

# The Cauchy problem for the quantum Boltzmann equation for bosons at very low temperature

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## Abstract

The system that describes the dynamics of a Bose-Einstein Condensate consists of a quantum Boltzmann equation of the excitation distribution function and the Gross-Pitaevskii equation of the condensate wave function. We solve the Cauchy problem for the quantum Boltzmann equation, that approximates the evolution of the distribution function of the excitations - thermal cloud, at the temperature regime which is very low compared to the Bose-Einstein Condensation critical temperature. Such an equation has a cubic kinetic transition probability kernel. We develop the existence and uniqueness result by means of abstract ODE's theory in Banach spaces by characterizing an invariant bounded, convex, closed subset  $\mathcal{S}$  of the positive cone associated with the Banach space  $C^1([0, \infty); L^1(|p|dp))$ . The subset  $\mathcal{S}$  depends on the kinetic transition probability kernel structure as well as the interaction law for bosons. It also depends on the shown propagation and creation of polynomial moments accounting for high energy tails in the sense of  $L^1$ . In addition, we show the scaled summability of polynomial moments by studying the propagation and generation of Mittag-Leffler moments. These estimates implies these solutions have exponential decaying high energy tails in the sense of  $L^1$ .

**Keywords** Quantum kinetic theory, low-temperature Bose particles, spin-Peierls model, Mittag-Leffler moments, abstract ODE theory.

**MSC:** 82C10, 82C22, 82C40.

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## 1 Introduction

After the first Bose-Einstein Condensate (BEC) was produced by Cornell, Wieman, and Ketterle, which led them to the 2001 Nobel Prize in Physics [3, 4], there has been an explosion of research on BECs and cold bosonic gases. Above the condensation temperature, the dynamic of a bose gas is determined by the Uehling-Uhlenbeck kinetic equation introduced in [46]; see for instance [20, 21] for interesting results and list of references. The first proof of BECs was done in [33]. Below the condensation temperature, the bosonic gas dynamics is governed by a system that couples a quantum Boltzmann and a Gross-Pitaevskii equations. In such a system, the wave function of the BEC follows the Gross-Pitaevskii equation and the quantum Boltzmann equation describes the evolution of the density function of the excitations. The coupled system was first derived by Kirkpatrick and

Dorfmann in [31, 32], using a Green function approach and was revisited by Zaremba-Nikuni-Griffin and Gardiner-Zoller et. al. in [26, 28, 30, 48]. It has then been developed and studied extensively in the last two decades by several authors (see [11, 27, 39, 44], and references therein). In [42], Spohn gives a heuristic derivation for the one-dimensional version of the coupled system, using an perturbation theory for the Uehling-Uhlenbeck equation. A formal derivation, for the full three-dimensional case, can be found at [41]. A mathematically rigorous derivation for the coupled system is still an open problem. A first step towards this direction may draw the ideas generated from the work in [16, 19], in combination with techniques from quantum field theory [41].

In this manuscript we study the excitations dynamics describe by a kinetic quantum Boltzmann model for low temperature condensates. More specifically, we are interested in the dynamics of dilute Bose gases at very low temperature under the assumption of reference [18, 22, 31, 32], that is, the BEC is very stable and contains a sizable number of atoms, the interaction between excited atoms is small, being the dominant interaction the one between excited atoms and the BEC. The evolution of the space homogeneous probability density distribution function  $f := f(t, p)$ , with  $(t, p) \in [0, \infty) \times \mathbb{R}^3$ , for  $p$  the momenta state variable, of such Bose gases can be described by the following bosonic quantum Boltzmann equation:

$$\frac{df}{dt} = n_c Q[f], \quad f(0, \cdot) = f_0, \quad (1.1)$$

where the interaction operator is defined as

$$\begin{aligned} Q[f] &:= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 [R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p)], \\ R(p, p_1, p_2) &:= \\ &|\mathcal{M}(p, p_1, p_2)|^2 \left[ \delta \left( \frac{\omega(p)}{k_B T} - \frac{\omega(p_1)}{k_B T} - \frac{\omega(p_2)}{k_B T} \right) \delta(p - p_1 - p_2) \right] \\ &\times [f(p_1)f(p_2)(1 + f(p)) - (1 + f(p_1))(1 + f(p_2))f(p)]. \end{aligned} \quad (1.2)$$

where  $\beta := \frac{1}{k_B T} > 0$  is the physical constant depending on the Boltzmann constant  $k_B$ , and the temperature of the quasiparticles  $T$  at equilibrium. The term  $\mathcal{M}(p, p_1, p_2)$  is the transition probability and the particle energy  $\omega(p)$  is given by the Bogoliubov dispersion law:

$$\omega(p) = \left[ \frac{gn_c}{m} |p|^2 + \left( \frac{|p|^2}{2m} \right)^2 \right]^{1/2}, \quad (1.3)$$

where  $p \in \mathbb{R}^3$  is the momenta,  $m$  is the mass of the particles,  $g$  is the interaction coupling constant and  $n_c = n_c(t) := |\Psi|^2(t)$  is the density of particles corresponding to the dynamics wave function  $\Psi(x, t)$  in the BEC.

As mentioned above,  $\Psi$  satisfies the cubic nonlinear Schrodinger equation and the evolution of the condensate density distribution  $n_c$ , under some further assumptions, follows the following differential equation (cf. [7, 42, 47])

$$\begin{cases} \frac{dn_c}{dt} &= -n_c \int_{\mathbb{R}^3} Q[f] dp, \\ n_c(0) &= n_0, \end{cases} \quad (1.4)$$

or, equivalently

$$\begin{cases} \frac{d}{dt} \log n_c &= - \int_{\mathbb{R}^3} Q[f] dp, \\ \log(n_c(0)) &= \log(n_0). \end{cases} \quad (1.5)$$

However, in the scope of our paper, we only focus on the study of the quantum Boltzmann equation and leave the coupling quantum Boltzmann equation - nonlinear Schrodinger equation topic for future research. We, therefore, impose the following condition on the density distribution of the condensate

$$n \in C^1[0, \infty), \quad \text{and there exists constants } \underline{n}_c, \bar{n}_c > 0 \text{ such that} \quad (1.6)$$

$$\underline{n}_c < n(t) < \bar{n}_c, \quad \forall t \in [0, \infty).$$

This assumption is physically meaningful. It says that the condensate does not vanish, and its density distribution is uniformly bounded from above and below in time.

The collision operator  $Q$  describes the interaction between the condensed and the excited atoms. The corresponding equilibrium distribution  $f_\infty$  of the collisional equation (1.1)-(1.2) has the form

$$f_\infty(p) = \frac{1}{e^{\beta\omega(p)} - 1}, \quad (1.7)$$

for  $\beta = (k_B T)^{-1}$ , as is usually referred as a Bose-Einstein distribution. In this work, we restrict the range of the temperature  $T$ , the condensate density  $n_c$ , and the interaction coupling constant  $g$  to values for which  $k_B T$  is much smaller than  $(gn_c/m)^{1/2}$ , i.e. a *cold gas regime*. Under this condition, the dispersion law  $\omega(p)$  in (1.3) is approximated by

$$\frac{1}{k_B T} \left[ \frac{gn_c}{m} |p|^2 + \left( \frac{|p|^2}{2m} \right)^2 \right]^{1/2} \approx \frac{c}{k_B T} |p|, \quad \text{where } c := \sqrt{\frac{gn_c}{m}},$$

when  $(gn_c/m)^{1/2}(k_B T)^{-1} = O(1)$  and  $k_B T \ll 1$ . In particular, the energy will be now defined by the classical phonon dispersion law (still using the same notation), see [15, 18, 29, 40]

$$\omega(p) = c|p|, \quad \text{for } c := c(t) = \sqrt{\frac{gn_c(t)}{m}}. \quad (1.8)$$

Under this *very cold gas regime*, the transition probability  $\mathcal{M}$  is approximated by

$$|\mathcal{M}|^2 = \kappa|p||p_1||p_2| \quad (1.9)$$

where

$$\kappa = \frac{9c}{64\pi^2 mn_c}. \quad (1.10)$$

We observe that  $O(\sqrt{\underline{c}}) \leq c(t) \leq O(\sqrt{\bar{c}})$  and  $O(\sqrt{\bar{c}}^{-1}) \leq \kappa \leq O(\sqrt{\underline{c}}^{-1})$  uniformly in time.

Different from previous mathematical works [7, 5, 8, 9], we do not truncate the transition probability  $|\mathcal{M}|^2$  from above, or assume that it is cut-off near the origin.

Thus, we perform the analysis in the whole momentum space, not in a piece of it or the torus [43], requiring a detailed control of the solution's tails.

Notice that in the pioneering experiments [3, 4, 10], one can observe the growth of the condensate after fast evaporative cooling. Equation (1.1)-(1.2) is the main term that leads to the growth of the BEC. Moreover, the kinetic equation (1.1)-(1.2) is also used to describe phonon interactions in anharmonic crystal lattices, first derived in this context by Peierls [37, 38], then by several other authors [15, 43].

In particular the linearization of the Quantum Boltzmann equation (1.1)-(1.2) about Bose-Einstein states is performed by setting

$$f(t, p) = f_\infty(p) + f_\infty(p)(1 + f_\infty(p))\Omega(t, p), \quad (1.11)$$

evaluated into collision operator in (1.2) and restricting the evaluation to the linear terms. The resulting linearized equation was obtained in [23]

$$f_\infty(p)(1 + f_\infty(p))\frac{\partial \Omega}{\partial t}(t, p) = -M(p)\Omega(t, p) + \int_{\mathbb{R}^3} dp' \mathcal{U}(p, p')\Omega(t, p'), \quad (1.12)$$

for some explicit function  $M(p)$  and measure  $\mathcal{U}(p, p')$ . The Cauchy problem and the convergence toward equilibrium of such linearized model (1.12) were

addressed in the aforementioned reference. The discrete theory of the equation, based on a dynamical system approach, was done in [17]. In reference [36], it has been proved that positive classical solutions of the model have a Gaussian in momenta barrier from below.

From now on, and without loss of generalization for the existence and uniqueness results as well as high energy tails behavior, we assume the temperature  $T \ll 1$ , such that  $0 < \underline{c}(k_B T)^{-1} < \bar{c}(k_B T)^{-1} < 1$  in the reduced phonon dispersion law (1.8), and so the quantum collisional integral (1.2) becomes

$$Q[f] := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 [R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p)], \quad (1.13)$$

$$R(p, p_1, p_2) := |p||p_1||p_2| [\delta(|p| - |p_1| - |p_2|) \delta(p - p_1 - p_2)] \\ \times [f(p_1)f(p_2)(1 + f(p)) - (1 + f(p_1))(1 + f(p_2))f(p)].$$

Clearly, from the interaction law  $p = p_1 + p_2$  and  $|p| = |p_1| + |p_2|$  modeled in the collision operator by the singular Dirac delta masses, this cubic collisional form (1.13) is reduced into a quadratic one, that can be split in the difference of two positive quadratic operators, as will be shown in the existence result.

In addition the low temperature quantum collisional form (1.13) can be split into *gain* and *loss* operator forms

$$Q[f](t, p) = Q^+[f](t, p) - Q^-[f](t, p) \\ = Q^+[f](t, p) - f(t, p) \nu[f](t, p), \quad (1.14)$$

as is done with the classical Boltzmann operator acting on an  $f(t, v)$ , for binary elastic interactions, when the transition probability (or collision kernel) is an integrable function with respect to the scattering angle as much as is integrable respect with a velocity  $v_*$ , the interacting with the velocity  $v$  in the binary process.

Here, the gain operator is also defined by the positive contributions in the total rate of change in time of the collisional form  $Q(f)(t, p)$  in (1.13),

$$Q^+[f](t, p) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 |p||p_1||p_2| \delta(p - p_1 - p_2) \\ \times \delta(|p| - |p_1| - |p_2|) f(t, p_1) f(t, p_2) + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 |p||p_1||p_2| \\ \times \delta(p_1 - p - p_2) \delta(|p_1| - |p| - |p_2|) [2f(t, p) f(t, p_1) + f(t, p_1)]. \quad (1.15)$$

In analog, the loss operator models the negative contributions in the total rate of change in time of same collisional form  $Q(f)(t, p)$ . It is local in  $f(t, p)$  and so written  $Q^-[f] := f \nu[f]$ , where  $\nu[f](t, p)$ , referred as the *collision frequency or attenuation coefficient*, defined by

$$\begin{aligned} \nu[f](t, p) := & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 |p| |p_1| |p_2| \delta(p - p_1 - p_2) \\ & \times \delta(|p| - |p_1| - |p_2|) [2f(t, p_1) + 1] + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 |p| |p_1| |p_2| \\ & \times \delta(p_1 - p - p_2) \delta(|p_1| - |p| - |p_2|) f(t, p_2), \end{aligned} \quad (1.16)$$

is nonlocal in  $f(t, p)$ .

**Remark 1.1** *In order to grant the split of the collision operator in gain and loss parts, it is necessary that  $\nu[f](t, p)$  is well defined. This is secured whenever solutions have at least the second moment finite throughout the evolution. This property will be automatically granted by the proofs of creation and propagation of statistical moments in Section 4 and the corresponding existence theorem in Section 5.*

Thus, our goal is to study the Cauchy problem of radial solutions for the quantum Boltzmann gas model at low temperature (1.1)-(1.13), or equivalently by (1.14, 1.15, 1.16). In addition we will show that the unique solutions of this Cauchy problem have exponential decaying tails in the sense of  $L^1(\mathbb{R}^3)$ , which we referred as Mittag-Leffler moments. This is the first step to solve an equation of the kind without cut-off assumptions in the transition probability kernels.

The existence and uniqueness arguments are based on techniques developed in the last few years for the classical Boltzmann equation in [2, 12, 24, 25, 45]. We point out that the propagation of polynomial moments enable us to find a natural space to show existence and uniqueness of solutions for equation using abstract ODE theory, without need of bounded initial entropy.

A technical difficulty in the analysis is the fact that the natural conservation law for the model is energy conservation, that is, the solution's first moment, whereas the homogeneity of the kinetic potential kernel in the model is 3. Due to this fact, it is essential to perform high moment analysis which, in contrast, it is not central for the Cauchy problem in the classical Boltzmann equation, refer to [6, 25, 35].

The organization of the paper is as follows. In Section 2 we present the weak formulation and recall the main conservation laws as well entropy estimate and corresponding analog to an  $h$ -Theorem for (1.1) with the low temperature regime collisional form (1.13).

The next three sections regard the Cauchy problem and high energy tail behavior, which will be fully developed in context of radially symmetric solutions. Section 3 is devoted to a key *a priori* estimate on the moments of equation (1.1)-(1.13) which will be used several times along the paper, Proposition 3.1. Using Proposition 3.1, we prove the creation and propagation of polynomial moments, Theorem 4.1 in Section 4. Then, using the *a priori* estimates of Section 4, we prove, in Section 5, existence and uniqueness of solutions of radially symmetric solutions for equation (1.1)-(1.13) under natural conditions. Existence is based on a Hölder estimate and a condition of the sub-tangent type for  $Q$ , see Theorem 5.2. Uniqueness is based on a one-side Lipschitz estimate.

Theorems 6.1 and 6.2 are the main results of Section 6. They address the propagation and creation of Mittag-Leffler moments for such solutions to low temperature quantum collision evolution given by (1.1)-(1.13).

## 2 Conservation of energy and momentum

For notational convenience, we will usually omit the time variable  $t$  unless some stress is necessary in the context.

The following properties hold for the the low temperature quantum collisional form (1.13).

**Proposition 2.1 (Weak Formulation)** *For any suitable test function  $\varphi$ , the following weak formulation holds for the collision operator (1.13)*

$$\begin{aligned} \int_{\mathbb{R}^3} dp n_c Q[f](p) \varphi(p) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp dp_1 dp_2 n_c |p| |p_1| |p_2| \delta(p - p_1 - p_2) \\ &\times \delta(|p| - |p_1| - |p_2|) [f(p_1) f(p_2) - f(p_1) f(p) - f(p_2) f(p) - f(p)] \\ &\times [\varphi(p) - \varphi(p_1) - \varphi(p_2)]. \end{aligned} \tag{2.1}$$

**Proof.** In this proof we use the short-hand  $\int := \int_{\mathbb{R}^9} dp dp_1 dp_2$ . First,

observe that

$$\begin{aligned}
& \int_{\mathbb{R}^3} dp n_c Q[f](p) \varphi(p) = \\
& \int n_c |p| |p_1| |p_2| \delta(p - p_1 - p_2) \delta(|p| - |p_1| - |p_2|) R(p, p_1, p_2) \varphi(p) \\
& - \int n_c |p| |p_1| |p_2| \delta(p - p_1 - p_2) \delta(|p| - |p_1| - |p_2|) R(p_1, p, p_2) \varphi(p) \\
& - \int n_c |p| |p_1| |p_2| \delta(p - p_1 - p_2) \delta(|p| - |p_1| - |p_2|) R(p_2, p_1, p) \varphi(p).
\end{aligned} \tag{2.2}$$

Second, interchanging variables  $p \leftrightarrow p_1$  and  $p \leftrightarrow p_2$ ,

$$\int n_c |p| |p_1| |p_2| R(p_1, p, p_2) \varphi(p) = \int n_c |p| |p_1| |p_2| R(p, p_1, p_2) \varphi(p_1), \tag{2.3}$$

and

$$\int n_c |p| |p_1| |p_2| R(p_2, p_1, p) \varphi(p) = \int n_c |p| |p_1| |p_2| R(p, p_1, p_2) \varphi(p_2). \tag{2.4}$$

Finally, combining (2.2), (2.3), (2.4), we get (2.1). ■

**Corollary 2.1 (Conservation laws)** *If  $f$  is a solution of (1.1)-(1.13), it formally conserves momentum and energy*

$$\int_{\mathbb{R}^3} dp f(t, p) p = \int_{\mathbb{R}^3} dp f_0(p) p, \tag{2.5}$$

$$\int_{\mathbb{R}^3} dp f(t, p) |p| = \int_{\mathbb{R}^3} dp f_0(p) |p|. \tag{2.6}$$

**Remark 2.1** *Since  $f$  is the density function of the thermal cloud, the mass is not conserved due to the fact that atoms could move in and out of the condensate. In other words, the total mass of the system thermal cloud - condensate is unchanged as time evolves, but the mass of each component of the system the thermal cloud and the condensate is not conserved.*

Now, let us look at the system that couples the two equations (1.1) and (1.4). Integrating Equation (1.1) in  $p$  and taking the sum with the second equation (1.5), we obtain

$$\frac{d}{dt} \left( \int_{\mathbb{R}^3} f(t, p) dp + n_c(t) \right) = 0, \tag{2.7}$$

which confirms that the total mass of the whole system is conserved.

**Corollary 2.2 (H-Theorem)** *If  $f(t, p)$  is a solution of (1.1)-(1.13), then*

$$\frac{d}{dt} \int_{\mathbb{R}^3} dp \left[ f(p) \log f(p) - (1 + f(p)) \log (1 + f(p)) \right] \leq 0.$$

*A radially symmetric equilibrium of the equation has the following form*

$$f(p) = \frac{1}{e^{\alpha \omega(p)} - 1}, \quad \text{for some } \alpha > 0. \quad (2.8)$$

**Proof.** We observe that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} dp \left[ f(p) \log f(p) - (1 + f(p)) \log (1 + f(p)) \right] = \\ \int_{\mathbb{R}^3} dp \partial_t f(p) \log \left( \frac{f(p)}{f(p) + 1} \right). \end{aligned}$$

In addition, we can rewrite

$$\begin{aligned} \int_{\mathbb{R}^3} dp n_c Q[f](p) \varphi(p) = \int_{\mathbb{R}^9} n_c |p| |p_1| |p_2| \delta(p - p_1 - p_2) \delta(|p| - |p_1| - |p_2|) \\ \times (1 + f(p)) (1 + f(p_1)) (1 + f(p_2)) \\ \times \left( \frac{f(p_1)}{f(p_1) + 1} \frac{f(p_2)}{f(p_2) + 1} - \frac{f(p)}{f(p) + 1} \right) \left[ \varphi(p) - \varphi(p_1) - \varphi(p_2) \right] dp dp_1 dp_2. \end{aligned}$$

Choosing  $\varphi(p) = \log \left( \frac{f(p)}{f(p)+1} \right)$  we obtain, in the case of equality, that

$$\frac{f(p_1)}{f(p_1) + 1} \frac{f(p_2)}{f(p_2) + 1} - \frac{f(p)}{f(p) + 1} = 0,$$

or equivalently, putting  $h(p) = \log \left( \frac{f(p)}{f(p)+1} \right)$ , we get

$$h(p_1) + h(p_2) = h(p). \quad (2.9)$$

The fact that  $h(\cdot)$  is radially symmetric yields  $h(p) = -\alpha \omega(p)$ , for all  $p \in \mathbb{R}^3$  and some positive constant  $\alpha$ . This proves the claim.  $\blacksquare$

The rest manuscript concerns the existence, uniqueness and high energy tail behavior of radially symmetric solutions.

### 3 *A priori* estimates on a solution's moments

$$f(t, p) = f(t, |p|).$$

Furthermore, we consider solutions of (1.1)-(1.13) that lie in  $\mathcal{C}([0, \infty); L^1(\mathbb{R}^3, |p|^k dp))$  where

$$L^1(\mathbb{R}^3, |p|^k dp) := \left\{ f \text{ measurable} \mid \int_{\mathbb{R}^3} dp |f(p)| |p|^k < \infty, k \geq 1 \right\}.$$

That is, in sections 3 and 4 the *a priori* estimates *assume* the existence of a radially symmetric solution enjoying time continuity in such Lebesgue spaces for  $k$  sufficiently large. Define the solution's moment of order  $k$  as

$$\mathcal{M}_k \langle f \rangle (t) := \int_{\mathbb{R}^3} dp f(t, |p|) |p|^k. \quad (3.1)$$

Using spherical coordinates, the integral with respect to  $dp$  on  $\mathbb{R}^3$  can be reduced to an integral on  $\mathbb{R}_+$  with respect to  $d|p|$ . Therefore, we also use the *line-moment* on  $\mathbb{R}_+$

$$m_k \langle f \rangle (t) := \int_0^\infty d|p| f(t, |p|) |p|^k. \quad (3.2)$$

We are going to use the definition of moments in two contexts: In one hand, in sections 3, 4 and 6 we always consider the moment applied to a given *radial solution of the equation*. Thus, there is no harm to omit the function dependence and just write  $\mathcal{M}_k(t)$ ,  $\mathcal{M}_k$ ,  $m_k(t)$  or  $m_k$  to denote moments and line-moments for simplicity. In the other hand, in section 5 we will use moments as norms of the spaces  $L^1(\mathbb{R}^3, |p|^k dp)$ , as a consequence, the functional dependence will be important. In addition, time dependence will not be key in this section, thus, we will write line-moments as  $m_k \langle \cdot \rangle$ . Note that, for radially symmetric functions,  $\mathcal{M}_k$  and  $m_{k+2}$  are equivalent. Then, according to the conservation law (2.6) and assuming initial energy finite, the following estimate hold

$$\mathcal{M}_1(t) = \mathcal{M}_1(0) < \infty, \quad m_3(t) = m_3(0) < \infty.$$

**Proposition 3.1 (Line-Moment Ordinary Differential Inequalities)**

For  $1/k \leq \gamma \leq 1$ ,  $k > 1$ , we have the following *a priori* estimate on the

moments valid with some universal constants  $C_1$  and  $C_2$

$$\begin{aligned} & \frac{d}{dt} m_{k\gamma+2}(t) \\ & \leq C_1 \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} (m_{i\gamma+4} m_{3+(k-i)\gamma} + m_{i\gamma+3} m_{4+(k-i)\gamma})(t) - C_2 m_{k\gamma+8}(t). \end{aligned} \quad (3.3)$$

In order to prove Proposition 3.1, we first need the following lemmata.

**Lemma 3.1** *For  $k > 3$ , we have the following equation for  $m_k$*

$$\begin{aligned} \frac{d}{dt} m_k(t) &= C(\pi) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 n_c(r_1 + r_2) r_1^3 r_2^3 [f(t, r_1) f(t, r_2) \\ & - 2f(t, r_1) f(t, r_1 + r_2) - f(t, r_1 + r_2)] \times [r_1 + r_2]^{k-2} - r_1^{k-2} - r_2^{k-2}. \end{aligned} \quad (3.4)$$

**Proof.** For simplicity we omit the  $t$ -time variable in this proof. Using  $|p|^{k-2}$  as a test function in (1.1)-(1.13) and recalling that the line-moment  $m_k$  is equivalent to  $\mathcal{M}_{k-2}$ , we obtain

$$\begin{aligned} \frac{d}{dt} m_k &= C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp dp_1 dp_2 |p| |p_1| |p_2| \delta(p - p_1 - p_2) \\ & \times \delta(|p| - |p_1| - |p_2|) [f(t, p_1) f(t, p_2) - f(t, p_1) f(t, p) \\ & - f(t, p_2) f(t, p) - f(t, p)] \times [p]^{k-2} - |p_1|^{k-2} - |p_2|^{k-2}, \end{aligned}$$

where  $C$  is some positive constant varying from line to line. The above integral, thanks to the Dirac measure  $\delta(p - p_1 - p_2)$ , can be reduced from an integral on  $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$  of  $dp dp_1 dp_2$  to an integral on  $\mathbb{R}^3 \times \mathbb{R}^3$  of  $dp_1 dp_2$

$$\begin{aligned} \frac{d}{dt} m_k &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dp_1 dp_2 n_c |p_1 + p_2| |p_1| |p_2| \delta(|p_1 + p_2| - |p_1| - |p_2|) \\ & \times [f(t, p_1) f(t, p_2) - f(t, p_1) f(t, p_1 + p_2) - f(t, p_2) f(t, p_1 + p_2) \\ & - f(t, p_1 + p_2)] \times [p_1 + p_2]^{k-2} - |p_1|^{k-2} - |p_2|^{k-2}. \end{aligned}$$

Using spherical coordinates one has  $dp_2 = |p_2|^2 \sin \gamma d|p_2| d\gamma d\rho$ , with  $\gamma \in [0, \pi]$ ,  $\rho \in [0, 2\pi]$ , and

$$\delta(|p_1 + p_2| - |p_1| - |p_2|) = \delta(1 - \cos \gamma).$$

Thus, we can reduce the integral of  $dp_2$  on  $\mathbb{R}^3$  to an integral of  $d|p_2|$  on  $\mathbb{R}_+$ .

$$\begin{aligned} \frac{d}{dt} m_k &= C(\pi) \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} dp_1 d|p_2| n_c |p_1 + p_2| |p_1| |p_2| [f(t, p_1) f(t, p_2) \\ & - f(t, p_1) f(t, p_1 + p_2) - f(t, p_2) f(t, p_1 + p_2) - f(t, p_1 + p_2)] \\ & \times [p_1 + p_2]^{k-2} - |p_1|^{k-2} - |p_2|^{k-2} |p_2|^2. \end{aligned}$$

This implies, by a similar change of variables, that one is able to reduce  $dp_1$  to  $d|p_1|$ . More specifically,

$$\begin{aligned} \frac{d}{dt}m_k &= C(\pi) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} d|p_1| d|p_2| n_c(|p_1| + |p_2|) |p_1|^3 |p_2|^3 \times \\ & [f(t, |p_1|)f(t, |p_2|) - f(t, |p_1|)f(t, |p_1| + |p_2|) - f(t, |p_2|)f(t, |p_1| + |p_2|) \\ & - f(t, |p_1| + |p_2|)] \times [|p_1 + p_2|^{k-2} - |p_1|^{k-2} - |p_2|^{k-2}]. \end{aligned}$$

This estimate completes the proof of this Lemma 3.1.  $\blacksquare$

**Lemma 3.2 (From Ref. [13])** *Assume that  $k > 1$ , let  $\lceil \frac{k+1}{2} \rceil$  denote the integer part of  $\frac{k+1}{2}$ . Then for all  $a, b > 0$ , the following inequality holds*

$$\begin{aligned} & \sum_{i=1}^{\lceil \frac{k+1}{2} \rceil - 1} \binom{k}{i} (a^i b^{k-i} + a^{k-i} b^i) \\ & \leq (a+b)^k - a^k - b^k \leq \sum_{i=1}^{\lceil \frac{k+1}{2} \rceil} \binom{k}{i} (a^i b^{k-i} + a^{k-i} b^i). \end{aligned} \quad (3.5)$$

**Proof. (of Proposition 3.1)** For simplicity we omit  $t$ , the time variable, in the argument of this proof. From (3.4), we eliminate the negative term  $-2f(t, r_1)f(t, r_1 + r_2)$  and take into account the fact that

$$|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma} > 0,$$

to get

$$\begin{aligned} \frac{d}{dt}m_{k\gamma+2}(t) &\leq C(\pi) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 n_c(r_1 + r_2) r_1^3 r_2^3 [f(t, r_1)f(t, r_2) - \\ & - f(t, r_1 + r_2)] \times [|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma}]. \end{aligned} \quad (3.6)$$

By applying the inequality

$$|r_1 + r_2|^{k\gamma} \leq (|r_1|^\gamma + |r_2|^\gamma)^k, \quad (3.7)$$

with  $1/k \leq \gamma \leq 1$  into (3.6), it yields

$$\begin{aligned} \frac{d}{dt}m_{k\gamma+2}(t) &\leq C(\pi, \bar{n}_c) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 f(t, r_1)f(t, r_2) \times \\ & [ (|r_1|^\gamma + |r_2|^\gamma)^k - r_1^{k\gamma} - r_2^{k\gamma} ] - C(\pi, \underline{n}_c) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \\ & \times f(t, r_1 + r_2) [|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma}]. \end{aligned} \quad (3.8)$$

In order to obtain (3.3), we estimate the two terms on the right hand side of (3.8). Using Lemma 3.2 with  $a = r_1^\gamma$  and  $b = r_2^\gamma$ , the first term can be estimated as follows

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \left[ (|r_1|^\gamma + |r_2|^\gamma)^k - r_1^{k\gamma} - r_2^{k\gamma} \right] f(t, r_1) f(t, r_2) \\ & \leq \int_{\mathbb{R}_+^2} dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( r_1^{i\gamma} r_2^{(k-i)\gamma} + r_1^{(k-i)\gamma} r_2^{i\gamma} \right) f(t, r_1) f(t, r_2), \end{aligned}$$

which, by a simple expansion process, can be bounded by

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( r_1^{i\gamma+4} r_2^{3+(k-i)\gamma} + r_1^{i\gamma+3} r_2^{4+(k-i)\gamma} \right. \\ & \quad \left. + r_1^{(k-i)\gamma+4} r_2^{i\gamma+3} + r_1^{(k-i)\gamma+3} r_2^{i\gamma+4} \right) f(t, r_1) f(t, r_2) \quad (3.9) \\ & \leq 2 \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( m_{i\gamma+4} m_{3+(k-i)\gamma} + m_{i\gamma+3} m_{4+(k-i)\gamma} \right) (t). \end{aligned}$$

Note that in the above inequality, we only use the definition of  $m_{i\gamma+3}$ ,  $m_{i\gamma+4}$ ,  $m_{(k-i)\gamma+3}$ , and  $m_{(k-i)\gamma+4}$ . Regarding the second term on the right side of (3.8), we rewrite it using the change of variables  $r_1 + r_2 \rightarrow r$  and  $r_1 \rightarrow r - r_2$

$$\begin{aligned} & - \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \left[ |r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma} \right] f(t, r_1 + r_2) \\ & = \int_0^\infty \int_0^r dr_2 dr r (r - r_2)^3 r_2^3 \left[ |r - r_2|^{k\gamma} + r_2^{k\gamma} - |r|^{k\gamma} \right] f(t, r). \quad (3.10) \end{aligned}$$

Set

$$I := \int_0^r dr_2 r (r - r_2)^3 r_2^3 \left[ |r - r_2|^{k\gamma} + r_2^{k\gamma} - |r|^{k\gamma} \right].$$

Then, by (3.7),  $I \leq 0$ . By the change of variables  $r_2 \rightarrow r - r_2$ , one gets the following identity

$$\int_0^r dr_2 (r - r_2)^{3+k\gamma} r_2^3 = \int_0^r dr_2 (r - r_2)^3 r_2^{3+k\gamma},$$

which implies the equality

$$I = \int_0^r dr_2 (r - r_2)^3 r_2^3 \left[ 2r_2^{k\gamma} - r^{k\gamma} \right]. \quad (3.11)$$

Develop  $(r - r_3)^3$  in the above integral, the following equality holds

$$\begin{aligned}
I &= \int_0^r dr_2 (r - r_2)^3 r_2^3 [2r_2^{k\gamma} - r^{k\gamma}] \\
&= \int_0^r dr_2 [r^3 - 3r_2 r^2 + 3r_2^2 r - r_2^3] [2r_2^{k\gamma+3} - r^{k\gamma} r_2^3] \\
&= \int_0^r dr_2 [2r_2^{k\gamma+3} r^3 - 6r_2^{k\gamma+4} r^2 + 6r_2^{k\gamma+5} r - 2r_2^{k\gamma+6} \\
&\quad - r^{k\gamma+3} r_2^3 + 3r^{k\gamma+2} r_2^4 - 3r^{k\gamma+1} r_2^5 + r_2^6 r^{k\gamma}] = -C r^{k\gamma+7},
\end{aligned} \tag{3.12}$$

where the last equality follows by evaluating the integral of  $dr_2$  in  $(0, r)$ . Since  $I \leq 0$ , the constant  $C$  is explicit and positive. Combining (3.10), (3.11), (3.12), we get the following equation for the second term on the right hand side of (3.8)

$$\begin{aligned}
& - \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 [|r_1 + r_2|^{k\gamma} - r_1^{k\gamma} - r_2^{k\gamma}] f(t, r_1 + r_2) \\
& = -C \int_0^\infty r^{k\gamma+8} f(t, r) dr = -C m_{k\gamma+8}.
\end{aligned} \tag{3.13}$$

Putting together (3.6), (3.9) and (3.13), we obtain the ordinary differential line-moments inequality

$$\frac{d}{dt} m_{k\gamma+2} \leq C \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( m_{i\gamma+4} m_{3+(k-i)\gamma} + m_{i\gamma+3} m_{4+(k-i)\gamma} \right) - C' m_{k\gamma+8}.$$

that shows inequality (3.3). Thus, the proof of Proposition 3.1 is now complete. ■

## 4 Creation and propagation of polynomial moments

Let us write the main result of this section.

**Theorem 4.1** *Suppose that  $f_0(p) = f_0(|p|)$ ,  $m_3(0) < \infty$  and  $m_k(t)$  defined in (3.2). Then, there exists a constant  $C_k(\mathfrak{h}_3)$  that depends only on  $\mathfrak{h}_3 := \mathfrak{h}_3(m_3(0))$ , and on  $k$  such that we have the following creation of the  $k^{\text{th}}$  line moment*

$$m_k(t) \leq C_k(\mathfrak{h}_3) (1 - e^{-C_k t})^{-\frac{k-3}{6}}, \quad \forall k > 3. \tag{4.1}$$

Moreover, if  $m_k(0) < \infty$ , we have the following propagation of the  $k^{\text{th}}$  line moment

$$m_k(t) \leq \max \{m_k(0), C_k(\mathfrak{h}_3)\}. \quad (4.2)$$

**Lemma 4.1 (Moment interpolation)** *The line-moment  $m_k = m_k(t)$  satisfies*

$$m_\rho \leq m_{\rho_1}^\gamma m_{\rho_2}^{1-\gamma}, \quad (4.3)$$

where the positive constants  $\rho, \rho_1, \rho_2, \gamma$  satisfy  $0 < \rho_1 \leq \rho \leq \rho_2$ ,  $0 < \gamma < 1$ , and  $\rho = \gamma\rho_1 + (1 - \gamma)\rho_2$ .

**Proof.** The proof of this statement is straightforward. Indeed, Hölder's inequality imply

$$\begin{aligned} m_{\rho_1}^\gamma m_{\rho_2}^{1-\gamma} &= \left( \int_{\mathbb{R}_+} dr |r|^{\rho_1} f(r) \right)^\gamma \left( \int_{\mathbb{R}_+} dr |r|^{\rho_2} f(r) \right)^{1-\gamma} \\ &\geq \int_{\mathbb{R}_+} dr |r|^{\rho_1\gamma + \rho_2(1-\gamma)} f(r) \geq \int_{\mathbb{R}_+} dr |r|^\rho f(r) \geq m_\rho. \end{aligned}$$

■

**Proof. (of Theorem 4.1)** In this proof, we will use Lemma 3.1 with  $\gamma = 1$  which reduces to

$$\frac{d}{dt} m_{k+2} \leq C_1 \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( m_{i+4} m_{3+(k-i)} + m_{i+3} m_{4+(k-i)} \right) - C_2 m_{k+8},$$

where  $C_1$  and  $C_2$  are some universal positive constants. For the sake of simplicity, we shift  $k+2 \rightarrow k$  in the above inequality to get

$$\frac{d}{dt} m_k \leq C_1 \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-2}{i} \left( m_{i+4} m_{1+(k-i)} + m_{i+3} m_{2+(k-i)} \right) - C_2 m_{k+6}. \quad (4.4)$$

From (4.4), our goal is to construct a differential inequality for  $m_k = m_k(t)$  from which the boundedness of  $m_k$  could be deduced. In order to do that, we will estimate the right hand side of (4.4) by some function of  $m_k$ , which leads to a uniform in time upper bound of  $m_k$ . First, let us start bounding the right hand side of (4.4) by estimating the term  $m_{i+4} m_{1+k-i}$  with Hölder's inequality,

$$m_{i+4} \leq m_3^{\frac{k+2-i}{k+3}} m_{k+6}^{\frac{i+1}{k+3}} = C m_{k+6}^{\frac{i+1}{k+3}},$$

where we notice that, by the conservation of energy (2.6),  $m_3$  and  $m_3^{\frac{k+1-i}{k+2}}$  are constants. Multiplying  $m_{i+4}$  by  $m_{1+k-i}$  and using Young's inequality

$$m_{i+4}m_{1+k-i} \leq C m_{k+6}^{\frac{i+1}{k+3}} m_{1+k-i} \leq \frac{m_{k+6}^{\frac{(i+1)p}{k+3}} \epsilon^p}{p} + \frac{m_{1+k-i}^q}{q\epsilon^q}. \quad (4.5)$$

We set  $q = \frac{k+3}{k+2-i}$  and  $p = \frac{k+3}{i+1}$  and choose  $\epsilon > 0$  in the sequel. The quantity  $m_{1+k-i}$  could be bounded by Hölder's inequality again

$$m_{1+k-i} \leq m_k^{\frac{k-i-2}{k-3}} m_3^{\frac{i-1}{k-3}}.$$

Therefore, from (4.5) and the aforementioned bound on  $m_{1+k-i}$ , we obtain the estimate for the term  $m_{i+4}m_{1+k-i}$  on the right side of (4.4)

$$m_{i+4}m_{1+k-i} \leq \frac{m_{k+6}\epsilon^p}{p} + \frac{m_k^{\frac{(k+3)(k-i-2)}{(k+2-i)(k-3)}}}{q\epsilon^q}. \quad (4.6)$$

Since

$$\frac{1}{2} < \frac{(k+3)(k-i-2)}{(k+2-i)(k-3)} < \frac{k-1}{k-3},$$

an interpolation argument applied to inequality (4.6) leads to

$$m_{i+4}m_{1+k-i} \leq \frac{m_{k+6}\epsilon^p}{p} + C \frac{m_k^{1/2}}{q\epsilon^q} + C \frac{m_k^{\frac{k-1}{k-3}}}{q\epsilon^q}, \quad (4.7)$$

where  $C$  is some positive constant that can vary from line to line. Second, we continue estimating the right side of (4.4) by controlling the term  $m_{i+3}m_{2+k-i}$ . We consider two cases: (1)  $i \geq 2$  (then  $2+k-i \leq k$ ), and (2)  $i = 1$  (then  $i+3 = 4 \leq k$ ). Let us start with the latter.

Case (2). Using Hölder inequality (4.3) and the conservation of momentum on  $m_3$

$$m_{2+k-i} \leq m_3^{\frac{4+i}{k+3}} m_{k+6}^{\frac{k-i-1}{k+3}} = C m_{k+6}^{\frac{k-i-1}{k+3}}.$$

Multiplying this inequality by  $m_{i+3}$  and employing Hölder's inequality again, we have

$$m_{i+3}m_{2+k-i} \leq C m_{i+3} m_{k+6}^{\frac{k-i-1}{k+3}} \leq \frac{m_{i+3}^r}{r\epsilon^r} + \frac{m_{k+6}^{\frac{s(k-i-1)}{k+3}} \epsilon^s}{s}, \quad (4.8)$$

where we set  $s = \frac{k+3}{k-1-i}$  and  $r = \frac{k+3}{i+4}$ . Since  $i+3 \leq k$ , we can use Hölder's inequality

$$m_{i+3} \leq m_k^{\frac{i}{k-3}} m_3^{\frac{k-3-i}{k-3}}.$$

One concludes that

$$m_{i+3} m_{2+k-i} \leq \frac{m_{k+6} \epsilon^s}{s} + \frac{m_k^{\frac{i}{k-3} \frac{k+3}{i+4}}}{r \epsilon^r} = \frac{m_{k+6} \epsilon^s}{s} + \frac{m_k^{\frac{k+3}{5(k-3)}}}{r \epsilon^r}. \quad (4.9)$$

For Case (1) a similar argument is made to conclude that

$$m_{i+3} \leq m_3^{\frac{k+3-i}{k+3}} m_{k+6}^{\frac{i}{k+3}} = C m_{k+6}^{\frac{i}{k+3}}.$$

Multiplying  $m_{i+3}$  by  $m_{2+k-i}$  and using Young's inequality

$$m_{i+3} m_{2+k-i} \leq C m_{k+6}^{\frac{i}{k+3}} m_{2+k-i} \leq \frac{m_{k+6}^{\frac{is'}{k+3}} \epsilon^{s'}}{s'} + \frac{m_{2+k-i}^{r'}}{r' \epsilon^{r'}},$$

where we set  $r' = \frac{k+3}{k+3-i}$  and  $s' = \frac{k+3}{i}$ . The quantity  $m_{2+k-i}$  can be bounded as

$$m_{2+k-i} \leq m_k^{\frac{k-i-1}{k-3}} m_3^{\frac{i-2}{k-3}}.$$

Therefore, we obtain the estimate for the term  $m_{i+3} m_{2+k-i}$  for the right side of (4.4)

$$m_{i+3} m_{2+k-i} \leq \frac{m_{k+6} \epsilon^p}{p} + \frac{m_k^{\frac{(k+3)(k-i-1)}{(k+3-i)(k-3)}}}{q \epsilon^q}.$$

Since

$$\frac{1}{2} < \frac{(k+3)(k-i-1)}{(k+3-i)(k-3)} < \frac{k-1}{k-3},$$

we can interpolate to conclude that

$$m_{i+3} m_{2+k-i} \leq \frac{m_{k+6} \epsilon^{s'}}{s'} + C \frac{m_k^{\frac{1}{2}}}{r' \epsilon^{r'}} + C \frac{m_k^{\frac{k-1}{k-3}}}{r' \epsilon^{r'}}. \quad (4.10)$$

Combining (4.4), (4.5), (4.9) and (4.10), we get

$$\frac{d}{dt} m_k \leq C(\epsilon) m_{k+6} + C'(\epsilon) \left[ m_k^{\frac{k-1}{k-3}} + m_k^{\frac{k+3}{5(k-3)}} + m_k^{\frac{1}{2}} \right] - C'' m_{k+6}, \quad (4.11)$$

where  $C(\epsilon)$  and  $C'(\epsilon)$  are positive constants satisfying  $C(\epsilon) \rightarrow 0$  and  $C'(\epsilon) \rightarrow \infty$  as  $\epsilon \rightarrow 0$ , and  $C''$  is a positive constant depending only on  $\mathfrak{h}_3 := m_3(0)$ .

Notice also that  $C(\epsilon)$  and  $C'(\epsilon)$  also depend on  $k$ . For  $\epsilon > 0$  sufficiently small, the constant  $C(\epsilon)$  is absorbed by  $C''$  and we infer from (4.11) that

$$\frac{d}{dt}m_k \leq C_k \left[ m_k^{\frac{k-1}{k-3}} + m_k^{\frac{k+3}{5(k-3)}} + m_k^{\frac{1}{2}} \right] - \frac{C''}{2} m_{k+6}, \quad (4.12)$$

for some  $C_k > 0$  depending only on  $k > 3$ . In order to obtain a differential inequality for  $m_k$ , it remains to estimate  $m_{k+6}$ . Indeed, using Hölder's inequality (4.3)

$$m_{k+6}^{\frac{k-3}{k+3}} m_3^{\frac{6}{k+3}} \geq m_k,$$

which implies  $m_{k+6} \geq m_k^{\frac{k+3}{k-3}}$ . As a consequence, from (4.12) we finally arrive to

$$\frac{d}{dt}m_k(t) \leq C_k \left[ m_k^{\frac{k-1}{k-3}} + m_k^{\frac{k+3}{5(k-3)}} + m_k^{\frac{1}{2}} \right](t) - \frac{C''}{2} m_k^{\frac{k+3}{k-3}}(t). \quad (4.13)$$

By Young inequality, there are positive constants  $C(\epsilon)$  and  $\epsilon$  such that

$$m_k^{\frac{k-1}{k-3}} \leq \epsilon m_k^{\frac{k+3}{k-3}}(t) + C(\epsilon), \quad m_k^{\frac{k+3}{5(k-3)}} \leq \epsilon m_k^{\frac{k+3}{k-3}}(t) + C(\epsilon),$$

and by Cauchy inequality

$$m_k^{\frac{1}{2}} \leq \frac{1}{2} m_k + \frac{1}{2}.$$

Combining the above inequalities, for  $\epsilon$  small, with (4.13) we conclude that there are positive constants, still denoted by  $C_k$  and  $C''/2$ , such that

$$\frac{d}{dt}m_k(t) \leq C_k(1 + m_k(t)) - C'' m_k^{\frac{k+3}{k-3}}(t). \quad (4.14)$$

By comparing (4.14) with the solution of the Bernoulli equation

$$\frac{d}{dt}Y(t) \leq C_k Y(t) - C'' Y^{\frac{k+3}{k-3}}(t),$$

which is

$$\begin{aligned} Y(t) &= \left[ (Y(0)e^{-C_k t})^{-\frac{6}{k-3}} + \frac{C''}{C_k} (1 - e^{-\frac{C_k 6t}{k-3}}) \right]^{-\frac{k-3}{6}} \\ &\leq C_k(\mathfrak{h}_3) (1 - e^{-\frac{C_k 6t}{k-3}})^{-\frac{k-3}{6}}, \end{aligned}$$

where  $C_k(\mathfrak{h}_3) := (C_k/C'')^{\frac{k-3}{6}}$  is a constant depending linearly on  $\sqrt{n_c}$  and  $\sqrt{n_c}$ , since  $C''$  depends only on  $\mathfrak{h}_3 = m_3(0)$  and  $C_k$  only on  $k$ . Hence inequality (4.1) holds. In addition, if the initial  $k^{\text{th}}$  line-moment  $m_k(0)$  is finite, then clearly the bound may be improved at  $t = 0$ , and  $m_k(t)$  clearly satisfies inequality (4.2).  $\blacksquare$

## 5 The Cauchy Problem

This section is devoted to show existence and uniqueness of positive solutions of the initial value problem associated to equation (1.14), (1.15) and (1.16), which corresponds to solutions of the initial value problem for equation (1.1)-(1.13) where the collision operator has a transition probability given by  $|\mathcal{M}|^2 = \kappa|p||p_1||p_2|$  from (1.9) for  $p = p_1 + p_2$  and  $|p| = |p_1| + |p_2|$ .

The approach we use is based on an abstract framework for solving ODE's in Banach spaces applied in this context to find uniqueness of non-negative homogeneous radially symmetric solutions of the quantum Boltzmann equation for bosons at very low temperature in  $L^1(\mathbb{R}^3, |p|dp)$ , the set of measurable functions, integrable w.r.t. the measure  $|p|dp$ .

More specifically, we have the following theorem, whose proof can be found in the Appendix 7.

**Theorem 5.1** *Let  $E := (E, \|\cdot\|)$  be a Banach space,  $\mathcal{S}$  be a bounded, convex and closed subset of  $E$ , and  $Q : \mathcal{S} \rightarrow E$  be an operator satisfying the following properties:*

*Hölder continuity condition*

$$\|Q[f] - Q[g]\| \leq C\|f - g\|^\beta, \quad \beta \in (0, 1), \quad \forall f, g \in \mathcal{S}, \quad (5.1)$$

*Sub-tangent condition*

$$\liminf_{h \rightarrow 0^+} h^{-1} \text{dist}(f + hQ[f], \mathcal{S}) = 0, \quad \forall f \in \mathcal{S}, \quad (5.2)$$

*and, one-sided Lipschitz condition*

$$[Q[f] - Q[g], f - g] \leq C\|f - g\|, \quad \forall f, g \in \mathcal{S}, \quad (5.3)$$

where  $[\varphi, \phi] := \lim_{h \rightarrow 0^-} h^{-1}(\|\phi + h\varphi\| - \|\phi\|)$ .

Suppose that  $n = n(t)$  is a continuous function in  $C^1([0, \infty))$  and  $n$  is bounded uniformly from below and above by positive constants  $\underline{n}$  and  $\bar{n}$ .

Then the equation

$$\partial_t f = nQ[f] \text{ on } [0, \infty) \times E, \quad f(0) = f_0 \in \mathcal{S} \quad (5.4)$$

has a unique solution in  $C^1((0, \infty), E) \cap C([0, \infty), \mathcal{S})$ .

This theorem is an extension of Theorem A.1 proved in [14] by Bressan in the context of solving the elastic Boltzmann equation for hard spheres in 3 dimension. We point out that [14] does not properly show that (5.2) is satisfied in that case. For completeness of this manuscript we rewrite Bressan's unpublished proof in the Appendix. The Bressan's needed techniques can be found in [34]. Indeed, referring to the argument given in [1], using conditions (5.1) and (5.2) combined with [34, Theorem VI.2.2] one has that conditions (C1), (C2) and (C3) in [34, pg. 229] are satisfied and hence, together with (5.3), all needed conditions for the existence and uniqueness theorem [34, Theorem VI.4.3] for ODEs in Banach spaces are fulfilled.

For our particular case, we need to identify a suitable Banach space and a corresponding bounded, convex and closed subset  $\mathcal{S}$ .

Indeed, choosing  $E = L^1(\mathbb{R}^3, |p|dp)$ , the choice of the subspace  $\mathcal{S}$ , defined below in (5.5), specifically depend on the estimates to solutions of the quantum Boltzmann equation (1.14), (1.15) and (1.16), whose collisional operator satisfy conditions (5.1), (5.2) and (5.3) when the transition probability (1.9) is given by  $|\mathcal{M}|^2 = \kappa|p||p_1||p_2|$  for  $p = p_1 + p_2$  and  $|p| = |p_1| + |p_2|$ .

More specifically, such subset  $\mathcal{S} \subset L^1(\mathbb{R}^3, |p|dp)$  is characterized by the Hölder continuity and sub-tangent conditions (5.1) and (5.2), respectively, (to be shown next in subsection 5.2), and it is defined as follows:

$$\mathcal{S} := \left\{ f \in L^1(\mathbb{R}^3, |p|dp) \mid \begin{array}{l} \text{i. } f \text{ nonnegative \& radially symmetric,} \\ \text{ii. } m_3\langle f \rangle = \int_{\mathbb{R}_+} d|p| f(|p|)|p|^3 = \mathfrak{h}_3, \\ \text{iii. } m_{10}\langle f \rangle = \int_{\mathbb{R}_+} d|p| f(|p|)|p|^{10} \leq \mathfrak{h}_{10} \end{array} \right\}, \quad (5.5)$$

where  $\mathfrak{h}_3$  is an arbitrary initial energy, and the specific  $\mathfrak{h}_{10}$  is defined below in (5.29). We are now in conditions to state and prove the existence and uniqueness theorem.

**Theorem 5.2 (Existence and Uniqueness)** *Let  $f_0(p) = f_0(|p|) \in \mathcal{S}$ . Then, equation (1.1)-(1.13) with (1.9) has a unique momentum and energy conservative solution*

$$0 \leq f(t, p) = f(t, |p|) \in \mathcal{C}([0, \infty); \mathcal{S}) \cap \mathcal{C}^1((0, \infty); L^1(\mathbb{R}^3, |p|dp)). \quad (5.6)$$

**Proof.** The proof of this theorem consists of verifying the three conditions (5.1), (5.2), and (5.3) in Subsections 5.1, 5.2, and 5.3, respectively. We start first with the Hölder continuity condition.

## 5.1 Hölder Estimate for $Q$

Recall the definition of  $m_k\langle f \rangle$ , the  $k^{\text{th}}$ -line-moment of a radially symmetric  $f(p) := f(|p|)$

$$m_k\langle f \rangle := \int_{\mathbb{R}_+} dp f(|p|) |p|^k, \quad k \geq 0, \quad (5.7)$$

and observe that  $m_3\langle |f| \rangle$  is equivalent to the usual norm for a radially symmetric function in  $L^1(\mathbb{R}^3, |p|dp)$ .

**Lemma 5.1 (Hölder continuity)** *The collision operator*

$$Q : \mathcal{S} \rightarrow L^1(\mathbb{R}^3, |p|dp)$$

*is Hölder continuous, with the following Hölder estimate*

$$m_3\langle |Q[f] - Q[g]| \rangle \leq A_1 m_3\langle |f - g| \rangle^{\frac{1}{7}} + A_2 m_3\langle |f - g| \rangle, \quad (5.8)$$

*valid for all  $f, g \in \mathcal{S}$ . The constants  $A_i$ , for  $i = \{1, 2\}$ , depend only on  $\mathfrak{h}_3$  and  $\mathfrak{h}_{10}$ .*

**Proof.** We first observe that for any  $f \in \mathcal{S}$ , properties **i.** and **ii.** in (5.5) yield the interpolation estimates shown in (4.3) for moments  $m_5\langle f \rangle \leq \mathcal{C}_5$  and  $m_6\langle f \rangle \leq \mathcal{C}_6$ , with  $\gamma = \frac{2}{7}$  and  $\gamma = \frac{3}{7}$  and positive constants depending only on  $\mathfrak{h}_3$  and  $\mathfrak{h}_{10}$ , respectively.

Next, in order to estimate the  $L^1(\mathbb{R}^3, |p|dp)$ -norm of the difference of the collision operator on any pair of functions  $f$  and  $g$  in  $\mathcal{S}$ , we use the weak formulation shown in Proposition 2.1 applied to the test function  $\varphi(p) = \text{sign}(Q[f] - Q[g])(p)$ , yielding the identity

$$\begin{aligned} \int_{\mathbb{R}^3} dp |Q[f] - Q[g]|(p) |p| &= \int_{\mathbb{R}^3} dp (Q[f] - Q[g])(p) \text{sign}(Q[f] - Q[g])(p) |p| \\ &= \int_{\mathbb{R}^9} dp dp_1 dp_2 |p p_1 p_2| \delta(p - p_1 - p_2) \delta(|p| - |p_1| - |p_2|) \\ &\quad \times \left[ f(p_1) f(p_2) - 2f(p_2) f(p) - f(p) - g(p_1) g(p_2) + 2g(p_2) g(p) + g(p) \right] \\ &\quad \times \left[ |p| \text{sign}(Q[f] - Q[g])(p) - |p_1| \text{sign}(Q[f] - Q[g])(p_1) \right. \\ &\quad \left. - |p_2| \text{sign}(Q[f] - Q[g])(p_2) \right]. \end{aligned}$$

So, using the triangle inequality, it follows

$$\int_{\mathbb{R}^3} dp |Q[f] - Q[g]|(p) |p|$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^9} dp dp_1 dp_2 |p p_1 p_2| \delta(p - p_1 - p_2) \delta(|p| - |p_1| - |p_2|) \\
&\quad \times \left| f(p_1) f(p_2) - 2f(p_2) f(p) - f(p) - g(p_1) g(p_2) + 2g(p_2) g(p) + g(p) \right| \\
&\quad \times \left[ |p| + |p_1| + |p_2| \right]. \tag{5.9}
\end{aligned}$$

Hence, using the same change of coordinates (3.10) used to obtain the a priori moment's estimates, now applied to the above inequality (5.9), yields

$$\begin{aligned}
&\int_{\mathbb{R}_+} dr |Q[f] - Q[g]|(r) r^3 \leq \\
&C \int_0^\infty \int_0^r dr_2 dr |r - r_2|^3 |r_2|^3 r |f(r - r_2) f(r_2) - 2f(r_2) f(r) - f(r) \\
&\quad - g(r - r_2) g(r_2) + 2g(r_2) g(r) + g(r)| (|r| + |r - r_2| + |r_2|), \tag{5.10}
\end{aligned}$$

where  $C$  is an explicit positive constant that varies from line to line. Now, since  $|r| + |r - r_2| + |r_2| = 2r$  in the  $0 \leq r_2 \leq r$  domain of integration, the simplified expression follows

$$\begin{aligned}
&\int_{\mathbb{R}_+} dr |Q[f] - Q[g]|(r) r^3 \leq \\
&C \int_0^\infty \int_0^r dr dr_2 r^2 |r - r_2|^3 |r_2|^3 |f(r - r_2) f(r_2) - 2f(r_2) f(r) - f(r) \\
&\quad - g(r - r_2) g(r_2) + 2g(r_2) g(r) + g(r)| \\
&= Q_1 + Q_2 + Q_3, \tag{5.11}
\end{aligned}$$

where the  $Q_i$ , with  $i \in \{1, 2, 3\}$ , are defined by

$$\begin{aligned}
&Q_1[f, g] := \\
&C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |f(r - r_2) f(r_2) - g(r - r_2) g(r_2)|, \tag{5.12}
\end{aligned}$$

$$Q_2[f, g] := C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |f(r_2) f(r) - g(r_2) g(r)|, \tag{5.13}$$

and

$$Q_3[f, g] := C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |f(r) - g(r)|. \tag{5.14}$$

Therefore, the proof of the Hölder estimate for the collision operator follows from estimating these three terms.

**Estimating  $Q_1$ .** First, splitting  $f(r - r_2)f(r_2) - g(r - r_2)g(r_2)$  as the sum of  $f(r - r_2)(f(r_2) - g(r_2))$  and  $g(r_2)(f(r - r_2) - g(r - r_2))$  and applying the triangle inequality from (5.12) yields

$$\begin{aligned} Q_1[f, g] &\leq C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |f(r - r_2)| |f(r_2) - g(r_2)| \\ &\quad + C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |g(r_2)| |f(r - r_2) - g(r - r_2)|. \end{aligned} \quad (5.15)$$

Exchanging variables  $r - r_2 \rightarrow r_1$ , the right side of (5.15) is bounded by

$$\begin{aligned} \int_{\mathbb{R}_+} dr |Q_1[f] - Q_1[g]|(r) r^3 &\leq C \int_{\mathbb{R}_+^2} dr_1 dr_2 (r_1 + r_2)^2 r_1^3 r_2^3 |f(r_1)| |f - g|(r_2) \\ &\quad + C \int_{\mathbb{R}_+^2} dr_1 dr_2 (r_1 + r_2)^2 r_1^3 r_2^3 |g(r_2)| |f - g|(r_1). \end{aligned}$$

Next, using the inequality  $(r_1 + r_2)^2 \leq 2(r_1^2 + r_2^2)$ , the right hand side integral is simplifies to

$$\begin{aligned} Q_1[f, g] &\leq C \int_{\mathbb{R}_+^2} dr_1 dr_2 (r_1^5 r_2^3 + r_1^3 r_2^5) |f(r_1)| |f(r_2) - g(r_2)| \\ &\quad + C \int_{\mathbb{R}_+^2} dr_1 dr_2 (r_1^5 r_2^3 + r_1^3 r_2^5) |g(r_2)| |f(r_1) - g(r_1)| \quad (5.16) \\ &\leq C (\mathfrak{h}_3 + \mathcal{C}_5) \int_{\mathbb{R}_+} dr |f(r) - g(r)| (|r|^3 + |r|^5), \end{aligned}$$

where last inequality holds by the propagation of moments estimate

$$\int_{\mathbb{R}_+} dr r^3 \max\{f, g\}(r) \leq \mathfrak{h}_3, \quad \int_{\mathbb{R}_+} dr r^5 \max\{f, g\}(r) \leq \mathcal{C}_5. \quad (5.17)$$

Finally, using Hölder inequality

$$\begin{aligned} \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^5 &\leq \left( \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3 \right)^{1/3} \\ &\quad \times \left( \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^6 \right)^{2/3} \leq \mathcal{C}_6^{2/3} \left( \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3 \right)^{1/3}, \end{aligned}$$

leads to estimate for the term  $Q_1$  as follows,

$$\begin{aligned} Q_1[f, g] &\leq C \mathfrak{h}_3 \mathcal{C}_6^{2/3} \left( \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3 \right)^{1/3} \\ &\quad + C \mathcal{C}_5 \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3, \end{aligned} \quad (5.18)$$

where, we recall, the constants  $\mathcal{C}_5$  and  $\mathcal{C}_6$  are controlled by  $\mathfrak{h}_3$  and  $\mathfrak{h}_{10}$ .

**Estimating  $Q_2$ .** Expressing  $f(r_2)f(r) - g(r_2)g(r)$  as the sum of  $(f(r_2) - g(r_2))f(r)$  and  $g(r_2)(f(r) - g(r))$  we estimate (5.13) as

$$\begin{aligned} Q_2[f, g] &\leq C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |f(r_2) - g(r_2)| |f(r)| \\ &\quad + C \int_0^\infty \int_0^r dr_2 dr r^2 |r - r_2|^3 |r_2|^3 |f(r) - g(r)| |g(r_2)|. \end{aligned} \quad (5.19)$$

Since  $|r - r_2| \leq |r|$ , we obtain from (5.19) that

$$\begin{aligned} Q_2[f, g] &\leq C \int_0^\infty \int_0^r dr_2 dr |r|^5 |r_2|^3 |f(r_2) - g(r_2)| |f(r)| \\ &\quad + C \int_0^\infty \int_0^r dr_2 dr |r|^5 |r_2|^3 |f(r) - g(r)| |g(r_2)| \\ &\leq C \mathfrak{h}_3 \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^5 + C \mathcal{C}_5 \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3, \end{aligned} \quad (5.20)$$

where we have used in the last inequality (5.17). By the same argument as (5.18), we get

$$\begin{aligned} Q_2[f, g] &\leq C \mathfrak{h}_3 \mathcal{C}_6^{2/3} \left( \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3 \right)^{1/3} \\ &\quad + C \mathcal{C}_5 \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^3. \end{aligned} \quad (5.21)$$

**Estimating  $Q_3$ .** Integrating in  $r_2$ , we can rewrite (5.14) as an integral in  $r$  only

$$Q_3[f, g] = C \int_{\mathbb{R}_+} dr |f(r) - g(r)| |r|^9, \quad (5.22)$$

where  $C$  is some other universal constant. Thus, using Hölder inequality as in (4.3) on  $|f - g|(r)$  with  $\gamma = \frac{6}{7}$ , one obtains

$$\begin{aligned} C^{-1} Q_3[f, g] &= \int_{\mathbb{R}_+} dr |f - g|(r) |r|^9 \\ &\leq \left( \int_{\mathbb{R}_+} dr |f - g|(r) |r|^{10} \right)^{6/7} \times \left( \int_{\mathbb{R}_+} dr |f - g|(r) |r|^3 \right)^{1/7} \\ &\leq (2\mathfrak{h}_{10})^{6/7} \left( \int_{\mathbb{R}_+} dr |f - g|(r) |r|^3 \right)^{1/7}. \end{aligned} \quad (5.23)$$

Therefore, estimate (5.8) follows by gathering (5.18), (5.21) and (5.23).  $\blacksquare$

## 5.2 Sub-tangent condition

This condition, jointly with the Hölder continuity, characterize the subset  $\mathcal{S} \subset L^1(\mathbb{R}^3, |p|dp)$  defined in (5.5).

First, we show that the collision operator  $Q$  can be split as the sum of a gain and a loss operators, as mentioned earlier in (1.14)

$$Q[f] = Q^+[f] - f\nu[f],$$

provided  $\nu[f]$  is finite whenever  $f \in \mathcal{S}$ . Indeed, this property follows by the nature of the interaction law (i.e. the form of the singular mass term in the integrand) and transition probability  $\mathcal{M}$ , since

$$\begin{aligned} \nu[f](p) &= \int_{\mathbb{R}^3} dp_1 |p||p_1| |p - p_1| \delta(|p| - |p_1| - |p - p_1|) [2f(p_1) + 1] \\ &\quad + 2 \int_{\mathbb{R}^3} dp_2 |p||p + p_2| |p_2| \delta(|p + p_2| - |p| - |p_2|) f(p_2) \\ &= \int_0^{|p|} dr |p| r^3 (|p| - r) [2f(r) + 1] + 2 \int_{\mathbb{R}_+} dr |p| (|p| + r) r^3 f(r) \\ &\leq C |p| (m_3 \langle f \rangle^{\frac{5}{4}} + m_4 \langle f \rangle + |p|^5), \end{aligned} \tag{5.24}$$

and, therefore,

$$|\nu[f](p)| \leq C(\mathfrak{h}_3, \mathfrak{h}_{10}) |p| (1 + |p|^5), \quad \forall f \in \mathcal{S}. \tag{5.25}$$

The sub-tangent condition (5.2) follows as a corollary of next Proposition 5.1.

**Proposition 5.1** *Fix  $f \in \mathcal{S}$ . Then, for any  $\epsilon > 0$ , there exists  $h_1 := h_1(f, \epsilon) > 0$ , such that the ball centered at  $f + hQ[f]$  with radius  $h\epsilon > 0$  intersects  $\mathcal{S}$ , that is,*

$$B(f + hQ[f], h\epsilon) \cap \mathcal{S}, \text{ is non-empty for any } 0 < h < h_1.$$

**Proof.** First, set  $\chi_R(p)$  the characteristic function of the ball of radius  $R > 0$  and introduce the truncated function  $f_R(p) := \chi_R(p)f(p)$ , then set  $w_R := f + hQ[f_R]$ .

We can control  $w_R$  from below to show it is possible to find an  $h_1$  such that  $w_R$  remains non-negative for as long  $0 < h < h_1$ . Indeed, for any  $f \in \mathcal{S}$  its truncation  $f_R(p) \in \mathcal{S}$  as well, and since  $Q^+$  is a positive operator,

$$w_R = f + Q^+[f_R] - hf_R\nu[f_R] \geq f - hf_R\nu[f_R]$$

$$\geq f\left(1 - h C(\mathfrak{h}_3, \mathfrak{h}_{10})R(1 + |R|^5)\right) \geq 0 \quad (5.26)$$

for any  $0 < h < h_1 := 1/C(\mathfrak{h}_3, \mathfrak{h}_{10})R(1 + |R|^5)$ . In addition, since  $f_R \in \mathcal{S}$ ,  $Q[f_R] \in L^1(\mathbb{R}^3, |p|dp)$  by Lemma 5.1, and, as a consequence,  $w_R \in L^1(\mathbb{R}^3, |p|dp)$  as well. Moreover, by conservation of energy  $\int_{\mathbb{R}^3} dp Q[f_R]|p|^3 = 0$ , yielding

$$\begin{aligned} m_3\langle w_R \rangle &= \int_{\mathbb{R}^3} dp w_R(|p|)|p|^3 = \int_{\mathbb{R}^3} dp (f + hQ[f_R])|p|^3 \\ &= \int_{\mathbb{R}^3} dp f(|p|)|p|^3 = \mathfrak{h}_3, \end{aligned} \quad (5.27)$$

with  $\mathfrak{h}_3$  independent of the parameter  $R$ . In particular,  $w_R$  satisfies, uniformly in  $R$ , property **i.** in the characterization of the  $\mathcal{S}$  defined in (5.5).

Finally we need to show that  $w_R$  also satisfies property **ii.** in the set  $\mathcal{S}$ . First, recall the *a priori* estimate for developed in (4.13) for the line-moment inequalities, namely

$$\begin{aligned} \int_{\mathbb{R}^3} dp Q[f]|p|^k &\leq \mathcal{L}_k(m_k\langle f \rangle) := \\ &C_k \left[ m_k\langle f \rangle^{\frac{k-1}{k-3}} + m_k\langle f \rangle^{\frac{k+3}{5(k-3)}} + m_k\langle f \rangle^{\frac{1}{2}} \right] - \frac{C''}{2} m_k\langle f \rangle^{\frac{k+3}{k-3}}, \end{aligned} \quad (5.28)$$

holds for any  $k > 3$  and  $C_k$  only depending on  $k$ , and  $C''$  only depending on  $m_3\langle f \rangle = \mathfrak{h}^3$ . Note that the map  $\mathcal{L}_k : [0, \infty) \rightarrow \mathbb{R}$  has only one root, denoted as  $\mathfrak{h}_*^k$ , at which  $\mathcal{L}_k$  changes from positive to negative for any  $k > 3$ . Note that this root only depends on  $\mathfrak{h}^3$  and  $k$ . Thus, it is always the case that

$$\int_{\mathbb{R}^3} dp Q[f]|p|^k \leq \mathcal{L}_k(m_k\langle f \rangle) \leq \max_{0 \leq x \leq \mathfrak{h}_*^k} \{\mathcal{L}_k(x)\}, \quad f \in \mathcal{S}.$$

Fix  $k = 10$  and define

$$\mathfrak{h}_{10} := \mathfrak{h}_*^{10} + \max_{0 \leq x \leq \mathfrak{h}_*^{10}} \{\mathcal{L}_{10}(x)\}. \quad (5.29)$$

For any  $f \in \mathcal{S}$ , we have two possibilities:  $m_{10}\langle f \rangle \leq \mathfrak{h}_*^{10}$ , or  $m_{10}\langle f \rangle > \mathfrak{h}_*^{10}$ . For the former, it readily follows that

$$\begin{aligned} m_{10}\langle w_R \rangle &= \int_{\mathbb{R}^3} dp w_R(|p|)|p|^{10} = \int_{\mathbb{R}^3} dp (f + hQ[f_R])|p|^{10} \\ &\leq \mathfrak{h}_*^{10} + h \max_{0 \leq x \leq \mathfrak{h}_*^{10}} \{\mathcal{L}_{10}(x)\} \leq \mathfrak{h}_{10}, \end{aligned}$$

where in the last inequality we have assumed  $h \leq 1$  without loss of generality.

For the latter, we can choose  $R := R(f)$  sufficiently large such that  $m_{10}\langle f_R \rangle \geq \mathfrak{h}_*^{10}$ , and therefore,

$$\int_{\mathbb{R}^3} dp Q[f_R] |p|^{10} \leq \mathcal{L}_{10}(m_{10}\langle f_R \rangle) \leq 0.$$

As a consequence,

$$m_{10}\langle w_R \rangle = \int_{\mathbb{R}^3} dp (f + hQ[f_R]) |p|^{10} \leq \int_{\mathbb{R}^3} dp f |p|^{10} \leq \mathfrak{h}_{10}.$$

The conclusion is that for any  $f \in \mathcal{S}$ , it is always the case that

$$m_{10}\langle w_R \rangle \leq \mathfrak{h}_{10}, \quad (5.30)$$

which ensures that  $w_R$  satisfies property **ii.** of the set  $\mathcal{S}$  in (5.5). We infer, thanks to (5.26), (5.27) and (5.30), that  $w_R \in \mathcal{S}$  for any  $0 < h < h_*$  where

$$h_* = \min \left\{ 1, 1 / (C(\mathfrak{h}_3)R(f)(1 + |R(f)|^5)) \right\}. \quad (5.31)$$

The argument ends using the Hölder estimate from Lemma 5.1 to obtain

$$\begin{aligned} h^{-1} m_3 \langle |f + hQ[f] - w_R| \rangle &= m_3 \langle |Q[f] - Q[f_R]| \rangle \\ &\leq A_1 m_3 \langle |f - f_R| \rangle^{\frac{1}{7}} + A_2 m_3 \langle |f - f_R| \rangle \leq \epsilon, \end{aligned}$$

for  $R := R(\epsilon)$  sufficiently large. Then,  $w_R \in B(f + hQ[f], h\epsilon)$  for this choice. Thus, choosing  $R = \max\{R(f), R(\epsilon)\}$  and  $h_1 := h_1(f, \epsilon)$  as in (5.31) one concludes that  $w_R \in B(f + hQ[f], h\epsilon) \cap \mathcal{S}$ . Consequently,

$$h^{-1} \text{dist}(f + hQ[f], \mathcal{S}) \leq \epsilon, \quad \forall 0 < h < h_1.$$

The proof of Proposition 5.1 is now complete. ■

### 5.3 One-side Lipschitz condition

Using dominate convergence theorem one can show that

$$[\varphi, \phi] \leq \int_{\mathbb{R}^3} dp \varphi(p) \text{sign}(\phi) |p|.$$

Thus, the one-side Lipschitz condition is met after proving the following lemma showing a Lipschitz condition for quantum-Boltzmann operator. The

following proof, which yields a uniqueness results, is in the same spirit of the original Di Blassio [?] uniqueness proof for initial value problem to the homogeneous Boltzmann equation for hard spheres, using data with enough initial moments.

**Lemma 5.2 (Lipschitz condition)** *Assume  $f, g \in \mathcal{S}$ . Then, there exists constant  $C := C(\mathfrak{h}_3, \mathfrak{h}_{10}) > 0$  such that*

$$\int_{\mathbb{R}^3} dp (Q[f](p) - Q[g](p)) \text{sign}(f - g) (|p|^1 + |p|^2) \leq C m_3 \langle |f - g| \rangle.$$

**Proof.** We start with the identity valid for radial functions  $f := f(|p|)$  and  $\varphi := \varphi(|p|)$

$$\begin{aligned} \int_{\mathbb{R}^3} dp Q[f](p) \varphi(p) &= 2(2\pi)^2 \int_0^\infty \int_0^\infty dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \\ &\quad \times [\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)] R(f)(r_1, r_2), \end{aligned}$$

where

$$R(f)(r_1, r_2) := f(r_1)f(r_2) - 2f(r_1)f(r_1 + r_2) - f(r_1 + r_2).$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^3} dp (Q[f](p) - Q[g](p)) \varphi(p) &= 2(2\pi)^2 \int_0^\infty \int_0^\infty dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \\ &\quad \times [\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)] (R(f)(r_1, r_2) - R(g)(r_1, r_2)), \end{aligned} \quad (5.32)$$

where, by definition

$$\begin{aligned} R(f)(r_1, r_2) - R(g)(r_1, r_2) &= (f(r_1)f(r_2) - g(r_1)g(r_2)) \\ &\quad - 2(f(r_1)f(r_1 + r_2) - g(r_1)g(r_1 + r_2)) - (f(r_1 + r_2) - g(r_1 + r_2)). \end{aligned}$$

Now, let us particularize for  $\varphi := \varphi_k = |\cdot|^k \text{sign}(f - g)$ , with  $k \in \{1, 2\}$ , and control each of the natural 3 terms appearing in the right side of (5.32). For the first, use simply  $|\varphi_k| \leq |\cdot|^k$  to obtain

$$\begin{aligned} &(f(r_1)f(r_2) - g(r_1)g(r_2)) [\varphi_k(r_1 + r_2) - \varphi_k(r_1) - \varphi_k(r_2)] \\ &\leq (|f(r_1) - g(r_1)| |f(r_2) + g(r_1)| |f(r_2) - g(r_2)|) [|r_1 + r_2|^k + |r_1|^k + |r_2|^k]. \end{aligned}$$

Since  $|r_1 + r_2|^k + |r_1|^k + |r_2|^k \leq 2(r_1 + r_2)^k$ , it readily follows that

$$\begin{aligned} & \int_0^\infty \int_0^\infty dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \\ & \quad \times [\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)] (f(r_1)f(r_2) - g(r_1)g(r_2)) \\ & \leq 2^{k+1} m_3 \langle f + g \rangle m_{k+4} \langle |f - g| \rangle + 2^{k+1} m_{k+4} \langle f + g \rangle m_3 \langle |f - g| \rangle. \end{aligned} \quad (5.33)$$

Similar argument for the second term, together with the change of variable  $r_1 + r_2 \rightarrow r_2$ , leads to

$$\begin{aligned} & -2 \int_0^\infty \int_0^\infty dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 \\ & \quad \times [\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)] (f(r_1)f(r_1 + r_2) - g(r_1)g(r_1 + r_2)) \\ & \leq 2 m_3 \langle g \rangle m_{k+4} \langle |f - g| \rangle + 2 m_{k+4} \langle f \rangle m_3 \langle |f - g| \rangle. \end{aligned} \quad (5.34)$$

Now, the absorption (third) term is nonpositive for  $k = 1$  since

$$\begin{aligned} & -(f(r_1 + r_2) - g(r_1 + r_2)) [\varphi_1(r_1 + r_2) - \varphi_1(r_1) - \varphi_1(r_2)] \\ & \leq |f(r_1 + r_2) - g(r_1 + r_2)| [|r_1| + |r_2| - |r_1 + r_2|] = 0. \end{aligned}$$

In addition, for  $k = 2$  it follows that

$$\begin{aligned} & -(f(r_1 + r_2) - g(r_1 + r_2)) [\varphi_2(r_1 + r_2) - \varphi_2(r_1) - \varphi_2(r_2)] \\ & \leq |f - g|(r_1 + r_2) [|r_1|^2 + |r_2|^2 - |r_1 + r_2|^2] = -2r_1 r_2 |f - g|(r_1 + r_2). \end{aligned}$$

In turn, this leads to

$$\begin{aligned} & - \int_0^\infty \int_0^\infty dr_1 dr_2 (r_1 + r_2) r_1^3 r_2^3 [\varphi(r_1 + r_2) - \varphi(r_1) - \varphi(r_2)] (f - g)(r_1 + r_2) \\ & \leq -2 \int_0^\infty \int_0^\infty dr_1 dr_2 (r_1 + r_2) r_1^4 r_2^4 |f(r_1 + r_2) - g(r_1 + r_2)| \\ & = -2 \int_0^\infty dr r |f - g|(r) \int_0^r dr_1 r_1^4 (r - r_1)^4 = -C m_{10} \langle |f - g| \rangle, \end{aligned} \quad (5.35)$$

for some universal  $C > 0$ . Gathering (5.33), (5.34) and (5.35) we conclude that for  $f, g \in \mathcal{S}$

$$\begin{aligned} & \int_{\mathbb{R}^3} dp (Q[f](p) - Q[g](p)) (|p|^1 + |p|^2) \text{sign}(f - g) \leq c_1 m_3 \langle |f - g| \rangle \\ & \quad + c_2 m_5 \langle |f - g| \rangle + c_3 m_6 \langle |f - g| \rangle - C m_{10} \langle |f - g| \rangle \leq c_4 m_3 \langle |f - g| \rangle, \end{aligned}$$

where the constants  $c_i$ , with  $i \in \{1, 2, 3, 4\}$ , depend on  $\mathfrak{h}_3$  and  $\mathfrak{h}_{10}$ . The last inequality follows noticing that  $c_1 r^3 + c_2 r^5 + c_3 r^6 - C r^{10} \leq c_4 r^3$  for any  $r \geq 0$ .  $\blacksquare$

The proof of Theorem 5.2 is now completed, as an application of Theorem 5.4, where the three conditions (5.1), (5.2), and (5.3) have been verified in Subsections 5.1, 5.2, and 5.3, respectively.  $\blacksquare$

## 6 Mittag-Leffler moments

### 6.1 Propagation of Mittag-Leffler tails

In this section we are interested in studying the propagation and creation of Mittag-Leffler moments of order  $a \in [1, \infty)$  and rate  $\alpha > 0$ . In terms infinite sums, see [45], this is equivalent to control the integral

$$\int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_a(\alpha^a |p|) = \sum_{k=1}^{\infty} \frac{\mathcal{M}_k(t) \alpha^{ak}}{\Gamma(ak + 1)}, \quad (6.1)$$

where

$$\mathcal{E}_a(x) := \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(ak + 1)} \approx e^{x^{1/a}} - 1, \quad x \gg 1. \quad (6.2)$$

We have excluded the term  $k = 0$  to account for the fact that equation (1.1)-(1.13) does not conserve mass. For convenience define for any  $\alpha > 0$  and  $a \in [1, \infty)$  the partial sums

$$\mathcal{E}_a^n(\alpha, t) := \sum_{k=1}^n \frac{\mathcal{M}_k(t) \alpha^{ak}}{\Gamma(ak + 1)} \quad \text{and} \quad \mathcal{I}_{a,\rho}^n(\alpha, t) := \sum_{k=1}^n \frac{\mathcal{M}_{k+\rho}(t) \alpha^{ak}}{\Gamma(ak + 1)}, \quad \rho > 0.$$

This notation will be of good use throughout this section.

**Theorem 6.1 (Propagation of Mittag-Leffler tails)** *Let  $f$  be a solution of (1.13) in  $\mathcal{S}$  associated to the initial condition  $f_0 \geq 0$ ,  $a \in [1, \infty)$ , and suppose that there exists positive  $\alpha_0$  such that*

$$\int_{\mathbb{R}^3} dp f_0(p) \mathcal{E}_a(\alpha_0^a |p|) \leq 1.$$

*Then, there exists positive constant  $\alpha := \alpha(\mathcal{M}_1(0), \alpha_0, a)$  such that*

$$\int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_a(\alpha^a |p|) \leq 2, \quad \forall t \geq 0. \quad (6.3)$$

**Lemma 6.1 (From Ref. [45])** *Let  $k \geq 3$ , then for any  $a \in [1, \infty)$ , we have*

$$\sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} B(ai+1, a(k-i)+1) \leq C_a (ak)^{-1-a},$$

for some constant  $C_a$  depending on  $a$ .

**Lemma 6.2 (Moment interpolation)**

$$\mathcal{M}_\rho \leq \mathcal{M}_{\rho_1}^\gamma \mathcal{M}_{\rho_2}^{1-\gamma}, \quad (6.4)$$

where the positive constants  $\rho, \rho_1, \rho_2, \gamma$  satisfy  $0 < \rho_1 \leq \rho \leq \rho_2$ ,  $0 < \gamma < 1$ , and  $\rho = \gamma\rho_1 + (1-\gamma)\rho_2$ .

**Remark 6.1** *Contrary to section 4, we will work in this section with the moments  $\mathcal{M}_k$  rather than work with the line-moments  $m_k$ . It turns out to be clearer in terms of notation.*

**Lemma 6.3** *Let  $\alpha > 0$ ,  $a \in [1, \infty)$ . Then, the following estimate holds*

$$\begin{aligned} & \sum_{k=k_0}^n \sum_1^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) \frac{\alpha^{ak}}{\Gamma(ak+1)} \\ & \leq C_a \frac{ak_0+1}{(ak_0)^{1+a}} \mathcal{E}_a^n \mathcal{I}_{a,3}^n, \quad n \geq k_0 \geq 1, \end{aligned} \quad (6.5)$$

with universal constant  $C_a$  depending only on  $a$ .

**Proof.** First, we estimate the sum of the left side of (6.5) by controlling the sum  $\mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)}$  with  $2\mathcal{M}_i \mathcal{M}_{k-i+3}$  for any  $i \geq 3$ . This can be done using Hölder's inequality (6.4)

$$\mathcal{M}_{i+2} \leq \mathcal{M}_i^{\frac{k+1-2i}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{2}{k+3-2i}} \quad \text{and} \quad \mathcal{M}_{1+(k-i)} \leq \mathcal{M}_i^{\frac{2}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{k+1-2i}{k+3-2i}}.$$

Thus, the product of these terms is controlled by

$$\mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} \leq \mathcal{M}_i \mathcal{M}_{k-i+3}.$$

Similarly, from (6.4), the following inequalities also hold

$$\mathcal{M}_{i+1} \leq \mathcal{M}_i^{\frac{k-2i+2}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{1}{k+3-2i}} \quad \text{and} \quad \mathcal{M}_{2+(k-i)} \leq \mathcal{M}_i^{\frac{1}{k+3-2i}} \mathcal{M}_{k-i+3}^{\frac{k-2i+2}{k+3-2i}},$$

which lead to the estimate

$$\mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \leq \mathcal{M}_i\mathcal{M}_{k-i+3}.$$

As a consequence,

$$\mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \leq 2\mathcal{M}_i\mathcal{M}_{3+(k-i)}.$$

Therefore, it readily follows that

$$\begin{aligned} \mathcal{J} &:= \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( \mathcal{M}_{i+2}\mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1}\mathcal{M}_{2+(k-i)} \right) \frac{\alpha^{ak}}{\Gamma(ak+1)} \\ &\leq 2 \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \mathcal{M}_i\mathcal{M}_{3+(k-i)} \frac{\alpha^{ak}}{\Gamma(ak+1)}. \end{aligned} \quad (6.6)$$

Using the following identities for the Beta and Gamma functions

$$\begin{aligned} &B(ai+1, a(k-i)+1) \\ &= \frac{\Gamma(ai+1)\Gamma(a(k-i)+1)}{\Gamma(a(i+1)+a(k-i)+1)} = \frac{\Gamma(ai+1)\Gamma(a(k-i)+1)}{\Gamma(ak+2)}, \end{aligned}$$

and the identity  $\alpha^{ak} = \alpha^{\alpha i} \alpha^{a(k-i)}$ , we deduce from (6.6) that

$$\begin{aligned} \mathcal{J} &\leq 2 \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \frac{\mathcal{M}_i \alpha^{\alpha i}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \\ &\quad \times B(ai+1, a(k-i)+1) \frac{\Gamma(ak+2)}{\Gamma(ak+1)}. \end{aligned} \quad (6.7)$$

Since  $\Gamma(ak+2) = (ak+1)\Gamma(ak+1)$ , the term  $\frac{\Gamma(ak+2)}{\Gamma(ak+1)}$  in (6.7) can be reduced to  $ak+1$ . That is,

$$\begin{aligned} \mathcal{J} &\leq 2 \sum_{k=k_0}^n (ak+1) \\ &\quad \times \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \frac{\mathcal{M}_i \alpha^{\alpha i}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} B(ai+1, a(k-i)+1). \end{aligned} \quad (6.8)$$

Also, each component in the sum on the right side of (6.8) can be bounded as

$$\begin{aligned} & \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} B(ai+1, a(k-i)+1) \\ & \leq \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{j} B(aj+1, a(k-j)+1), \end{aligned}$$

which implies, by Lemma 6.1, that

$$\begin{aligned} & \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} B(ai+1, a(k-i)+1) \\ & \leq \frac{C_a}{(ak)^{1+a}} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}. \end{aligned} \quad (6.9)$$

Combining (6.8) and (6.9) yields the estimate on  $\mathcal{J}$

$$\mathcal{J} \leq 2C_a \sum_{k=k_0}^n \frac{ak+1}{(ak)^{1+a}} \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)}. \quad (6.10)$$

Notice that  $\frac{ak+1}{(ak)^{1+a}}$  decreases towards 0 as  $k$  increases to infinity. Therefore, from (6.10) one concludes that

$$\begin{aligned} & \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) \frac{\alpha^{ak}}{\Gamma(ak+1)} \\ & \leq 2C_a \frac{ak_0+1}{(ak_0)^{1+a}} \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \frac{\mathcal{M}_{k-i+3} \alpha^{a(k-i)}}{\Gamma(a(k-i)+1)} \\ & \leq 2C_a \frac{ak_0+1}{(ak_0)^{1+a}} \sum_{i=1}^n \frac{\mathcal{M}_i \alpha^{ai}}{\Gamma(ai+1)} \sum_{i=1}^n \frac{\mathcal{M}_{i+3} \alpha^{ai}}{\Gamma(ai+1)} \leq C_a \frac{ak_0+1}{(ak_0)^{1+a}} \mathcal{E}_a^n \mathcal{I}_{a,3}^n. \end{aligned} \quad (6.11)$$

■

**Lemma 6.4** *The following control is valid for any  $\alpha > 0$  and  $a \in [1, \infty)$*

$$\mathcal{I}_{a,6}^n(\alpha, t) \geq \frac{1}{\alpha^3} \mathcal{E}_a^n(\alpha, t) - \frac{1}{\alpha^{5/2}} \mathcal{M}_1 \mathcal{E}_a(a-1/2). \quad (6.12)$$

**Proof.** Observe that

$$\mathcal{I}_{a,6}^n(\alpha, t) = \sum_{k=1}^n \frac{\mathcal{M}_{k+6}(t)\alpha^{ak}}{\Gamma(ak+1)} \geq \sum_{k=1}^n \int_{\{|p| \geq \frac{1}{\sqrt{\alpha}}\}} dp \frac{|p|^{k+6}\alpha^{ak}}{\Gamma(ak+1)} f(t, p).$$

Note that in the set  $\{|p| \geq \frac{1}{\sqrt{\alpha}}\}$  one has  $|p|^{k+6} \geq \frac{|p|^k}{\alpha^3}$ , therefore

$$\begin{aligned} \mathcal{I}_{a,6}^n(\alpha, t) &\geq \frac{1}{\alpha^3} \sum_{k=1}^n \int_{\{|p| \geq \frac{1}{\sqrt{\alpha}}\}} dp \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t, p) \\ &= \frac{1}{\alpha^3} \left( \sum_{k=1}^n \int_{\mathbb{R}^3} dp \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t, p) - \sum_{k=1}^n \int_{\{|p| < \frac{1}{\sqrt{\alpha}}\}} dp \frac{|p|^k \alpha^{ak}}{\Gamma(ak+1)} f(t, p) \right). \end{aligned}$$

In the set  $\{|p| < \frac{1}{\sqrt{\alpha}}\}$  one has  $|p|^k < |p| \alpha^{-(k-1)/2}$ , consequently

$$\begin{aligned} \mathcal{I}_{a,6}^n(\alpha, t) &\geq \frac{1}{\alpha^3} \left( \mathcal{E}_a^n(t) - \sum_{k=1}^n \int_{\mathbb{R}^3} dp \frac{\alpha^{-(k-1)/2} \alpha^{ak}}{\Gamma(ak+1)} f(t, p) |p| \right) \\ &= \frac{1}{\alpha^3} \mathcal{E}_a^n(t) - \frac{\mathcal{M}_1}{\alpha^{5/2}} \sum_{k=1}^n \frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)}. \end{aligned}$$

Since

$$\sum_{k=1}^n \frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)} \leq \sum_{k=1}^{\infty} \frac{\alpha^{(a-1/2)k}}{\Gamma(ak+1)} = \mathcal{E}_a(a-1/2),$$

estimate (6.12) follows.  $\blacksquare$

**Proof. (of Theorem 6.1)** The proof consists in showing that for any  $a \in [1, \infty)$ , there exists positive constant  $\alpha$  such that

$$\mathcal{E}_a^n(\alpha, t) \leq 2, \quad \forall t \geq 0, \forall n \in \mathbb{N} \setminus \{0\}. \quad (6.13)$$

For this purpose we define for sufficiently small  $\alpha > 0$ , chosen in the sequel, the sequence of times

$$T_n := \sup \{t \geq 0 \mid \mathcal{E}_a^n(\alpha, \tau) \leq 2, \forall \tau \in [0, t]\}$$

and prove that  $T_n = +\infty$ . This sequence of times is well-defined and positive. Indeed, for any  $\alpha \leq \alpha_0$

$$\mathcal{E}_a^n(\alpha, 0) = \sum_{k=1}^n \frac{\mathcal{M}_k(0)\alpha^{ak}}{\Gamma(ak+1)} \leq \sum_{k=1}^n \frac{\mathcal{M}_k(0)\alpha_0^{ak}}{\Gamma(ak+1)} = \int_{\mathbb{R}^3} dp f_0(p) \mathcal{E}_a(\alpha_0^a |p|) \leq 1.$$

Since each term  $\mathcal{M}_k(t)$  is continuous in  $t$ , the partial sum  $\mathcal{E}_a^n(\alpha, t)$  is also continuous in  $t$ . Therefore,  $\mathcal{E}_a^n(\alpha, t) \leq 2$  in some nonempty interval  $(0, t_n)$  and, thus,  $T_n$  is well-defined and positive for every  $n \in \mathbb{N}$ .

Now, let us establish a differential inequality for the partial sums that implies  $T_n = +\infty$ . Note that (3.3), with  $\gamma = 1$ , implies that

$$\frac{d}{dt} \mathcal{M}_k \leq C_1 \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) - C_2 \mathcal{M}_{k+6}.$$

Multiplying the above inequality by  $\frac{\alpha^k}{\Gamma(ak+1)}$  and summing with respect to  $k$  in the interval  $k_0 \leq k \leq n$ , with  $k_0 \geq 1$  to be chosen later on sufficiently large,

$$\begin{aligned} \frac{d}{dt} \sum_{k=k_0}^n \frac{\mathcal{M}_k \alpha^k}{\Gamma(ak+1)} &\leq C_1 \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} \right. \\ &\quad \left. + \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) \frac{\alpha^k}{\Gamma(ak+1)} - C_2 \sum_{k=k_0}^n \frac{\mathcal{M}_{k+6} \alpha^k}{\Gamma(ak+1)}. \end{aligned} \quad (6.14)$$

We observe that the sum on the left side of (6.14) will become  $\frac{d}{dt} \mathcal{E}_a^n(\alpha, t)$  after adding

$$\frac{d}{dt} \sum_{k=1}^{k_0-1} \frac{\mathcal{M}_k \alpha^k}{\Gamma(ak+1)} \leq C(k_0, \alpha_0, a) < \infty \quad (6.15)$$

to this expression. The latter inequality holds due to the choice  $\alpha \leq \alpha_0$  and the control of moments (3.3). Therefore, from (6.14) and (6.15), we obtain the differential inequality

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_a^n(\alpha, t) &\leq C_1 \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \right. \\ &\quad \left. \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) \frac{\alpha^k}{\Gamma(ak+1)} - C_2 \sum_{k=k_0}^n \frac{\mathcal{M}_{k+6} \alpha^k}{\Gamma(ak+1)} + C(k_0, \alpha_0, a). \end{aligned} \quad (6.16)$$

Let us now estimate the sum on the right side of (6.16). We deduce from Theorem 4.1 that

$$\sum_{k=1}^{k_0} \frac{\mathcal{M}_{k+6} \alpha^k}{\Gamma(ak+1)} \leq \sum_{k=1}^{k_0} \frac{\mathcal{M}_{k+6} \alpha_0^k}{\Gamma(ak+1)} \leq C(k_0, \alpha_0, a),$$

which leads to the following estimate for (6.16)

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_a^n(\alpha, t) \leq C_1 \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} & \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \right. \\ & \left. \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) \frac{\alpha^k}{\Gamma(ak+1)} - C_2 \sum_{k=1}^n \frac{M_{k+6} \alpha^k}{\Gamma(ak+1)} + C(k_0, \alpha_0, a). \end{aligned} \quad (6.17)$$

By the definition of  $\mathcal{I}_{a,6}^n$

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_a^n(\alpha, t) \leq C_1 \sum_{k=k_0}^n \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{i} & \left( \mathcal{M}_{i+2} \mathcal{M}_{1+(k-i)} + \right. \\ & \left. \mathcal{M}_{i+1} \mathcal{M}_{2+(k-i)} \right) \frac{\alpha^k}{\Gamma(ak+1)} - C_2 \mathcal{I}_{a,6}^n + C(k_0, \alpha_0, a). \end{aligned} \quad (6.18)$$

Thus, thanks to Lemma 6.3, we have the control on (6.18)

$$\frac{d}{dt} \mathcal{E}_a^n \leq C_a \frac{ak_0 + 1}{(ak_0)^{1+a}} \mathcal{E}_a^n \mathcal{I}_{a,3}^n - C_2 \mathcal{I}_{a,6}^n + C(k_0, \alpha_0, a). \quad (6.19)$$

We now estimate the right hand side of (6.19) starting with the term  $\mathcal{I}_{a,3}^n$ . Using Cauchy inequality  $|p|^3 \leq \frac{1}{2} + \frac{1}{2}|p|^6$ , then

$$\mathcal{M}_{k+3} \leq \frac{1}{2} \mathcal{M}_k + \frac{1}{2} \mathcal{M}_{k+6}, \quad k \geq 0.$$

Multiplying this inequality with  $\frac{\alpha^{ak}}{\Gamma(ak+1)}$  and summing with respect to  $k$  in the interval  $0 \leq k \leq n$  yields

$$\mathcal{I}_{a,3}^n \leq \frac{1}{2} \mathcal{E}_a^n + \frac{1}{2} \mathcal{I}_{a,6}^n.$$

Since we are considering  $t \in [0, T_n]$  one has  $\mathcal{E}_a^n \leq 2$  and, as a result, the following inequality is valid

$$\mathcal{I}_{a,3}^n \leq 1 + \frac{1}{2} \mathcal{I}_{a,6}^n.$$

This implies from (6.19) the estimate on

$$\frac{d}{dt} \mathcal{E}_a^n \leq 2C_a \frac{ak_0 + 1}{(ak_0)^{1+a}} \left( 1 + \frac{1}{2} \mathcal{I}_{a,6}^n \right) - C_2 \mathcal{I}_{a,6}^n + C(k_0, \alpha_0, a). \quad (6.20)$$

Choosing  $k_0$  sufficiently large, the term  $2C_a \frac{ak_0+1}{2(ak_0)^{1+a}} \mathcal{I}_{a,6}^n$  is absorbed by  $\frac{C_2}{2} \mathcal{I}_{a,6}^n$ . Thus,

$$\frac{d}{dt} \mathcal{E}_a^n \leq -\frac{C_2}{2} \mathcal{I}_{a,6}^n + C(\mathcal{M}_1, \alpha_0, a). \quad (6.21)$$

Recall that  $C_2$  only depends on the energy  $\mathcal{M}_1 = \mathcal{M}_1(0)$ , thus,  $k_0$  only depends on the initial energy and  $a$ . Let us estimate the right side of (6.21) in terms of  $\mathcal{E}_a^n$ . Lemma 6.4 provides a lower bound on  $\mathcal{I}_{a,6}^n$  in terms of  $\mathcal{E}_a^n$  which can be used in (6.21) to obtain

$$\frac{d}{dt} \mathcal{E}_a^n \leq -\frac{C_2}{2\alpha^3} \mathcal{E}_a^n + \frac{C_2}{2\alpha^{5/2}} \mathcal{M}_1 \mathcal{E}_a(a-1/2) + C(\mathcal{M}_1, \alpha_0, a).$$

Integrating the differential inequality

$$\mathcal{E}_a^n \leq 1 + \frac{2\alpha^3}{C_2} \left( \frac{C_2}{2\alpha^{5/2}} \mathcal{M}_1 \mathcal{E}_a(a-1/2) + C(\mathcal{M}_1, \alpha_0, a) \right) < 2, \quad t \in [0, T_n], \quad (6.22)$$

provided that  $\alpha := \alpha(\mathcal{M}_1, \alpha_0, a) > 0$  is such that

$$\frac{2\alpha^3}{C_2} \left( \frac{C_2}{2\alpha^2} \mathcal{M}_1 \mathcal{E}_a(a-1/2) + C(\mathcal{M}_1, \alpha_0, a) \right) < 1.$$

Given the continuity of  $\mathcal{E}_a^n(\alpha, t)$  with respect to  $t$ , estimate (6.22) contradicts the maximality of  $T_n$ , unless  $T_n = +\infty$ . Therefore,  $\mathcal{E}_a^n(\alpha, t) \leq 2$  for  $t \in [0, \infty)$  and  $n \in \mathbb{N} \setminus \{0\}$ . Now taking the limit as  $n \rightarrow \infty$  and using the definition of Mittag-Leffler moments of order  $a \in [1, \infty)$  and rate  $\alpha > 0$ , as defined in (6.1), yields

$$\int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_a(\alpha^a |p|) = \lim_{n \rightarrow \infty} \mathcal{E}_a^n(\alpha, t) \leq 2.$$

This concludes the argument. ■

## 6.2 Creation of exponential tails

**Theorem 6.2** *Let  $f$  be a positive solution of (1.1)-(1.13) in  $\mathcal{S}$ . Then, there exists constant  $\alpha > 0$  depending only on  $m_3(0)$  such that*

$$\int_{\mathbb{R}^3} dp f(t, p) |p| e^{\alpha \min\{1, t^{\frac{1}{6}}\} |p|} \leq \frac{1}{2\alpha}, \quad \forall t \geq 0. \quad (6.23)$$

**Proof.** Thanks to equation (4.1) we have the control

$$m_k(t) \leq C_k(\mathfrak{h}_3)(1 - e^{-C_k t})^{-\frac{k-3}{6}}, \quad \forall k > 3.$$

This implies that

$$\mathcal{E}_1^n(t^{\frac{1}{6}}\alpha, t) = \int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_1^n(t^{\frac{1}{6}}\alpha | p) \leq C_n(\alpha) t^{\frac{1}{6}}, \quad \alpha > 0. \quad (6.24)$$

Fix parameters  $\alpha, \vartheta \in (0, 1]$  and define

$$T_n := \sup \left\{ t \in [0, 1] \mid \mathcal{E}_1^n(t^{\frac{1}{6}}\alpha, t) \leq t^{\frac{1-\vartheta}{6}} \right\}.$$

We proof that for sufficiently small  $\alpha > 0$  depending only on  $m_3(0)$ ,  $T_n = 1$  for all  $n \in \mathbb{N}$  and  $\vartheta \in (0, 1]$ . One notices first that  $T_n > 0$  for each  $n$  thanks to (6.24). Also, for  $n \geq k_0 \geq 1$  we have

$$\frac{d}{dt} \sum_{k=k_0}^n \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{k!} = \sum_{k=k_0}^n \mathcal{M}'_k(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{k!} + \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_0}^n \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!}. \quad (6.25)$$

Observe that for the last term in the right side of (6.25)

$$\begin{aligned} & \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_0}^n \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!} \\ &= \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_0+6}^n \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!} + \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_0}^{k_0+5} \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!} \\ &= \frac{\alpha^6}{6} \sum_{k=k_0}^{n-6} \mathcal{M}_{k+6}(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{(k+5)!} + \frac{\alpha}{6t^{\frac{5}{6}}} \sum_{k=k_0}^{k_0+5} \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^{k-1}}{(k-1)!} \\ &\leq \frac{\alpha^6}{6} \sum_{k=k_0}^n \mathcal{M}_{k+6}(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{k!} + \frac{\alpha^{k_0}}{t^{\frac{5}{6}}} C(k_0, m_3(0)). \end{aligned}$$

Thus, arguing as in (6.14)-(6.19) we conclude that for the quantities

$$\mathcal{E}_1^n := \mathcal{E}_1^n(t^{\frac{1}{6}}\alpha, t), \quad \mathcal{I}_{1,6}^n := \mathcal{I}_{1,6}^n(t^{\frac{1}{6}}\alpha, t),$$

it follows that

$$\frac{d}{dt} \mathcal{E}_1^n \leq \frac{C}{k_0} \mathcal{E}_1^n \mathcal{I}_{1,3}^n - \left(C_2 - \frac{\alpha^6}{6}\right) \mathcal{I}_{1,6}^n + \frac{\alpha}{t^{\frac{5}{6}}} C(k_0, m_3(0)), \quad (6.26)$$

for a universal constant  $C > 0$  and constant  $C_2 > 0$  depending only  $m_3(0)$ . Using that

$$\mathcal{I}_{1,3}^n \leq \frac{\mathcal{E}_1^n}{2} + \frac{\mathcal{I}_{1,6}^n}{2}$$

and the definition of  $T_n$ , it follows from (6.26)

$$\frac{d}{dt}\mathcal{E}_1^n \leq \frac{C}{2k_o} - \left(C_2 - \frac{\alpha^6}{6} - \frac{C}{2k_o}\right)\mathcal{I}_{1,6}^n + \frac{\alpha}{t^{\frac{5}{6}}}C(k_o, m_3(0)), \quad 0 < t \leq T_n. \quad (6.27)$$

Now fix  $k_o \in \mathbb{N}$  and  $\alpha \in (0, 1]$  such that

$$\frac{C}{2k_o} \leq \frac{C_2}{4}, \quad \frac{\alpha^6}{6} \leq \frac{C_2}{4},$$

to conclude from (6.27) that

$$\frac{d}{dt}\mathcal{E}_1^n \leq \frac{C}{2k_o} - \frac{C_2}{2}\mathcal{I}_{1,6}^n + \frac{\alpha}{t^{\frac{5}{6}}}C(k_o, m_3(0)), \quad 0 < t \leq T_n. \quad (6.28)$$

Also observe that

$$\begin{aligned} \mathcal{I}_{1,6}^n &= \sum_{k=1}^n \mathcal{M}_{k+6}(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{k!} \\ &= \frac{1}{t\alpha^6} \sum_{k=7}^{n+6} \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{(k-6)!} \geq \frac{1}{t\alpha^6} \sum_{k=7}^n \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{k!} \\ &= \frac{1}{t\alpha^6} \mathcal{E}_1^n - \frac{1}{t\alpha^6} \sum_{k=1}^6 \mathcal{M}_k(t) \frac{(t^{\frac{1}{6}}\alpha)^k}{k!} \geq \frac{1}{t\alpha^6} \mathcal{E}_1^n - \frac{C(m_3(0))}{t^{\frac{5}{6}}\alpha^5}. \end{aligned}$$

Together with (6.28), this leads finally to

$$\frac{d}{dt}\mathcal{E}_1^n \leq \frac{C}{2k_o} + \frac{C(k_o, m_3(0))}{t^{\frac{5}{6}}\alpha^5} - \frac{C_2}{2t\alpha^6}\mathcal{E}_1^n, \quad 0 < t \leq T_n.$$

Thus, using a comparison principle for ode's, we can choose  $\alpha > 0$  sufficiently small, say

$$\alpha := C_2 \left[ \frac{C}{k_o} + 2C(k_o, m_3(0)) \right]^{-1}$$

to deduce that  $\mathcal{E}_1^n < t^{\frac{1}{6}}$ . That is,

$$\int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_1^n(t^{\frac{1}{6}} \alpha |p|) < t^{\frac{1}{6}}, \quad 0 \leq t \leq T_n.$$

Time continuity of  $\mathcal{E}_1^n$  and the maximality of  $T_n$  imply that  $T_n = 1$  for all  $n \geq 1$  and  $\vartheta \in (0, 1]$ . In particular, sending  $\vartheta \rightarrow 0$  and, then,  $n \rightarrow \infty$  one arrives to

$$\int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_1(t^{\frac{1}{6}} \alpha |p|) \leq t^{\frac{1}{6}}, \quad 0 \leq t \leq 1.$$

Furthermore, this estimate shows that

$$\int_{\mathbb{R}^3} dp f(1, p) \mathcal{E}_1(\alpha |p|) \leq 1.$$

Then, using Theorem 6.1, the exponential moment propagates for  $t > 1$ , and choosing  $\alpha > 0$  sufficiently small

$$\int_{\mathbb{R}^3} dp f(t, p) \mathcal{E}_1(\alpha |p|) \leq 1, \quad t \geq 1.$$

The result follows after noticing that

$$\mathcal{E}_1(t^{\frac{1}{6}} \alpha |p|) \geq t^{\frac{1}{6}} \alpha |p| e^{t^{\frac{1}{6}} \frac{\alpha}{2} |p|}, \quad 0 \leq t \leq 1.$$

■

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## 7 Appendix: Proof of Theorem 5.1

Our proof follows the same lines of the argument of Bressan's proof of Theorem A.1 in [14]. The proof is divided into three steps:

**Step 1.** Since  $\mathcal{S}$  is bounded, there exists a uniform bound  $C_Q$  of  $Q(u)$ , for all  $u$  in  $\mathcal{S}$ . Let  $\tau$  be in  $[0, \infty)$  and  $u$  be in  $\mathcal{S}$ , there exists  $h_{u,\tau} > 0$  such that for  $0 < h < h_{u,\tau}$  and for all  $\epsilon > 0$  sufficiently small,

- the intersection  $B(u+n(\tau)hQ(u), \epsilon) \cap \mathcal{S} \setminus \{u+n(\tau)hQ(u)\}$  is non-empty;
- and, from properties on  $n(t)$  as stated in Theorem 5.1,

$$|n(\tau + s) - n(\tau)| \leq \frac{\epsilon}{2C_Q}, \forall s \in [0, h]. \quad (7.1)$$

In addition, since  $n = n(t) \leq \bar{n}$ , then we can estimate  $n(\tau)\|Q(u) - Q(v)\| \leq \frac{\epsilon}{4}$ , if  $\|u - v\| \leq \bar{n}(C_Q + 1)h$ . Hence, take  $w$  to be a point inside  $B(u + n(\tau)hQ(u), \epsilon) \cap \mathcal{S} \setminus \{u + n(\tau)hQ(u)\}$  satisfying

$$\|w - u - n(\tau)hQ(u)\| \leq \frac{\epsilon h}{4}.$$

We consider the linear map

$$s \mapsto \rho(s) = u + \frac{s(w - u)}{h}, \quad s \in [0, h].$$

By the convexity of  $\mathcal{S}$ ,  $\rho(s) \in \mathcal{S}$  for all  $s$  in  $[0, h]$ . Moreover, since  $\dot{\rho}(s) = \frac{w-u}{h}$ ,

$$\|\dot{\rho}(s) - n(\tau)Q(u)\| \leq \frac{\epsilon}{4}.$$

Now, we can see that

$$\|\rho(s) - u\| = \left\| \frac{s(w - u)}{h} \right\| \leq \|w - u\| \leq n(\tau)h\|Q(u)\| + \frac{\epsilon h}{4} < \bar{n}(C_Q + 1)h,$$

which implies

$$n(\tau)\|Q(\rho(s)) - Q(u)\| \leq \frac{\epsilon}{4}, \quad \forall s \in [0, h].$$

Therefore,

$$\|\dot{\rho}(s) - n(\tau)Q(\rho(s))\| \leq \frac{\epsilon}{2}, \quad \forall s \in [0, h]. \quad (7.2)$$

Using (7.1), we deduce that

$$\|\dot{\rho}(s) - n(s)Q(\rho(s))\| \leq \epsilon, \quad \forall s \in [0, h]. \quad (7.3)$$

A consequence of this fact is that

$$\|\dot{\rho}(s)\| \leq 1 + \bar{n} C_Q \quad (7.4)$$

for all  $s$  in  $[0, h]$  and  $\epsilon < 1$ .

**Step 2.** From Step 1, we have proved the existence of solution  $\rho$  to the equation (7.3) on an interval  $[0, h]$ . From this solution, we carry on the following process.

- (1) We start with the solution  $\rho$ , defined on  $[0, h]$  of (7.3).
- (2) Suppose that the solution  $\rho$  of (7.3) is constructed on  $[0, \tau]$ . Since  $\rho(\tau) \in \mathcal{S}$ , by the same process as in Step 1, the solution  $\rho$  could be extended to  $[\tau, \tau + h_\tau]$ .
- (3) Suppose that the solution  $\rho$  of (7.3) is constructed on a series of intervals  $[0, \tau_1]$ ,  $[\tau_1, \tau_2]$ ,  $\dots$ ,  $[\tau_n, \tau_{n+1}]$ ,  $\dots$ . Moreover, suppose the increasing sequence  $\{\tau_n\}$  is bounded. Set

$$\tau = \lim_{n \rightarrow \infty} \tau_n.$$

Since  $G(\rho)$  is bounded by  $C_G$  on  $[\tau_n, \tau_{n+1}]$  for all  $n \in \mathbb{N}$ ,  $\dot{\rho}$  is bounded by  $\epsilon + C_G$  on  $[0, \tau]$ . Therefore, we can define  $\rho(\tau)$  satisfying

$$\rho(\tau) = \lim_{n \rightarrow \infty} \rho(\tau_n), \quad \dot{\rho}(\tau) = \lim_{n \rightarrow \infty} \dot{\rho}(\tau_n),$$

which implies that  $\rho$  is a solution of (7.3) on  $[0, \tau]$ .

By (3) of this process, we can see that if the solution  $\rho$ , constructed as above, is defined on  $[0, T)$ , it could be extended to  $[0, T]$ . Suppose that  $[0, T]$  is the maximal closed interval that  $\rho$  could be constructed, by Step 2 of the process,  $\rho$  could be extended to a larger interval  $[T, T + T_h]$ , which means that  $\rho$  can be constructed on the whole interval  $[0, \infty)$ .

**Step 3.** Let us now consider two sequences of approximate solutions  $u^\epsilon$ ,  $w^\epsilon$ , where  $\epsilon$  tends to 0. From Step 1 and Step 2, one can see that the time interval  $[0, T]$  can be decomposed into

$$\left( \bigcup_{\gamma} I_{\gamma} \right) \cup \mathfrak{N},$$

where  $I_\gamma$  are countably many open intervals and  $\mathfrak{N}$  is of measure 0. Taking the derivative of the difference  $\|u^\epsilon(t) - w^\epsilon(t)\|$  gives

$$\begin{aligned} \frac{d}{dt}\|u^\epsilon(t) - w^\epsilon(t)\| &= \left[ u^\epsilon - w^\epsilon, \dot{u}^\epsilon(t) - \dot{w}^\epsilon(t) \right]_- \\ &\leq \left[ u^\epsilon - w^\epsilon, \dot{u}^\epsilon(t) - \dot{w}^\epsilon(t) \right]_- + 2\epsilon \\ &\leq L\|u^\epsilon(t) - w^\epsilon(t)\| + 2\epsilon, \end{aligned}$$

which yields

$$\|u^\epsilon(t) - w^\epsilon(t)\| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0,$$

and we have the convergence  $u^\epsilon \rightarrow u$  uniformly on  $[0, T]$ . The function  $u$  is, then, a solution of our equation.

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