w,w_n,\nabla w_n) + a(w) - a(w_n)]\varphi dx.\end{aligned}$$

By $(A0)$ and $(F2)$, it follows that

$$\int_\Omega \sum_{i=1}^N a_i(x,w,\nabla w)\frac{\partial \varphi}{\partial x_i}dx \le \int_\Omega f(x,w,w,\nabla w)\varphi dx.$$

Thus w is a subsolution of (2). By Lemma 3.5, there exists v in $\mathfrak{S}_0$ such that $\underline{u} \le w \le v \le \overline{u}$ and $\forall \varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \sum_{i=1}^N a_i(x,v,\nabla v)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega [f(x,w,v,\nabla v) + a(w) - a(v)]\varphi dx.$$

Therefore, v is an upper bound of $\{w_n\}$ in $\mathfrak{S}_0$. By Theorem 1.1, the operator H_0 has a fixed point w^* in $\mathfrak{S}_0 \subset [\underline{u},\overline{u}]$. It follows that for any φ in $W_0^{1,p}(\Omega)$

$$\int_\Omega \sum_{i=1}^N a_i(x,w^*,\nabla w^*)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega f(x,w^*,w^*,\nabla w^*)\varphi dx.$$

Let w^{**} be a solution of (13) in $[\underline{u},\overline{u}]$ such that $w^* \le w^{**}$, then $w^{**} \in \mathfrak{S}_0$. By Theorem 1.1, we have $w^* = w^{**}$ and get the theorem. ■

Remark 3.7. Theorem 3.2 have been studied in [11] if $a_i(x,u,\nabla u) = A_i(x,\nabla u)$ and there is a positive real number c such that

$$[a(r_1) - a(r_2)](r_1 - r_2) \geq c|r_1 - r_2|^p \quad \forall r_1, r_2 \in \mathbb{R}. \tag{28}$$

In our results we only need the following condition (see (F2))

$$[a(r_1) - a(r_2)](r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, r_1 \neq r_2.$$

Remark 3.8. If $1 < p < 2$, we show that the condition (28) is never satisfied by any a. Indeed, suppose that such a function exists. Put $x_n = \sum_1^n \frac{1}{m^{1/(p-1)}}$. We see that $\{x_n\}$ is an increasing sequence converging to a real number x, thus $a(x) \geq \sup_{n\in\mathbb{N}} a(x_n)$. Since $a(x_n) - a(x_{n-1}) \geq c(x_n - x_{n-1})^{p-1} = \frac{c}{n}$, then $a(x_n) - a(x_1) \geq \sum_2^n \frac{c}{m}$, which tends to infinity when n goes to infinity. Hence $a(x) = \infty$, which is a contradiction.

Moreover our result only partially needs conditions on compactness, ellipticity and coercivity.

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